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The effect of tangential drifts on neoclassical transport in stellarators close to omnigeneity

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Abstract. In general, the orbit-averaged radial magnetic drift of trapped particles in stellarators is non-zero due to the three-dimensional geometry of the magnetic field. Stellarators in which the orbit-averaged radial magnetic drift vanishes are called omnigeneous, and they exhibit neoclassical transport levels comparable to those of axisymmetric tokamaks. However, the fulfilment of the omnigeneity condition requires such a precise alignment of the external coils that the effect of deviations from omnigeneity cannot be neglected in practice. The effect of such deviations is more deleterious at small collisionalities. In stellarator reactor conditions the ion collision frequency is expected to be sufficiently low (in particular below the values that define the $1/\nu$ regime) for the terms of the drift-kinetic equation involving the components of the drifts tangential to the flux surface to become relevant. This article focuses on the study of such collisionality regimes in stellarators close to omnigeneity. It is found that at those collisionality values transport is determined by two small collisional layers located at different regions of phase space. One of these layers corresponds to the so-called $\sqrt{\nu}$ regime and the other to the so-called superbanana-plateau regime. A formula for the ion energy flux that includes both regimes is given. Finally, we explain why below a certain collisionality value, that we estimate, new regimes can appear and it is expected that this formula will cease to be valid.

1. Introduction

Stellarators [1] present some intrinsic advantages with respect to tokamaks, such as the possibility of steady-state operation and the absence of disruptions. However, the magnetic configuration of a stellarator has to be designed very carefully for it to have confinement properties comparable to those of an axisymmetric tokamak. In a generic stellarator, trapped particle orbits have non-zero secular radial drifts and they leave the device in a short time. The stellarator is called omnigeneous [2, 3, 4] if the magnetic configuration is chosen so that the secular radial drifts vanish.

Omnigeneity guarantees that the neoclassical transport level of the stellarator is similar to that in a tokamak. Define the normalized ion Larmor radius $\rho_{i*} := v_{ti}/\Omega_i L_0$, where v_{ti} and Ω_i are the ion thermal speed and the ion gyrofrequency, and L_0 is the typical length of variation of the magnetic field, which is assumed to be of the order of the system size. The gyrofrequency is $\Omega_i = Z_i e B/(m_i c)$, where $Z_i e$ is the charge of the ions, e is the elementary charge, B is the magnitude of the magnetic field, m_i is the ion mass, and c is the speed of light. Since $\rho_{i*} \ll 1$ in a strongly magnetized plasma, the drift-kinetic formalism [5] is appropriate. To lowest order in ρ_{i*}, the phase-space distribution function $f_i(\mathbf{r}, \mathbf{v})$ is a Maxwellian f_{Mi} with density n_i and temperature T_i that are constant on flux surfaces.

If we denote by f_{i1} the $O(\rho_{i*} f_{Mi})$ perturbation to the Maxwellian distribution, i.e. $f_i = f_{Mi} + f_{i1} + O(\rho_i^{*2} f_{Mi})$, the radial ion energy flux Q_i reads

$$Q_i = \int \mathrm{d}^2 S \int \mathrm{d}^3 v \frac{m_i v^2}{2} \mathbf{v}_d \cdot \hat{\mathbf{n}}\, f_{i1}. \tag{1}$$

Here, $\mathbf{v}_d$ is the drift velocity, $\hat{\mathbf{n}}$ is the unit vector normal to the flux surface, $\mathbf{v}_d \cdot \hat{\mathbf{n}} \sim \rho_{i*} v_{ti}$, and the integrals are performed over velocity space and over the flux surface. In a perfectly omnigeneous stellarator [6]

$$Q_i \sim \nu_{i*} \rho_{i*}^2 n_i T_i v_{ti} S_\psi, \tag{2}$$

where $\nu_{i*} := \nu_{ii} L_0 / v_{ti}$ is the ion collisionality and ν_{ii} is the ion-ion collision frequency. The area of the flux surface is denoted by S_ψ, with ψ the radial coordinate.

The proof of Cary and Shasharina [2, 3] for the existence of omnigeneous magnetic fields implies, at the end of the day, that exact omnigeneity throughout the plasma requires non-analiticity. Let us explain this in more detail. As shown in references [2] and [3], there exist omnigeneous magnetic fields that are analytic. These configurations coincide with the set of quasisymmetric magnetic fields [7, 8]. To the virtues of omnigeneity, quasisymmetry adds the vanishing of neoclassical damping in the quasisymmetric direction. Therefore, in quasisymmetric stellarators larger flow velocities can be attained. In principle, this makes the stellarator plasma prone to develop large flow shear, that is known to reduce turbulent transport [9]. However, the quasisymmetry condition is incompatible with the magnetohydrodynamic equilibrium equations in the whole plasma [10], and the stellarator can be made quasisymmetric only in a limited radial region.

The mathematical obstructions to achieve quasisymmetry do not exist for omnigeneity. This is why we said above that a necessary condition for exact omnigeneity is non-analiticity; specifically, the discontinuity of some derivatives of second or higher order. However, designing and aligning coils that create a magnetic field with discontinuous derivatives at certain points in space is probably technically impossible.

Therefore, even in optimized magnetic fields, the effect of deviations from the desired omnigeneous configuration cannot be neglected. It is therefore necessary to study magnetic fields of the form $\mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}_1$, where $\mathbf{B}_0$ is omnigeneous and $\delta\mathbf{B}_1$ is a perturbation, with $0 \le \delta \ll 1$ and $\mathbf{B}_1 \sim \mathbf{B}_0$. The effect of these deviations is more dangerous at small collisionalities, when the details of particle orbits are more relevant. If $\nu_{i*} \ll 1$ and the stellarator is non-omnigeneous, the non-omnigeneous piece of f_{i1} becomes large, so that $f_{i1} \gg \rho_{i*} f_{Mi}$ and the energy flux can be much larger than the estimation (2) even if δ is small. The quantification of this effect for

$$\rho_{i*} \ll \nu_{i*} \ll 1, \tag{3}$$

that defines the $1/\nu$ regime, has been the subject of [11, 12, 13] for stellarators close to quasisymmetry‡ and is the subject of [14] for stellarators close to omnigeneity. However, this regime does not exhaust the low collisionality parameter space in stellarators. When

$$\nu_{i*} \lesssim \rho_{i*} \tag{4}$$

the terms of the drift-kinetic equation involving the components of the drifts tangential to the flux surface count [1, 15]. In this paper we begin the study of stellarators close to omnigeneity in the collisionality regime (4), which is actually relevant because in a stellarator reactor the ions can have such low collisionalities.

It is important to point out that the calculations in this paper do not rely on large aspect ratio approximations. Of course, if the stellarator close to omnigeneity under consideration has large aspect ratio, one can perform a subsidiary expansion in the inverse aspect ratio and refine the results obtained in what follows, and also the definitions of the collisionality regimes (3) and (4). This will be the subject of future work.

The rest of the paper is organized as follows.

In Section 2 we introduce a set of flux coordinates that is well-adapted to stellarator magnetic geometries. Then, we give the formal definition of omnigeneity.

In Section 3 we derive, starting from the complete drift-kinetic equation, the equation for the dominant component of the distribution function when $\delta \ll 1$ and $\nu_{i*} \lesssim \rho_{i*}$. In particular, we explain why the standard drift-kinetic expansion in ρ_{i*} breaks down for a generic stellarator when $\nu_{i*} \ll \rho_{i*}$. In brief, the reason is that f_{i1} becomes so large that $f_{i1} \sim f_{Mi}$. For stellarators close to omnigeneity, however, the ρ_{i*} expansion still makes sense when $\nu_{i*} \ll \rho_{i*}$ due to the presence of the additional small parameter δ. In addition, in a generic stellarator the drift-kinetic equation becomes

‡ Stellarators close to quasisymmetry have been studied in different regimes in [11, 12, 13], but with a special focus on the $1/\nu$ regime.

radially non-local when $\nu_{i*} \lesssim \rho_{i*}$. We will see that the condition $\delta \ll 1$ allows to derive a radially local drift-kinetic equation in this collisionality regime.

A precision must be made about the asymptotic expansion in δ carried out in this paper. When $\rho_{i*} \ll \nu_{i*} \ll 1$, it has been understood (in [11, 12, 13] for stellarators close to quasisymmetry and in [14] for stellarators close to omnigeneity) that the effect of the deviations (from quasisymmetry or omnigeneity) is very different depending on the size of the gradients on the surface of the magnetic field perturbation. For the regime (4), the case of deviations with small gradients and the case of deviations with large gradients also require different treatments, in principle. Here, we restrict to deviations with small gradients, and this restriction includes the expansion in $\delta \ll 1$ of Section 3. Let us be more specific. If $B_0 := |\mathbf{B}_0|$ and $B_1 := |\mathbf{B}_1|$, by 'small gradients' we mean that the characteristic variation length of B_0 and B_1 along the directions tangent to the flux surface is L_0.

In Section 4, the equation derived in Section 3 for the non-omnigeneous piece of the distribution function is solved when $\nu_{i*} \ll \rho_{i*}$. We find that Q_i is dominated by two collisional layers in phase space. One of the layers lies at the boundary between trapped and passing trajectories and produces an energy flux

$$Q_i \sim \delta^2 \frac{\nu_{ii}^{1/2}}{\omega_\alpha^{3/2}} \rho_{i*}^2 n_i T_i v_{ti}^2 L_0^{-1} S_\psi, \tag{5}$$

where ω_α, defined in Section 4, is the precession frequency in the direction α due to the tangential drifts. The other layer lies at the points of phase space where ω_α vanishes and yields Q_i independent of ν_{i*}. Namely,

$$Q_i \sim \delta^2 \rho_{i*} n_i T_i v_{ti} S_\psi. \tag{6}$$

The first layer is an analogue of the so-called $\sqrt{\nu}$ regime, found in certain models of stellarator geometry [16] where the inverse aspect ratio and the helical ripple are employed as expansion parameters. The second layer is an analogue of the superbanana-plateau regime, derived in [17] for finite aspect ratio tokamaks with broken symmetry. We will also use this nomenclature for the regimes giving (5) and (6). Typically, the contribution of one of the layers is much larger than the contribution of the other, but the formula for Q_i is additive in both regions of phase space§ and a general expression embracing both regimes can be provided.

In Section 5 we use the quasineutrality equation and the results of previous sections to give the equations that allow to solve for the radial electric field and for the lowest-order contribution to the electric field tangent to the flux surface. The potential importance of the latter for impurity transport is pointed out, for example, in [18, 19].

Finally, in Section 6, we point out that the results of Section 4 are not valid for arbitrarily small ν_{i*}. For each δ, there exists a value of the collisionality $\nu_{\delta*}$ such that if $\nu_{i*} < \nu_{\delta*}$ the collisional layers are not responsible anymore for the dominant contribution to Q_i. We explain this and estimate $\nu_{\delta*}$.

In Section 7 we summarize the conclusions of the paper.

§ Cases in which the two layers overlap are not treated here.

2. Omnigeneous stellarators

Throughout the paper, we deal with stellarators whose magnetic field configurations possess nested flux surfaces. In the first place, we define the spatial coordinates $\{\psi, \alpha, l\}$, adapted to the magnetic field, that will be employed. The coordinate ψ determines the flux surface, whereas α is an angular coordinate that labels a magnetic field line once ψ has been fixed. Finally l, the arc length over the magnetic field line, specifies the position along the line for fixed ψ and α. Denote by $\psi(\mathbf{r})$, $\alpha(\mathbf{r})$ and $l(\mathbf{r})$ the functions giving the value of these coordinates for each point $\mathbf{r}$ in the stellarator. The magnetic field can be written as

$$\mathbf{B} = \Psi_t'(\psi)\nabla\psi \times \nabla\alpha. \tag{7}$$

Here, Ψ_t is the toroidal magnetic flux over 2π and primes stand for differentiation with respect to ψ. In order to have unique pairs (α, l) associated to each point on a flux surface, we choose a curve $\mathcal{C}$ that closes poloidally‖. This curve can be parameterized by α. All points on the curve are assigned, by definition, the value $l = 0$. For each pair ψ and α, we take $l \in [0, L(\psi, \alpha)]$, where $L(\psi, \alpha)$ is found by integrating from $\mathcal{C}$ along the line until the curve $\mathcal{C}$ is hit again.

Let v be the magnitude of the velocity and $\lambda = v_\perp^2/(v^2 B)$ the pitch angle. Given a flux surface determined by ψ, particles are passing or trapped depending on the value of λ. Passing trajectories have $\lambda < 1/B_{\rm max}(\psi)$, where $B_{\rm max}(\psi)$ is the maximum value of B on the flux surface. Passing particles explore the entire flux surface and always have vanishing average radial magnetic drift. Particles with $\lambda > 1/B_{\rm max}(\psi)$ are trapped. For trapped particles, the radial magnetic drift averaged over the orbit is non-zero in a generic stellarator. A stellarator is called omnigeneous if the orbit-averaged radial magnetic drift is zero for all particles [2, 3, 4]. That is, if and only if the second adiabatic invariant $J = 2\int_{l_{b1}}^{l_{b2}} |v_{||}| \mathrm{d}l$ is a flux function, which means that

$$\partial_\alpha \int_{l_{b1}}^{l_{b2}} \sqrt{1 - \lambda B}\,\mathrm{d}l = 0 \tag{8}$$

must hold for every trapped trajectory. Here l_{b1} and l_{b2} are the bounce points, the solutions for l of $1 - \lambda B(\psi, \alpha, l) = 0$ for a particular trapped trajectory. Since (8) has to be satisfied for every λ, we can equivalently define omnigeneity by requiring that

$$\partial_\alpha \int_{l_{b1}}^{l_{b2}} \Lambda(\psi, v, \lambda, B(\psi, \alpha, l))\mathrm{d}l = 0 \tag{9}$$

for any function Λ that depends on α and l only through B. We will make use of this definition of omnigeneity several times along the article.

‖ To fix ideas one can think of α as a poloidal angle, but things work analogously if α has a different helicity.

3. Low-collisionality drift-kinetic equation in stellarators close to omnigeneity

As we said in the Introduction, the smallness of ρ_{i*} allows to employ the drift-kinetic approach [5, 20, 11]. It consists of a systematic way to average, order by order in ρ_{i*}, over the fast gyration of particles around magnetic field lines. This is achieved by finding a coordinate transformation on phase space that decouples the gyromotion from the comparatively slow motion in the other directions. The new coordinates are called drift-kinetic coordinates.

The form of the drif-kinetic equation is determined by the transformation from coordinates $\{\mathbf{r}, \mathbf{v}\}$ to drift-kinetic coordinates¶ $\{\mathbf{R}, \mathcal{E}, \mu, \sigma, \gamma\}$, where $\mathbf{R}$ is the position of the guiding center, $\mathcal{E}$ is the total energy per mass unit, μ is the magnetic moment, σ is the sign of the parallel velocity and γ is the gyrophase. Namely,

$$\begin{aligned}
&\mathbf{R} = \mathbf{r} - \frac{1}{\Omega_i}\hat{\mathbf{b}} \times \mathbf{v} + O(\rho_i^{*2} L_0),\\
&\mathcal{E} = v^2/2 + Z_i e\varphi/m_i,\\
&\mu = \frac{1}{2B}(v^2 - (\mathbf{v}\cdot\hat{\mathbf{b}})^2) + O(\rho_{i*} v_{ti}^2/B),\\
&\gamma = \arctan\left(\mathbf{v}\cdot\hat{\mathbf{e}}_2/\mathbf{v}\cdot\hat{\mathbf{e}}_1\right) + O(\rho_{i*}),
\end{aligned} \tag{10}$$

where $\hat{\mathbf{b}} = B^{-1}\mathbf{B}$, the right sides of the previous expressions are evaluated at $\mathbf{r}$ and φ is the electrostatic potential. The orthogonal unit vector fields $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ satisfy at each point $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{b}}$. The higher-order corrections in the definition of μ are determined by the fact that μ is the adiabatic invariant corresponding to the ignorable coordinate γ. Finally, $\sigma = v_{||}/|v_{||}|$ gives the sign of the parallel velocity, where the latter is viewed as a function of the other coordinates through the expression

$$v_{||} = \sigma\sqrt{2\left(\mathcal{E} - \mu B - \frac{Z_i e\varphi}{m_i}\right)}\ . \tag{11}$$

Denote by $F_i(\psi(\mathbf{R}), \alpha(\mathbf{R}), l(\mathbf{R}), \mathcal{E}, \mu, \sigma)$ the distribution function in drift-kinetic coordinates. We assume from the beginning that our distribution function does not depend on the gyrophase, which is true for all the calculations in this paper (see [11] for the proof that only pieces of the distribution function $O(\rho_i^{*2} f_{Mi})$ or smaller are gyrophase dependent). In these coordinates the drift-kinetic equation reads

$$\dot{\mathbf{R}}\cdot\nabla F_i = C_{ii}^{\mathcal{E}}[F_i, F_i]. \tag{12}$$

Here,

$$\dot{\mathbf{R}}\cdot\hat{\mathbf{b}}\hat{\mathbf{b}} = v_{||}\hat{\mathbf{b}} + O(\rho_{i*} v_{ti}) \tag{13}$$

¶ Even though we will end up employing the coordinates v and λ defined in Section 2, it is conceptually advisable to start using as independent coordinates the total energy per mass unit $\mathcal{E}$ and the magnetic moment μ.

and

$$\dot{\mathbf{R}} - \dot{\mathbf{R}}\cdot\hat{\mathbf{b}}\hat{\mathbf{b}} = \mathbf{v}_{M,i} + \mathbf{v}_E + O(\rho_i^{*2} v_{ti}), \tag{14}$$

with

$$\mathbf{v}_{M,i} = \frac{1}{\Omega_i}\hat{\mathbf{b}}\times\left(v_{||}^2\boldsymbol{\kappa} + \mu\nabla B\right), \tag{15}$$

$$\mathbf{v}_E = \frac{c}{B}\hat{\mathbf{b}}\times\nabla\varphi \tag{16}$$

and $\boldsymbol{\kappa} := \hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}$.

In (13) and (14) we have only shown the terms that will be needed later on. All the terms of $\dot{\mathbf{R}}$ up to $O(\rho_i^{*2} v_{ti})$ have been computed in [20]. In (12), an expansion in the mass ratio $\sqrt{m_e/m_i} \ll 1$ has been taken so that ion-electron collisions are neglected, and $C_{ii}^{\mathcal{E}}$ is the ion-ion Landau collision operator written in coordinates $\mathcal{E}$ and μ. Its explicit expression (see [21], for example) is not necessary for our purposes. From now on, we concentrate on ion transport.

We write

$$F_i(\psi,\alpha,l,\mathcal{E},\mu,\sigma) = F_{i0}^{\mathcal{E}}(\psi,\mathcal{E}) + F_{i1}(\psi,\alpha,l,\mathcal{E},\mu,\sigma) + O(\rho_i^{*2}F_{i0}^{\mathcal{E}}) \tag{17}$$

with $F_{i1}\sim\rho_{i*}F_{i0}^{\mathcal{E}}$, and we also expand the electrostatic potential,

$$\varphi = \varphi_0 + \varphi_1 + O(\rho_i^{*2}\varphi_0), \tag{18}$$

where $\varphi_0 \sim T_i/Z_i e$ and $\varphi_1\sim\rho_{i*}\varphi_0$. To lowest order in ρ_{i*} equation (12) imposes φ_0 to be a flux function [22] and

$$F_{i0}^{\mathcal{E}}(\psi,\mathcal{E}) = n_i(\psi)\left(\frac{m_i}{2\pi T_i(\psi)}\right)^{3/2}\exp\left(-\frac{m_i\mathcal{E} - Z_i e\varphi_0(\psi)}{T_i(\psi)}\right). \tag{19}$$

To $O(\rho_{i*}v_{ti}L_0^{-1}F_{i0}^{\mathcal{E}})$ equation (12) gives

$$v_{||}\partial_l F_{i1} + \Upsilon_i^{\mathcal{E}}\mathbf{v}_{M,i}\cdot\nabla\psi F_{i0}^{\mathcal{E}} = C_{ii}^{\mathcal{E},\ell}[F_{i1}], \tag{20}$$

where

$$\Upsilon_i^{\mathcal{E}} = \frac{n_i'}{n_i} + \frac{T_i'}{T_i}\left(\frac{m_i\mathcal{E} - Z_i e\varphi_0}{T_i} - \frac{3}{2}\right) + \frac{Z_i e\varphi_0'}{T_i} \tag{21}$$

and $C_{ii}^{\mathcal{E},\ell}$ is the linearization of $C_{ii}^{\mathcal{E}}$ around $F_{i0}^{\mathcal{E}}$. Namely,

$$C_{ii}^{\mathcal{E},\ell}[F_{i1}] = C_{ii}^{\mathcal{E}}[F_{i1},F_{i0}^{\mathcal{E}}] + C_{ii}^{\mathcal{E}}[F_{i0}^{\mathcal{E}},F_{i1}]. \tag{22}$$

3.1. Drift-kinetic equation when $\rho_{i*} \ll \nu_{i*} \ll 1$

If the collisionality is small, $\nu_{i*}\ll 1$, one can perform an expansion in ν_{i*} that is subsidiary with respect to the expansion in ρ_i^*. We write

$$F_{i1} = F_{i1}^{[-1]} + F_{i1}^{[0]} + O(\nu_{i*}\rho_{i*}F_{i0}), \tag{23}$$

where $F_{i1}^{[j]}\sim\nu_{i*}^j\rho_{i*}F_{i0}$. To lowest order in the ν_{i*} expansion, (20) yields

$$v_{||}\hat{\mathbf{b}}\cdot\nabla F_{i1}^{[-1]} = 0. \tag{24}$$

This implies that on an ergodic flux surface $F_{i1}^{[-1]}$ can be written as

$$F_{i1}^{[-1]} = H_i(\psi, \mathcal{E}, \mu, \sigma) + g_i(\psi, \alpha, \mathcal{E}, \mu), \qquad (25)$$

where g_i can be chosen such that it vanishes in the passing region of phase space and such that

$$\int_0^{2\pi} g_i \, \mathrm{d}\alpha = 0. \qquad (26)$$

The functions H_i and g_i are found from (20) to next order in ν_{i*},

$$v_{||}\hat{\mathbf{b}} \cdot \nabla F_{i1}^{[0]} + +\Upsilon_i^{\mathcal{E}} \mathbf{v}_{M,i} \cdot \nabla\psi F_{i0}^{\mathcal{E}} = C_{ii}^{\mathcal{E},\ell}[F_{i1}^{[-1]}]. \qquad (27)$$

For passing particles, we multiply (27) by $1/v_{||}$ and integrate over α and l, obtaining

$$\int_0^{2\pi} \mathrm{d}\alpha \int_0^{L(\psi,\alpha)} \mathrm{d}l \frac{1}{|v_{||}|} C_{ii}^{\mathcal{E},\ell}[F_{i1}^{[-1]}] = 0. \qquad (28)$$

For trapped particles, we multiply (27) by $1/v_{||}$ and integrate over the lowest order orbit. We get

$$\sum_\sigma \int_{l_{b1}}^{l_{b2}} \frac{1}{|v_{||}|} C_{ii}^{\mathcal{E},\ell}[F_{i1}^{[-1]}] \, \mathrm{d}l = \left(\sum_\sigma \int_{l_{b1}}^{l_{b2}} \frac{1}{|v_{||}|} \mathbf{v}_{M,i} \cdot \nabla\psi \, \mathrm{d}l \right) \Upsilon_i^{\mathcal{E}} F_{i0}^{\mathcal{E}}. \qquad (29)$$

Equations (28) and (29) allow to determine H_i and g_i in the so called $1/\nu$ regime.

The electrostatic potential φ_1 is found from the quasineutrality equation, which to lowest order in $\sqrt{m_e/m_i} \ll 1$ reads

$$\left(\frac{Z_i}{T_i} + \frac{1}{T_e} \right) \varphi_1 = \frac{1}{en_i} \int F_{i1} \, \mathrm{d}v^3, \qquad (30)$$

with $\mathrm{d}^3 v \equiv \sum_\sigma B/|v_{||}| \mathrm{d}\mathcal{E}\mathrm{d}\mu\mathrm{d}\gamma$. Here, T_e is the electron temperature. From (30) and the fact that $F_{i1} \sim \nu_{i*}^{-1}\rho_{i*}F_{i0}$, one realizes that $\varphi_1 \sim \nu_{i*}^{-1}\rho_{i*}T_i/e$. The $1/\nu$ regime exists for any stellarator. The analysis of the $1/\nu$ regime in stellarators close to omnigeneity is carried out in depth in reference [14]. However, we explain in the next subsection that the $1/\nu$ regime is restricted to the range $\rho_{i*} \ll \nu_{i*} \ll 1$, and that it ceases to exist for values of the collisionality $\nu_{i*} \lesssim \rho_{i*}$.

3.2. Drift-kinetic equation when $\nu_{i*} \lesssim \rho_{i*}$ in stellarators close to omnigeneity

In this paper, we would like to understand what happens at values of the collisionality below the $1/\nu$ regime. It is easy to realize that as long as ν_{i*} becomes comparable to ρ_{i*}, the ρ_{i*} expansion breaks down because F_{i1} becomes as large as F_{i0} and φ_1 becomes as large as φ_0. Equivalently, some terms that are nominally $O(\rho_i^{*2})$ in (12) cannot be neglected. This is an important point: in a generic stellarator the drift-kinetic expansion does not make sense when $\nu_{i*} \lesssim \rho_{i*}$. Furthermore, the drift-kinetic equation becomes radially non-local because there is no reason, in principle, to drop terms like $\mathbf{v}_{M,i}\cdot\nabla\psi\partial_\psi F_i$ in (12). That is, instead of (20), in a generic stellarator one should solve

$$(\mathbf{v}_{M,i} + \mathbf{v}_E) \cdot \nabla\psi\partial_\psi F_{i1} + (\mathbf{v}_{M,i} + \mathbf{v}_E) \cdot \nabla l \partial_l F_{i1}$$
$$+(\mathbf{v}_{M,i} + \mathbf{v}_E) \cdot \nabla\alpha\partial_\alpha F_{i1} + v_{||}\partial_l F_{i1} + \Upsilon_i^{\mathcal{E}}(\mathbf{v}_{M,i} + \mathbf{v}_E) \cdot \nabla\psi F_{i0}^{\mathcal{E}} = C_{ii}^{\mathcal{E},\ell}[F_{i1}] \qquad (31)$$

when $\nu_{i*} \lesssim \rho_{i*}$. However, we show next that when the stellarator is close to omnigeneity the drift-kinetic expansion is still meaningful and the derived drift-kinetic equation is radially local, all due to the existence of an additional small parameter, δ, in the theory.

We assume that the non-omnigeneous piece of the distribution function dominates transport. If

$$\delta\partial_l B_1 \sim \delta L_0^{-1} B_0,$$
$$\delta\partial_\alpha B_1 \sim \delta B_0, \tag{32}$$

it is proven in [14] that an expansion in integer powers of δ is consistent. If the perturbation B_1 has large helicities so that $\delta\partial_l B_1 \sim L_0^{-1}B_0$ or $\delta\partial_\alpha B_1 \sim B_0$, the asymptotic expansion in δ of the drift kinetic equation is remarkably more complicated. This kind of perturbations will not be treated in this paper, but left for future work. Then, we assume (32) and define the expansions

$$F_{i1} = \delta F_i^{(1)} + \ldots \tag{33}$$

and

$$\varphi_1 = \delta\varphi^{(1)} + \ldots \tag{34}$$

The equation derived in this subsection will be valid for stellarators close to omnigeneity in the $1/\nu$ regime and also when $\nu_{i*} \lesssim \rho_{i*}$. The sizes of $F_i^{(1)}$ and $\varphi^{(1)}$ are given by $F_i^{(1)} \sim \nu_{i*}^{-1}\rho_{i*}F_{i0}$ and $\varphi^{(1)} \sim \nu_{i*}^{-1}\rho_{i*}\varphi_0$ in the $1/\nu$ regime. When $\nu_{i*} \lesssim \rho_{i*}$ the correct ansatz is $F_{i1}^{(1)} \sim F_{i0}$ and $\varphi^{(1)} \sim \varphi_0$, as will be shown in Section 4.

The largest term in (31) is the parallel streaming. Imposing $\partial_l F_i^{(1)} = 0$ implies

$$F_i^{(1)} = H_i^{(1)}(\psi, \mathcal{E}, \mu, \sigma) + g_i^{(1)}(\psi, \alpha, \mathcal{E}, \mu), \tag{35}$$

where $g_i^{(1)}$ can be chosen such that it vanishes in the passing region of phase space and such that

$$\int_0^{2\pi} g_i^{(1)} \mathrm{d}\alpha = 0. \tag{36}$$

The $O(\delta)$ piece of the second term in (31) equals zero due to (35). In order to find an equation for $F_i^{(1)}$ we take averages of the remaining terms in (31) expanded to $O(\delta)$. For passing particles we multiply (31) by $1/|v_{||}^{(0)}|$ and integrate on the flux surface, obtaining

$$\int_0^{2\pi} \mathrm{d}\alpha \int_0^{L(\psi,\alpha)} \frac{1}{|v_{||}^{(0)}|} C_{ii}^{\mathcal{E},\ell(0)}[H_i^{(1)}]\mathrm{d}l = 0, \tag{37}$$

where

$$v_{||}^{(0)}(\psi, \alpha, l, \mathcal{E}, \mu) = \sigma\sqrt{2\left(\mathcal{E} - \mu B_0(\psi, \alpha, l) - \frac{Z_i e}{m_i}\varphi_0(\psi)\right)} \tag{38}$$

and in $C_{ii}^{\mathcal{E},\ell(0)}$ the superindex (0) indicates that only B_0 has been kept in the kernel that defines the linearized collision operator. In order to get (37) we have used that in the passing region $F_i^{(1)} = H_i^{(1)}$, and is therefore independent of α. We have also employed that for passing trajectories

$$\int_0^{2\pi} \mathrm{d}\alpha \int_0^{L(\psi,\alpha)} \frac{1}{|v_{||}^{(0)}|}(\mathbf{v}_{M,i} + \mathbf{v}_E) \cdot \nabla\psi \mathrm{d}l = 0 \tag{39}$$

and finally the fact that

$$\int_0^{2\pi} \mathrm{d}\alpha \int_0^{L(\psi,\alpha)} \frac{1}{|v_{||}^{(0)}|} C_{ii}^{\mathcal{E},\ell(0)}[g_i^{(1)}]\mathrm{d}l = 0 \tag{40}$$

due to (36).

For trapped particles we multiply (31) by $1/v_{||}^{(0)}$ and integrate over the orbit, arriving at

$$-\partial_\psi J^{(0)} \partial_\alpha F_i^{(1)} + \partial_\alpha J^{(1)} \Upsilon_i^{\mathcal{E}} F_{i0}^{\mathcal{E}} = \sum_\sigma \frac{Z_i e \Psi_t'}{m_i c} \int_{l_{b10}}^{l_{b20}} \frac{\mathrm{d}l}{|v_{||}^{(0)}|} C_{ii}^{\mathcal{E},\ell(0)}[F_i^{(1)}]. \tag{41}$$

Equation (41) has conveniently been expressed in terms of the second adiabatic invariant J, defined by

$$J(\psi, \alpha, \mathcal{E}, \mu) := 2\int_{l_{b_1}}^{l_{b_2}} |v_{||}|\mathrm{d}l. \tag{42}$$

For that, we have employed the relations

$$2\int_{l_{b_1}}^{l_{b_2}} \frac{1}{|v_{||}|}(\mathbf{v}_{M,i} + \mathbf{v}_{E,0} + \mathbf{v}_{E,1}) \cdot \nabla\psi \,\mathrm{d}l = \frac{m_i c}{Z_i e \Psi_t'} \partial_\alpha J \tag{43}$$

and

$$2\int_{l_{b_1}}^{l_{b_2}} \frac{1}{|v_{||}|}(\mathbf{v}_{M,i} + \mathbf{v}_{E,0} + \mathbf{v}_{E,1}) \cdot \nabla\alpha \,\mathrm{d}l = -\frac{m_i c}{Z_i e \Psi_t'} \partial_\psi J, \tag{44}$$

that are derived in Appendix A, and we have expanded J as

$$J = J^{(0)} + \delta J^{(1)} + \ldots, \tag{45}$$

with

$$J^{(0)} = 2\int_{l_{b_{10}}}^{l_{b_{20}}} |v_{||}|^{(0)}\mathrm{d}l,$$

$$J^{(1)} = -2\int_{l_{b_{10}}}^{l_{b_{20}}} \frac{1}{|v_{||}|^{(0)}} \left(\mu B_1(\alpha, l) + \frac{Z_i e}{m_i}\varphi_1(\alpha, l)\right) \mathrm{d}l. \tag{46}$$

Here $l_{b_{10}}$ and $l_{b_{20}}$ are the bounce points for the orbits of $\mathbf{B}_0$; i.e. the solutions of $1 - \lambda B_0(\psi, \alpha, l) = 0$. In order to write (41) we have used that $\partial_\alpha J^{(0)} = 0$ because $\mathbf{B}_0$ is omnigeneous (recall the definition (8)).

Now, we integrate (41) over α, which gives

$$\sum_\sigma \int_0^{2\pi} \mathrm{d}\alpha \int_{l_{b10}}^{l_{b20}} \frac{1}{|v_{||}^{(0)}|} C_{ii}^{\mathcal{E},\ell(0)}[H_i^{(1)}]\mathrm{d}l = 0. \tag{47}$$

Here, we have used that $\sum_\sigma \int_{l_{b10}}^{l_{b20}} |v_{||}^{(0)}|^{-1} C_{ii}^{\mathcal{E},\ell(0)}[\,\cdot\,]\mathrm{d}l$ is an operator with coefficients independent of α when acting on functions independent of l, as can be deduced by invoking the definition of omnigeneity (9). Multiplying (37) and (47) by $-H_i^{(1)}/F_{i0}$, integrating over velocity space and applying an entropy-production argument, we find that $H_i^{(1)}$ has to be a Maxwellian distribution with zero flow. Thus, it can be absorbed in the definition of F_{i0} and, from here on, we can assume $H_i^{(1)} \equiv 0$ without loss of generality.

Then, we only need to determine $g_i^{(1)}$, which is found from (41) by setting $H_i^{(1)}$ equal to zero. Namely,

$$-\partial_\psi J^{(0)} \partial_\alpha g_i^{(1)} + \partial_\alpha J^{(1)} \Upsilon_i^{\mathcal{E}} F_{i0}^{\mathcal{E}} = \sum_\sigma \frac{Z_i e \Psi_t'}{m_i c} \int_{l_{b10}}^{l_{b20}} \frac{\mathrm{d}l}{|v_{||}^{(0)}|} C_{ii}^{\mathcal{E},\ell(0)}[g_i^{(1)}], \tag{48}$$

It is obvious, but still worth pointing out, that when $\rho_{i*} \ll \nu_{i*} \ll 1$ the first term in (48) can be neglected and one recovers the equation for the dominant piece of the distribution function in the $1/\nu$ regime in a stellarator close to omnigeneity where the non-omnigeneous perturbation to the omnigeneous magnetic field has small gradients.

Note that the orbit integrations in (46) and (48) only involve B_0 and φ_0. In particular, the integrations are actually taken by keeping constant the kinetic energy $m_i v^2/2$ instead of the total energy $m_i\mathcal{E}$ (recall that φ_0 is a flux function and ψ does not change along the integration path). This is a consequence of assumption (32), that allowed to expand the drift-kinetic equation in integer powers of δ, and it turned out that to lowest order only the omnigeneous orbits count. In general (that is, if B_1 has large helicities), however, the modifications of the orbits by φ_1 might be non-negligible, and the distinction between integrating keeping constant $\mathcal{E}$ or keeping $v^2/2$ might be important. A detailed discussion on the asymptotic expansion of the second adiabatic invariant in a stellarator close to omnigeneity is provided in reference [23]. As we have said above, we restrict ourselves to perturbations satisfying (32).

In what follows we employ the more common coordinates

$$\begin{aligned} v &= \sqrt{2(\mathcal{E} - Z_i e\varphi_0/m_i)}, \\ \lambda &= \frac{\mu}{\mathcal{E} - Z_i e \varphi_0/m_i}. \end{aligned} \tag{49}$$

We will not change the names of the functions $v_{||}^{(0)}$, $\partial_\psi J^{(0)}$, $\partial_\alpha J^{(1)}$ and $g_i^{(1)}$ but assume that they are expressed in coordinates v and λ. Let us be explicit to avoid any confusion. From now on, by $\partial_\psi J^{(0)}$ and $\partial_\alpha J^{(1)}$ we understand

$$\partial_\psi J^{(0)} = -\int_{l_{b_{10}}}^{l_{b_{20}}} \frac{\lambda v \partial_\psi B_0 + 2Z_i e/(m_i v)\partial_\psi \varphi_0}{\sqrt{1-\lambda B_0}} \mathrm{d}l, \tag{50}$$

$$\partial_\alpha J^{(1)} = -\int_{l_{b_{10}}}^{l_{b_{20}}} \frac{\lambda v \partial_\alpha B_1 + 2Z_i e/(m_i v)\partial_\alpha \varphi_1}{\sqrt{1-\lambda B_0}} \mathrm{d}l. \tag{51}$$

In these coordinates, the equation for $g_i^{(1)}$ reads

$$-\partial_\psi J^{(0)} \partial_\alpha g_i^{(1)} + \partial_\alpha J^{(1)} \Upsilon_i F_{i0} = \sum_\sigma \frac{Z_i e \Psi_t'}{m_i c} \int_{l_{b10}}^{l_{b20}} \frac{\mathrm{d}l}{|v_{||}^{(0)}|} C_{ii}^{\ell(0)}[g_i^{(1)}], \tag{52}$$

where

$$F_{i0}(\psi, v) = n_i(\psi)\left(\frac{m_i}{2\pi T_i(\psi)}\right)^{3/2} \exp\left(-\frac{m_i v^2}{2T_i(\psi)}\right), \tag{53}$$

$$\Upsilon_i = \frac{n_i'}{n_i} + \frac{T_i'}{T_i}\left(\frac{m_i v^2}{2T_i} - \frac{3}{2}\right) + \frac{Z_i e \varphi_0'}{T_i} \tag{54}$$

and $C_{ii}^{\ell(0)}$ is the linearized collision operator corresponding to B_0 written in coordinates v and λ.

The energy flux can be written as

$$Q_i = \frac{\pi m_i^2 c \delta^2}{2Z_i e}\int_0^\infty \mathrm{d}v v^5 \int_{1/B_{0,\max}}^{1/B_{0,\min}} \mathrm{d}\lambda \int_0^{2\pi} \mathrm{d}\alpha\, \partial_\alpha J^{(1)}\, g_i^{(1)}, \tag{55}$$

where $B_{0,\min}$ and $B_{0,\max}$ are the minimum and maximum values of B_0 on the flux surface, respectively. In Section 4 we solve (52) when $\nu_{i*} \ll \rho_{i*}$ and give the expressions that allow to compute (55) in such collisionality regimes.

4. Solution of the drift-kinetic equation (52) when $\nu_{i*} \ll \rho_{i*}$

Let us define the precession frequency

$$\omega_\alpha(\psi, v, \lambda) := \frac{m_i c}{Z_i e \Psi_t' \tau_b^{(0)}} \partial_\psi J^{(0)}, \tag{56}$$

where

$$\tau_b^{(0)}(\psi, v, \lambda) = \frac{2}{v}\int_{l_{b_{10}}}^{l_{b_{20}}} \frac{\mathrm{d}l}{\sqrt{1 - \lambda B_0(\psi, \alpha, l)}} \tag{57}$$

is the time that a particle trapped in $\mathbf{B}_0$ takes to complete its orbit. Note that $\tau_b^{(0)}$ does not depend on α due to property (9), and therefore ω_α is also independent of α.

Typically, $\omega_\alpha \sim \rho_{i*} v_{ti}/L_0$, and equation (52) is solved by expanding in $\nu_{ii}/\omega_\alpha \sim \nu_{i*}/\rho_{i*} \ll 1$. We use the notation

$$g_i^{(1)} = g_0 + g_1 + O((\nu_{ii}/\omega_\alpha)^2 F_{i0}), \tag{58}$$

where $g_1/g_0 \sim O(\nu_{ii}/\omega_\alpha)$ and $g_0 \sim F_{i0}$.

To lowest order in the ν_{ii}/ω_α expansion, equation (52) gives

$$\partial_\alpha g_0 = \frac{\partial_\alpha J^{(1)}}{\partial_\psi J^{(0)}} \Upsilon_i F_{i0}. \tag{59}$$

The solution of (59), choosing $\int_0^{2\pi} g_0\, \mathrm{d}\alpha = 0$, is

$$g_0 = \frac{1}{\partial_\psi J^{(0)}}\left(J^{(1)} - \frac{1}{2\pi}\int_0^{2\pi} J^{(1)} \mathrm{d}\alpha\right)\Upsilon_i F_{i0}. \tag{60}$$

It is easy to realize that (60) does not contribute to (55). The next order terms of (52) in the ν_{ii}/ω_α expansion yield

$$\partial_\alpha g_1 = -\frac{1}{\omega_\alpha}\overline{C_{ii}^{\ell(0)}[g_0]}, \tag{61}$$

where we have introduced a convenient notation for the orbit average,

$$\overline{(\cdot)} = \frac{1}{\tau_b^{(0)} v}\sum_\sigma \int_{l_{b_{10}}}^{l_{b_{20}}} (\cdot) \frac{\mathrm{d}l}{\sqrt{1-\lambda B_0(\psi,\alpha,l)}}. \quad (62)$$

Thus,

$$g_1 = -\frac{1}{\omega_\alpha}\int^\alpha \overline{C_{ii}^{\ell(0)}[g_0]}\mathrm{d}\alpha', \quad (63)$$

where the lower limit of the integral is selected so that $\int_0^{2\pi} g_1 \mathrm{d}\alpha = 0$.

When plugged into (60), this piece of the distribution function gives a scaling

$$Q_i \sim \delta^2 \frac{\nu_{ii}}{\omega_\alpha}\rho_{i*} n_i T_i v_{ti} S_\psi \quad (64)$$

for the particle flux. The interesting point is that this is not the dominant contribution to Q_i. It turns out that the energy flux is dominated by two small collisional layers. We prove this and solve the equations in the layers in subsections 4.1 and 4.2.

4.1. Layer around the boundary between trapped and passing particles: the $\sqrt{\nu}$ regime

Recall that $g_i^{(1)} \equiv 0$ in the passing region. The value of $g_i^{(1)}$ at the boundary of the trapped region is given by[+] $g_+ := g_0(\lambda_c) \neq 0$, with $\lambda_c = 1/B_{0,\mathrm{max}}$ and g_0 given by (60). Then, the distribution function is not continuous. This discontinuity comes from an incorrect treatment of the region around the interface between passing and trapped particles. More specifically, it is the consequence of dropping the collision term in that region. Usually, this indicates [25] that there is a small layer in a neighborhood of λ_c where the distribution function develops large variations in λ, and neglecting the collision term is not correct. In the standard language of boundary-layer theory g_0 is the outer solution, and the inner solution of the boundary layer, that we will denote by g_{bl}, remains to be found.

We have to replace (58) by

$$g_i^{(1)} = g_0 + g_{\mathrm{bl}} + \ldots, \quad (65)$$

where g_{bl} satisfies the equation

$$\omega_\alpha \partial_\alpha g_{\mathrm{bl}} + \overline{C_{ii}^{\ell(0)}[g_{\mathrm{bl}}]} = -\overline{C_{ii}^{\ell(0)}[g_0]}. \quad (66)$$

The collision operator acting on g_0 has been included on right-hand side of the previous equation because very close to λ_c the function g_0 varies fast with λ, and the right side of (66) actually diverges at λ_c, as we will see below.

Rigorously speaking, we need to find an interpretation of (66) as an equation on $\lambda \in [\lambda_c, \infty)$. The boundary conditions will be such that g_{bl} vanishes when $\lambda - \lambda_c \to \infty$, and such that $g_{\mathrm{bl}}(\lambda_c) = -g_+$.

If the two terms on the left side of (66) are to be comparable in size, then $g_{\mathrm{bl}} \sim g_0$ and the support of g_{bl} (that is, the size of the boundary layer) is $\Delta\lambda B_0 \sim (\nu_{ii}/\omega_\alpha)^{1/2}$.

[+] Sometimes, in order to ease the notation, we will omit some of the arguments of the functions. For example, in this section it will be common to display only the dependences on λ.

The latter estimation allows to keep only the pitch angle scattering piece of the collision operator on the left side of (66),

$$C_{ii}^{\ell(0)}[g_{\rm bl}] = \frac{\nu_\lambda v_{||}^{(0)}}{v^2 B_0}\partial_\lambda\left(v_{||}^{(0)}\lambda\partial_\lambda g_{\rm bl}\right) + \dots \tag{67}$$

where

$$\nu_\lambda(v) = \frac{8\pi n_i Z_i^4 e^4 \ln\Lambda}{m_i^2 v^3}\left[\mathrm{erf}(v/v_{ti}) - \chi(v/v_{ti})\right] \tag{68}$$

is the pitch angle scattering frequency, $\ln\Lambda$ is the Coulomb logarithm, $\chi(x) = [\mathrm{erf}(x) - (2x/\sqrt{\pi})\exp(-x^2)]/(2x^2)$ and $\mathrm{erf}(x) = (2/\sqrt{\pi})\int_0^x \exp(-t^2)\,\mathrm{d}t$ is the error function.

In the boundary layer the pitch-angle scattering operator in the right side of (67) is dominated by the piece that involves $\partial_\lambda^2 g_{\rm bl}$. The same happens for the right side of (66) close to λ_c, as will be justified below. Therefore, (66) can be approximated by

$$\omega_\alpha\partial_\alpha g_{\rm bl} + \nu_\lambda\lambda\,\overline{B_0^{-1}(1-\lambda B_0)}\,\partial_\lambda^2 g_{\rm bl} = -\nu_\lambda\lambda\,\overline{B_0^{-1}(1-\lambda B_0)}\,\partial_\lambda^2 g_0, \tag{69}$$

where the coefficient multiplying $\partial_\lambda^2 g_{\rm bl}$ and $\partial_\lambda^2 g_0$ does not depend on α due to (9), again.

The smallness of the boundary layer allows to approximate this equation further by writing

$$\partial_\psi J^{(0)}\partial_\alpha g_{\rm bl} + \nu_\lambda\xi\partial_\lambda^2 g_{\rm bl} = -\nu_\lambda\xi\partial_\lambda^2 g_0, \tag{70}$$

where

$$\xi(\psi, v) := \frac{Z_i e\Psi_t'}{m_i c}\frac{2\lambda_c}{v}\int_{l_{b_{10}}}^{l_{b_{20}}} B_0^{-1}\sqrt{1-\lambda_c B_0(\psi,\alpha,l)}\,\mathrm{d}l. \tag{71}$$

The dependence of $\partial_\psi J^{(0)}$ on λ cannot be neglected because $\partial_\psi J^{(0)}(\psi, v, \lambda)$ diverges when $\lambda \to \lambda_c$, so that λ_c is a singular point of the differential equation (70) and requires a careful analysis. The right-hand side of (70) also diverges at λ_c, as pointed out above. Now, we proceed to explain how these divergences emerge.

In Appendix B we show that the asymptotic expansion of $\partial_\psi J^{(0)}$ for small $\lambda - \lambda_c$ (with $\lambda > \lambda_c$) is of the form

$$\partial_\psi J^{(0)} = a_1 \ln(B_{0,\max}(\lambda-\lambda_c)) + a_2 + O(v_{ti}L_0 B_0(\lambda-\lambda_c)/\psi), \tag{72}$$

where

$$a_1 = \sqrt{\frac{1}{2\lambda_c}}\sum_{k=1}^{2}\frac{\lambda_c v\partial_\psi B_0(l_{M,k}) + 2Z_i e/(m_i v)\partial_\psi\varphi_0}{\sqrt{|\partial_l^2 B_0(l_{M,k})|}}. \tag{73}$$

and the values $l_{M,k}$, for $k = 1, 2$, locate two consecutive absolute maxima of B_0 when moving along the field line; in particular, $B_0(\psi, \alpha, l_{M,k}) \equiv B_{0,\max}(\psi)$ for $k = 1, 2$, does not depend on α. Then, the coefficient a_2 can be computed from the relation

$$a_2 = \lim_{\lambda\to\lambda_c}\left(\partial_\psi J^{(0)} - a_1\ln(B_{0,\max}(\lambda-\lambda_c))\right). \tag{74}$$

By rewriting a_2 as

$$a_2 = a_1 \ln\left(B_{0,\max}^{-1}\tilde{a}_2\right) \tag{75}$$

one can recast (72) in the more convenient way

$$\partial_\psi J^{(0)} = a_1 \ln(\tilde{a}_2(\lambda - \lambda_c)) + O(v_{ti} L_0 B_0 (\lambda - \lambda_c)/\psi). \tag{76}$$

Analogously, the asymptotic expansion of $J^{(1)}$ yields

$$J^{(1)} = c_1 \ln(B_{0,\max}(\lambda - \lambda_c)) + c_2 + O(v_{ti} L_0 B_0 (\lambda - \lambda_c)), \tag{77}$$

where

$$c_1 = \sqrt{\frac{1}{2\lambda_c}} \sum_{k=1}^{2} \frac{\lambda_c v B_1(l_{M,k}) + 2Z_i e/(m_i v)\varphi_1(l_{M,k})}{\sqrt{|\partial_l^2 B_0(l_{M,k})|}} \tag{78}$$

and

$$c_2 = \lim_{\lambda\to\lambda_c} \left(J^{(1)} - c_1 \ln(B_{0,\max}(\lambda - \lambda_c)) \right). \tag{79}$$

Then, we rewrite (77) as

$$J^{(1)} = c_1 \ln(\tilde{c}_2(\lambda - \lambda_c)) + O(v_{ti} L_0 B_0 (\lambda - \lambda_c)), \tag{80}$$

with

$$c_2 = c_1 \ln\left(B_{0,\max}^{-1} \tilde{c}_2\right). \tag{81}$$

As we said above, we would like to formulate our boundary-layer equation as a differential equation in $[\lambda_c, \infty)$. We do this by replacing (70) by

$$\widehat{\partial_\psi J^{(0)}} \partial_\alpha g_{\rm bl} + \nu_\lambda \xi \partial_\lambda^2 g_{\rm bl} = -\nu_\lambda \xi \partial_\lambda^2 \widehat{g_0}, \tag{82}$$

where

$$\widehat{\partial_\psi J^{(0)}} = a_1 \ln(\tilde{a}_2(\lambda - \lambda_c)), \tag{83}$$

$$\widehat{g_0} = \frac{1}{\widehat{\partial_\psi J^{(0)}}} \left(\widehat{J^{(1)}} - \frac{1}{2\pi} \int_0^{2\pi} \widehat{J^{(1)}} \mathrm{d}\alpha \right) \Upsilon_i F_{i0} \tag{84}$$

and

$$\widehat{J^{(1)}} = c_1 \ln(\tilde{c}_2(\lambda - \lambda_c)). \tag{85}$$

That is, (82) is obtained from (70) by keeping only the dominant terms in the asymptotic expansions of $\partial_\psi J^{(0)}$ and $J^{(1)}$ near λ_c. The fact that the right side of (66) gets large in the neighborhood of λ_c (and only there) is clear by observing (83), (84) and (85). It is also obvious that, whereas both $\partial_\psi J^{(0)}$ and $J^{(1)}$ diverge at λ_c, $g_+ := g_0(\lambda_c)$ is finite, as it should be.

The solution of equation (82) is more easily found by first expanding $g_{\rm bl}$ in Fourier modes with respect to the coordinate α. Define $g_{{\rm bl},n}$ and $g_{+,n}$ by the relations

$$g_{\rm bl}(\lambda) = \sum_{n=-\infty}^{\infty} g_{{\rm bl},n}(\lambda) e^{in\alpha},$$

$$\widehat{g_0}(\lambda) = \sum_{n=-\infty}^{\infty} g_{0,n}(\lambda) e^{in\alpha},$$

$$g_+ = \sum_{n=-\infty}^{\infty} g_{+,n} e^{in\alpha}, \tag{86}$$

recalling that $g_{\mathrm{bl},0} = 0$ because of condition (36). Here, we have stressed the α and λ dependence although, obviously, g_{bl}, $\widehat{g_0}$ and g_+ also depend on ψ and v. Then, (82) transforms into the set of ordinary differential equations

$$in\,\widehat{\partial_\psi J^{(0)}} g_{\mathrm{bl},n} + \nu_\lambda \xi \partial_\lambda^2 g_{\mathrm{bl},n} = -\nu_\lambda \xi \partial_\lambda^2 g_{0,n}, \tag{87}$$

that must be solved with the boundary conditions

$$g_{\mathrm{bl},n}(\lambda_c) = -g_{+,n} \tag{88}$$

and

$$\lim_{\lambda\to\infty} g_{\mathrm{bl},n}(\lambda) = 0. \tag{89}$$

For each n, the boundary layer equation (87) has two irregular singular points [25], $\lambda = \lambda_c$ and $\lambda = \infty$. In Appendix C we prove that (87) possesses solutions compatible with (88) and (89).

Using the notation

$$\widehat{J^{(1)}}(\psi,\alpha,v,\lambda) = \sum_{n=-\infty}^{\infty} \widehat{J^{(1)}}_n(\psi,v,\lambda)e^{in\alpha} \tag{90}$$

and the solution for g_{bl}, we find that the contribution of the boundary layer to the right side of (55) is

$$Q_{i,\sqrt{\nu}} = \frac{\pi^2 m_i^2 c\delta^2}{Z_i e}\sum_{n=-\infty}^{\infty}(-in)\int_0^\infty \mathrm{d}v v^5 \int_{\lambda_c}^\infty \mathrm{d}\lambda\, \widehat{J^{(1)}}_{-n}\, g_{\mathrm{bl},n}. \tag{91}$$

The fact that (91) scales with the square root of the collisionality can be understood from a simple rescaling of the coordinate λ in (87). However, it is important to emphasize that the scaling is not exact but has logarithmic corrections. Even if these corrections do not change the qualitative features of this collisionality regime, they must be correctly accounted for in order to have accurate results for the neoclassical fluxes. Roughly, the size of $Q_{i,\sqrt{\nu}}$ is

$$Q_{i,\sqrt{\nu}} \sim \delta^2 \frac{\nu_{ii}^{1/2}}{\omega_\alpha^{3/2}} \rho_{i*}^2 n_i T_i v_{ti}^2 L_0^{-1} S_\psi. \tag{92}$$

4.2. Layer around points where $\omega_\alpha = 0$: the superbanana plateau regime

The outer solution (60) for the distribution function is correct everywhere except near the boundary between the passing and trapped regions (already treated in subsection 4.1) and in the neighborhood of points where $\omega_\alpha = 0$. Around these 'resonant points' the $\nu_{ii}/\omega_\alpha \ll 1$ expansion is not valid. This region of phase space is the subject of the present section.

In order to understand what happens in the vicinity of a point where $\omega_\alpha = 0$, we go back to equation (52) and do not carry out the $\nu_{ii}/\omega_\alpha \ll 1$ expansion. That is, we consider the equation

$$\omega_\alpha \partial_\alpha g_i^{(1)} + \overline{C_{ii}^{\ell(0)}[g_i^{(1)}]} = S \tag{93}$$

with

$$S(\psi, \alpha, v, \lambda) = \frac{m_i c}{Z_i e \Psi_t' \tau_b^{(0)}} \partial_\alpha J^{(1)} \Upsilon_i F_{i0}. \tag{94}$$

Assume for a moment that $\partial_\psi \varphi_0 = 0$. Given an omnigeneous magnetic field $\mathbf{B}_0$, ω_α may vanish for some values of λ or may not vanish for any value of λ. If ω_α never vanishes, we do not have to continue with the analysis. We assume that if $\partial_\psi \varphi_0 = 0$, then ω_α vanishes for a single value of λ, which is obviously independent of v. In such a case, for non-zero $\partial_\psi \varphi_0$ there exists a minimum value of v for which $\omega_\alpha = 0$ for some value of λ. We denote this value of v by $v_{\min}$. When $v \geq v_{\min}$, we denote by λ_r the value of λ such that $\omega_\alpha = 0$. Of course, λ_r is a function of ψ and v, $\lambda_r \equiv \lambda_r(\psi, v)$.

Around λ_r,

$$\omega_\alpha(\lambda) = \partial_\lambda \omega_\alpha(\lambda_r)(\lambda - \lambda_r) + O((\lambda - \lambda_r)^2), \tag{95}$$

where the dependence on ψ and v has been omitted for simplicity. The balance of the two terms on the left side of (93) implies that in a neighborhood of λ_r of size $\Delta\lambda$,

$$\partial_\lambda \omega_\alpha(\lambda_r) \Delta\lambda \sim \frac{\nu_{ii}}{B_0^2 \Delta\lambda^2}. \tag{96}$$

Since, typically, $\partial_\lambda \omega_\alpha(\lambda_r) \sim \rho_{i*} B_0 L_0^{-1} v_{ti}$, one finds

$$(B_0 \Delta\lambda)^3 \sim \frac{\nu_{i*}}{\rho_{i*}} \ll 1 \tag{97}$$

and therefore

$$\overline{C_{ii}^{\ell(0)}[g_{\rm rl}]} \sim \nu_{ii} (\rho_{i*}/\nu_{i*})^{2/3} g_{\rm rl}, \tag{98}$$

where $g_{\rm rl}$ is the distribution function in the 'resonant layer' of size $\Delta\lambda$. In particular, the pitch-angle scattering piece of the collision operator dominates in this layer,

$$\overline{C_{ii}^{\ell(0)}[g_{\rm rl}]} = \overline{\frac{\nu_\lambda v_{||}^{(0)}}{v^2 B_0} \partial_\lambda \left(v_{||}^{(0)} \lambda \partial_\lambda g_{\rm rl} \right)} + \ldots, \tag{99}$$

and we can actually keep only the term involving $\partial_\lambda^2 g_{\rm rl}$. Hence, in the resonant layer we can write the drift kinetic equation as

$$\partial_\lambda \omega_{\alpha,r} (\lambda - \lambda_r) \partial_\alpha g_{\rm rl} + \nu_\lambda \chi_r \partial_\lambda^2 g_{\rm rl} = S_r, \tag{100}$$

with

$$\chi_r(\psi, v) := \lambda_r \overline{B_0^{-1}(1 - \lambda_r B_0)}\ , \tag{101}$$

$$\partial_\lambda \omega_{\alpha,r}(\psi, v) := \partial_\lambda \omega_\alpha(\psi, v, \lambda_r(\psi, v)) \tag{102}$$

and

$$S_r(\psi, \alpha, v) := S(\psi, \alpha, v, \lambda_r(\psi, v)). \tag{103}$$

Define the Fourier expansions

$$g_{\rm rl} = \sum_{n=-\infty}^{\infty} g_{{\rm rl},n} e^{in\alpha},$$

$$S_r = \sum_{n=-\infty}^{\infty} S_{r,n} e^{in\alpha}, \tag{104}$$

noting that $g_{\mathrm{rl},0} = 0$ due to (36), and that $S_{r,0} = 0$ due to definition (94). Inserting the expansions in (100) and noting that $\partial_\lambda \omega_{\alpha,r}$ and χ_r do not depend on α, we find an ordinary differential equation for each mode $g_{\mathrm{rl},n}$,

$$in\partial_\lambda \omega_{\alpha,r}(\lambda - \lambda_r) g_{\mathrm{rl},n} + \nu_\lambda \chi_r \partial_\lambda^2 g_{\mathrm{rl},n} = S_{r,n}. \tag{105}$$

The solution of this equation is

$$g_{\mathrm{rl},n} = -\frac{S_{r,n}}{\partial_\lambda \omega_{\alpha,r} n^{2/3} \lambda_r \beta} \int_0^\infty \exp\left(i\frac{n^{1/3}}{\beta}\frac{\lambda - \lambda_r}{\lambda_r} z - \frac{1}{3}z^3 \right) \mathrm{d}z, \tag{106}$$

where

$$\beta := \left(\frac{\nu_\lambda \chi_r}{\partial_\lambda \omega_{\alpha,r} \lambda_r^3} \right)^{1/3} \ll 1 \tag{107}$$

gives the width of the layer.

Then, the contribution of resonant particles to (55) is

$$Q_{i,\mathrm{sb-p}} = -\frac{\pi^2 m_i^2 c \delta^2}{Z_i e} \sum_{n=-\infty}^{\infty} in \int_{v_{\min}}^{\infty} \mathrm{d}v v^5 \int_{-\infty}^{\infty} \mathrm{d}\lambda \, J_{-n}^{(1)} g_{\mathrm{rl},n}$$

$$= -\frac{\pi^2 m_i^3 c^2 \delta^2}{Z_i^2 e^2 \Psi_t'} \sum_{n=-\infty}^{\infty} \int_{v_{\min}}^{\infty} \mathrm{d}v v^5 \frac{n^{4/3}}{\partial_\lambda \omega_{\alpha,r} \tau_{b,r}^{(0)} \lambda_r} \Upsilon_i F_{i0} \int_{-\infty}^{\infty} \mathrm{d}\lambda \, |J_n^{(1)}|^2 \Bigg\{$$

$$\frac{1}{\beta} \int_0^\infty \exp\left(i\frac{n^{1/3}}{\beta}\frac{\lambda - \lambda_r}{\lambda_r} z - \frac{1}{3}z^3 \right) \mathrm{d}z \Bigg\}$$

$$= -\frac{2\pi^2 m_i^3 c^2 \delta^2}{Z_i^2 e^2 \Psi_t'} \sum_{n=1}^{\infty} \int_{v_{\min}}^{\infty} \mathrm{d}v v^5 \frac{n^{4/3}}{\partial_\lambda \omega_{\alpha,r} \tau_{b,r}^{(0)} \lambda_r} \Upsilon_i F_{i0} \int_{-\infty}^{\infty} \mathrm{d}\lambda \, |J_n^{(1)}|^2 \Bigg\{$$

$$\frac{1}{\beta} \int_0^\infty \cos\left(\frac{n^{1/3}}{\beta}\frac{\lambda - \lambda_r}{\lambda_r} z \right) \exp\left(-\frac{1}{3}z^3 \right) \mathrm{d}z \Bigg\}, \tag{108}$$

where we have defined

$$J^{(1)}(\psi, \alpha, v, \lambda_r) = \sum_{n=-\infty}^{\infty} J_n^{(1)}(\psi, v, \lambda_r(\psi, v)) e^{in\alpha} \tag{109}$$

and $\tau_{b,r}^{(0)} = \tau_b^{(0)}(\psi, v, \lambda_r(\psi, v))$.

Next, we prove that the right side of (108) has a non-zero limit when $\beta \to 0$. For this, we employ the identity

$$\lim_{\beta \to 0} \frac{1}{\beta} \int_0^\infty e^{-z^3/3} \cos\left(\frac{1}{\beta} x z \right) \mathrm{d}z = \pi \delta(x) \tag{110}$$

and the property $\delta(ax) = |a|^{-1}\delta(x)$, where $\delta(\cdot)$ is the Dirac delta distribution and a is a real number. Then, for $\beta \ll 1$, the asymptotically dominant term is

$$Q_{i,\mathrm{sb-p}} = -\frac{2\pi^3 m_i^3 c^2 \delta^2}{Z_i^2 e^2 \Psi_t'} \sum_{n=1}^{\infty} n \int_{v_{\min}}^{\infty} \mathrm{d}v v^5 \frac{1}{\partial_\lambda \omega_{\alpha,r} \tau_{b,r}^{(0)}} \Upsilon_i F_{i0} |J_n^{(1)}|^2. \tag{111}$$

If $v_{\min} \lesssim v_{ti}$, the size of the energy flux is

$$Q_{i,\mathrm{sb-p}} \sim \delta^2 \rho_{i*} n_i T_i v_{ti} S_\psi. \tag{112}$$

4.3. Formula for the ion energy flux when $\nu_{i*} \ll \rho_{i*}$

Since the layers studied in subsections 4.1 and 4.2 are small and, in general, they are located around different points of phase space, their contributions to transport are additive. This means that we can write, for $\nu_{i*} \ll \rho_{i*}$,

$$Q_i = Q_{i,\sqrt{\nu}} + Q_{i,\mathrm{sb-p}}, \tag{113}$$

where $Q_{i,\sqrt{\nu}}$ is given by (91) and $Q_{i,\mathrm{sb-p}}$ is given by (111). The weight of each term in (113) is determined by the value of $v_{\min}$. If $v_{\min} \lesssim v_{ti}$, then the superbanana-plateau regime (recall (111) and (112)) dominates over the $\sqrt{\nu}$ regime (recall (91) and (92)). If, on the contrary, $v_{\min} \gg v_{ti}$, then the superbanana-plateau regime will be subdominant with respect to the $\sqrt{\nu}$ regime because the integral in (111) will be taken over a region of phase space in which F_{i0} is very small.

Finally, we note that the value of $v_{\min}$ is influenced by the size of $\partial_\psi \varphi_0$, but also by the specific λ-dependence of ω_α when $\partial_\psi \varphi_0 = 0$.

5. Calculation of the electric field

The radial electric field, determined by φ_0', is one of the quantities that are routinely computed in standard neoclassical calculations. It is found by imposing that the radial electric current vanish, as recalled in subsection 5.2.

On the contrary, the components of the electric field tangent to the flux surface are usually neglected. However, we have seen that under our ordering $\varphi_1 = \delta\varphi^{(1)} + \ldots$ contributes to the energy transport of the main ion species. It is therefore necessary to explain how to compute it and we deal with this problem in subsection 5.1.

Perhaps the most obvious way (although not the only one) to calculate both, $\varphi^{(1)}$ and φ_0', is to use (115) to solve for $\varphi^{(1)}$ for a fixed value of φ_0', and then use (120) to find φ_0' in an iterative process.

5.1. Calculation of $\varphi^{(1)}$

The components of the electric field along the flux surface are given, to lowest order, by $\varphi^{(1)}$, which is found from (30). Expanding (30) in δ, we obtain

$$\left(\frac{Z_i}{T_i} + \frac{1}{T_e}\right)\varphi^{(1)} = \frac{2\pi}{en_i}\int_0^\infty \mathrm{d}v \int_{B_{0,\max}^{-1}}^{B^{-1}} \mathrm{d}\lambda \frac{v^3 B_0}{|v_{||}^{(0)}|} g_i^{(1)}, \tag{114}$$

where we have used that in $\{v, \lambda, \gamma\}$ coordinates $\mathrm{d}^3v \equiv \sum_\sigma v^3 B/(2|v_{||}|)\mathrm{d}v\mathrm{d}\lambda\mathrm{d}\gamma$ and that $g_i^{(1)}$ vanishes in the passing region, so the integral on the right side of (114) is taken only over trapped trajectories. The solution (60) does not contribute to transport, but it does contribute to (114). Note, however, that (60) diverges wherever $\partial_\psi J^{(0)} = 0$. Interestingly, in general the points λ_r where $\partial_\psi J^{(0)}$ vanishes contribute to (114) as much as the rest of phase space. Asymptotically, we can write

$$\left(\frac{Z_i}{T_i} + \frac{1}{T_e}\right)\varphi^{(1)} = \frac{2\pi}{en_i}\mathrm{P.V.}\int_0^\infty \mathrm{d}v \int_{B_{0,\max}^{-1}}^{B^{-1}} \mathrm{d}\lambda \frac{v^3 B_0}{|v_{||}^{(0)}|} g_0$$

$$+\frac{2\pi}{en_i}\int_{v_{\rm min}}^{\infty}\mathrm{d}v\left(\lim_{\beta\to 0}\int_{B_{0,\rm max}^{-1}}^{B^{-1}}\mathrm{d}\lambda\frac{v^3B_0}{|v_{||}^{(0)}|}g_{\rm rl}\right). \tag{115}$$

Here, P.V. stands for the principal value of the integral in λ.

Expanding $g_{\rm rl}$ in Fourier modes, and using (106) and (110), one gets

$$\left(\frac{Z_i}{T_i}+\frac{1}{T_e}\right)\varphi^{(1)} = \frac{2\pi}{en_i}\mathrm{P.V.}\int_0^{\infty}\mathrm{d}v\int_{B_{0,\rm max}^{-1}}^{B^{-1}}\mathrm{d}\lambda\frac{v^3B_0}{|v_{||}^{(0)}|}g_0$$

$$-\frac{2\pi^2}{en_i}\int_{v_{\rm min}}^{\infty}\mathrm{d}v\frac{v^3B_0}{|v_{||,r}^{(0)}|}\frac{1}{\partial_\lambda\omega_{\alpha,r}}\sum_{n=-\infty}^{\infty}\frac{1}{|n|}S_{r,n}e^{in\alpha}, \tag{116}$$

where

$$|v_{||,r}^{(0)}(\psi,\alpha,v)| := v\sqrt{1-\lambda_r(\psi,v)B(\psi,\alpha,l)}\,. \tag{117}$$

Employing (60) for g_0 and the definition (94) for S, and recalling the expression of $\partial_\alpha J^{(1)}$ in terms of $\varphi^{(1)}$ (see (46)), one gets a linear equation that allows to solve for the latter.

5.2. Calculation of φ_0'

Let us denote by Γ_i and Γ_e the radial fluxes of ions and electrons. The radial electric field is determined by imposing

$$Z_ie\Gamma_i - e\Gamma_e = 0, \tag{118}$$

and to lowest order in a mass ratio expansion $\sqrt{m_e/m_i}\ll 1$ this is equivalent to the condition

$$\Gamma_i = 0. \tag{119}$$

The calculation of Γ_i is completely analogous to that of Q_i. Hence, asymptotically, (119) amounts to the condition

$$\Gamma_{i,\sqrt{\nu}} + \Gamma_{i,\rm sb-p} = 0, \tag{120}$$

where

$$\Gamma_{i,\sqrt{\nu}} = \frac{2\pi^2 m_i c\delta^2}{Z_ie}\sum_{n=-\infty}^{\infty}(-in)\int_0^{\infty}\mathrm{d}v v^3\int_{\lambda_c}^{\infty}\mathrm{d}\lambda\,\widehat{J^{(1)}}_{-n}\,g_{{\rm bl},n} \tag{121}$$

and

$$\Gamma_{i,\rm sb-p} = -\frac{4\pi^3 m_i^2c^2\delta^2}{Z_i^2e^2\Psi_t'}\sum_{n=1}^{\infty}n\int_{v_{\rm min}}^{\infty}\mathrm{d}v v^3\frac{1}{\partial_\lambda\omega_{\alpha,r}\tau_{b,r}^{(0)}}\Upsilon_iF_{i0}|J_n^{(1)}|^2. \tag{122}$$

6. Estimation of $\nu_{\delta *}$

In Section 4 we have solved the drift-kinetic equation and computed Q_i when $\nu_{i*}\ll\rho_{i*}$. But we have advanced in the Introduction that our results are not valid for arbitrarily small ν_{i*}. There exists a value of the collisionality, that we call $\nu_{\delta *}$, below which equation (113) is expected to be incorrect because the drift-kinetic equation (52) is incorrect. Hence, it is more precise to say that our results in Section 4 are correct when $\nu_{\delta *} \ll \nu_{i*} \ll \rho_{i*}$. In this section we explain the reason for the existence of $\nu_{\delta *}$ and estimate its value.

The limitations of equation (52) for sufficiently small ν_{i*} are well understood by inspecting the drift-kinetic equation in coordinates u and μ, given in (D.3) of Appendix D. Equation (D.3) contains all terms that are needed to describe neoclassical transport in stellarators when $\nu_{i*} \ll 1$. In Appendix D we show how to obtain (52) from (D.3), and therefore we can deduce which terms are lacking in (52) at small enough collisionality. Tracking the derivation in Appendix D and employing the notation introduced there, it is clear that the term

$$u\boldsymbol{\kappa} \cdot \mathbf{v}_{\nabla B,i} \partial_u \widehat{F}_{i1} \tag{123}$$

is the key. It is easy to see that only the piece

$$u(\boldsymbol{\kappa} \cdot \mathbf{v}_{\nabla B,i})^{(0)} \partial_u \widehat{F}_{i1}, \tag{124}$$

corresponding to the omnigeneous magnetic field $\mathbf{B}_0$, enters (52). The effect of higher-order terms like

$$u(\boldsymbol{\kappa} \cdot \mathbf{v}_{\nabla B,i})^{(1)} \partial_u \widehat{F}_{i1} \tag{125}$$

have not been included. However, in Section 4 we learnt that transport is dominated by two collisional layers when $\nu_{i*} \ll \rho_{i*}$. In these layers, derivatives with respect to u (or, equivalently, with respect to λ) are large, and they grow as ν_{i*} decreases. Let us denote by Δu the width of the layer in the coordinate u. The term (125) becomes comparable with the pitch-angle scattering piece of the collision operator when

$$\frac{\delta \rho_{i*} v_{ti}}{L_0(\Delta u / v_{ti})} \sim \frac{\nu_{ii}}{(\Delta u / v_{ti})^2}. \tag{126}$$

If the stellarator is in the $\sqrt{\nu}$ regime, the boundary layer has a width $\Delta u / v_{ti} \sim \sqrt{\nu_{i*}/\rho_{i*}}$, and we get the estimation

$$\nu_{\delta *} \sim \delta^2 \rho_{i*}. \tag{127}$$

If the stellarator is in the superbanana-plateau regime, the size of the boundary layer is $\Delta u / v_{ti} \sim (\nu_{i*}/\rho_{i*})^{1/3}$ and we get

$$\nu_{\delta *} \sim \delta^{3/2} \rho_{i*}. \tag{128}$$

When $\nu_{i*} \lesssim \nu_{\delta *}$, effects like those described in [24] must be taken into account. We leave this for future work.

7. Conclusions

Omnigeneity is the property of stellarators that have been perfectly optimized regarding neoclassical transport. It has been argued in [14] and in the Introduction of the present paper that deviations from omnigeneity are likely to have a non-negligible effect on the neoclassical fluxes. It is natural to expect that this effect will be larger at low collisionality ν_{i*}.

The $1/\nu$ regime in stellarators close to omnigeneity is studied in [14]; this regime is defined by $\rho_{i*} \ll \nu_{i*} \ll 1$. In the core of hot stellarators, even lower collisionality regimes are relevant. The subject of this paper has been the study of the parameter range $\nu_{i*} \ll \rho_{i*}$, with the restriction (32) for the perturbations of the omnigeneous configuration (i.e. the helicities of the perturbations have to be small).

In this regime the terms in the drift-kinetic equation that involve the components of the drifts tangential to the flux surface have to be retained. The appropriate drift-kinetic equation to solve for the dominant non-omnigeneous piece of the distribution function has been derived in Section 3. In Section 4, the equation has been solved and an explicit formula for the ion energy flux Q_i has been provided in (113). The formula manifests, in particular, that when $\nu_{i*} \ll \rho_{i*}$ transport is determined by two small collisional layers on phase space. One of the layers is located around the boundary between trapped and passing particles and the other is located in the neighborhood of the points where the precession frequency (which is associated to the motion caused by the tangential drifts) vanishes. The former corresponds to the $\sqrt{\nu}$ regime and the latter to the superbanana-plateau regime.

In Section 5 we have given equations to determine the dominant contributions to the radial electric field and to the electric field tangent to the flux surface.

Finally, in Section 6 we have explained why the results of Section 4 are not valid below a certain value of the collisionality, that we call $\nu_{\delta *}$ and that we have estimated. The treatment of the regime $\nu_{i*} \lesssim \nu_{\delta *}$ in stellarators close to omnigeneity is left for future work.

Appendix A. Proof of relations (43) and (44)

A straightforward calculation shows that

$$2\int_{l_{b_1}}^{l_{b_2}} \frac{1}{|v_{||}|}(\mathbf{v}_{M,i} + \mathbf{v}_{E,0} + \mathbf{v}_{E,1}) \cdot \nabla\psi \, \mathrm{d}l =$$

$$\frac{2m_i c}{Z_i e \Psi_t'} \partial_\alpha \int_{l_{b_1}}^{l_{b_2}} |v_{||}| \mathrm{d}l - \frac{2m_i c}{Z_i e \Psi_t'} \int_{l_{b_1}}^{l_{b_2}} \partial_l (|v_{||}| \partial_\alpha \mathbf{r} \cdot \hat{\mathbf{b}}) \mathrm{d}l \qquad \text{(A.1)}$$

and

$$2\int_{l_{b_1}}^{l_{b_2}} \frac{1}{|v_{||}|}(\mathbf{v}_{M,i} + \mathbf{v}_{E,0} + \mathbf{v}_{E,1}) \cdot \nabla\alpha \, \mathrm{d}l =$$

$$-\frac{2m_i c}{Z_i e \Psi_t'} \partial_\psi \int_{l_{b_1}}^{l_{b_2}} |v_{||}| \mathrm{d}l + \frac{2m_i c}{Z_i e \Psi_t'} \int_{l_{b_1}}^{l_{b_2}} \partial_l (|v_{||}| \partial_\psi \mathbf{r} \cdot \hat{\mathbf{b}}) \mathrm{d}l. \qquad \text{(A.2)}$$

The last term in both (A.1) and (A.2) vanishes because $v_{||}$ equals zero at l_{b_1} and l_{b_2}. Finally, using the definition (42), we obtain (43) and (44).

Appendix B. Asymptotic expansion of $\partial_\psi J^{(0)}$ near the boundary between trapped and passing particles

We show that

$$\partial_\psi J^{(0)} = -\int_{l_{b_{10}}}^{l_{b_{20}}} \frac{\lambda v \partial_\psi B_0 + 2Z_i e/(m_i v)\partial_\psi \varphi_0}{\sqrt{1-\lambda B_0}} \mathrm{d}l \tag{B.1}$$

has the form (72) for small $\lambda - \lambda_c > 0$ by, first, using the trivial identity

$$\begin{aligned}\partial_\psi J^{(0)} = &-\sum_{k=1}^{2}\int_{l_{b_{10}}}^{l_{b_{20}}} \frac{\lambda v \partial_\psi B_0(l_{M,k}) + 2Z_i e/(m_i v)\partial_\psi \varphi_0}{\sqrt{(\lambda_c/2)|\partial_l^2 B_0(l_{M,k})|(l-l_{M,k})^2 - B_0(l_{M,k})(\lambda-\lambda_c)}}\mathrm{d}l\\
&-\int_{l_{b_{10}}}^{l_{b_{20}}}\Bigg(\frac{\lambda v \partial_\psi B_0(l) + 2Z_i e/(m_i v)\partial_\psi \varphi_0}{\sqrt{1-\lambda B_0(l)}}\\
&-\sum_{k=1}^{2}\frac{\lambda v \partial_\psi B_0(l_{M,k}) + 2Z_i e/(m_i v)\partial_\psi \varphi_0}{\sqrt{(\lambda_c/2)|\partial_l^2 B_0(l_{M,k})|(l-l_{M,k})^2 - B_0(l_{M,k})(\lambda-\lambda_c)}}\Bigg)\mathrm{d}l,\end{aligned} \tag{B.2}$$

which is well defined for sufficiently small $\lambda - \lambda_c$. Here, we have only displayed the dependence of B_0 on l. The values $l_{M,k}$, for $k = 1, 2$, locate two consecutive absolute maxima of B_0 when moving along the field line; in particular, $B_0(l_{M,k}) = B_{0,\max}$ for $k = 1, 2$. The second integral on the right side of (B.2) is finite when $\lambda \to \lambda_c$, and hence it contributes to a_2 and higher-order terms in (72). The first integral on the right side of (B.2) can be computed analytically; namely,

$$\begin{aligned}-\sum_{k=1}^{2}\int_{l_{b_{10}}}^{l_{b_{20}}} &\frac{\lambda v \partial_\psi B_0(l_{M,k}) + 2Z_i e/(m_i v)\partial_\psi \varphi_0}{\sqrt{(\lambda_c/2)|\partial_l^2 B_0(l_{M,k})|(l-l_{M,k})^2 - B_0(l_{M,k})(\lambda-\lambda_c)}}\mathrm{d}l = \\
&-\sum_{k=1}^{2}\frac{\lambda v \partial_\psi B_0(l_{M,k}) + 2Z_i e/(m_i v)\partial_\psi \varphi_0}{\sqrt{(\lambda_c/2)|\partial_l^2 B_0(l_{M,k})|}}\\
&\times\left[\ln\left(x+\sqrt{x^2 - \frac{2B_0(l_{M,k})}{\lambda_c|\partial_l^2 B_0(l_{M,k})|}(\lambda-\lambda_c)}\right)\right]_{l_{b_{10}}-l_{M,k}}^{l_{b_{20}}-l_{M,k}}.\end{aligned} \tag{B.3}$$

For small $\lambda - \lambda_c$,

$$l_{b_{10}} - l_{M,1} = \sqrt{\frac{2B_0(l_{M,k})(\lambda-\lambda_c)}{\lambda_c|\partial_l^2 B_0(l_{M,k})|}} + \ldots, \tag{B.4}$$

and

$$l_{b_{20}} - l_{M,2} = -\sqrt{\frac{2B_0(l_{M,k})(\lambda-\lambda_c)}{\lambda_c|\partial_l^2 B_0(l_{M,k})|}} + \ldots, \tag{B.5}$$

whereas $l_{b_{20}} - l_{M,1} = O(L_0)$ and $l_{b_{10}} - l_{M,2} = O(L_0)$. Using these results in (B.3), it is straightforward to deduce that

$$-\sum_{k=1}^{2}\int_{l_{b_{10}}}^{l_{b_{20}}} \frac{\lambda v \partial_\psi B_0(l_{M,k}) + 2Z_i e/(m_i v)\partial_\psi \varphi_0}{\sqrt{(\lambda_c/2)|\partial_l^2 B_0(l_{M,k})|(l-l_{M,k})^2 - B_0(l_{M,k})(\lambda-\lambda_c)}}\mathrm{d}l =$$

$$\sqrt{\frac{1}{2\lambda_c}}\sum_{k=1}^{2}\frac{\lambda_c v\partial_\psi B_0(l_{M,k})+2Z_ie/(m_iv)\partial_\psi\varphi_0}{\sqrt{|\partial_l^2 B_0(l_{M,k})|}}\ln(B_{0,\rm max}(\lambda-\lambda_c))\ +O(v_{ti}L_0/\psi), \quad \text{(B.6)}$$

from where (73) follows.

Appendix C. Analysis of the singular points of (87)

In this appendix we use the variable $x=\lambda-\lambda_c$ and rewrite (87) as

$$\partial_x^2 g_n + in\frac{a_1}{\nu_\lambda\xi}\ln(\tilde{a}_2 x)\, g_n = -\partial_x^2 g_{0,n}, \quad \text{(C.1)}$$

where $g_n(x)=g_{{\rm bl},n}(\lambda_c+x)$ and $g_{0,n}(x)=g_{0,n}(\lambda_c+x)$. The equations (87) for $n\neq 0$ (recall that $g_n(x)$ and $g_{0,n}(x)$ vanish) have two irregular singular points [25], $x=0$ and $x=\infty$.

We start by analyzing the point $x=0$. The standard methods do not work to study the behavior near $x=0$ of the solutions of the homogeneous equation

$$\partial_x^2 g_n + in\frac{a_1}{\nu_\lambda\xi}\ln(\tilde{a}_2 x)\, g_n = 0. \quad \text{(C.2)}$$

However, one can check that the ansatz

$$g_n = \sum_{m,p=0}^{\infty} a_{m,p}\, x^{2p+m}(\ln x)^p \quad \text{(C.3)}$$

is consistent, in the sense that by substitution in (C.2) one can find recurrence relations that determine all the coefficients $a_{m,p}$ except two of them. The free coefficients can be taken to be $a_{0,0}$ and $a_{1,0}$. This way one proves that there exist two linearly independent solutions of (C.2) that are finite at $x=0$.

It is easy to realize that the source term on the right side of (C.1) does not make g_n diverge at $x=0$. First, note that $g_{0,n}$ is finite for any value of x. If one takes $g_n=-g_{0,n}+f_n$, (C.1) gives the following equation for f_n:

$$\partial_x^2 f_n + in\frac{a_1}{\nu_\lambda\xi}\ln(\tilde{a}_2 x)\, f_n = \frac{in}{\nu_\lambda\xi}(d_n + c_{1,n}\ln x)\Upsilon_i F_{i0}, \quad \text{(C.4)}$$

with

$$c_1 - \frac{1}{2\pi}\int_0^{2\pi} c_1 {\rm d}\alpha = \sum_{n=-\infty}^{\infty} c_{1,n}e^{in\alpha} \quad \text{(C.5)}$$

and

$$c_1\ln(\tilde{c}_2) - \frac{1}{2\pi}\int_0^{2\pi} c_1\ln(\tilde{c}_2) {\rm d}\alpha = \sum_{n=-\infty}^{\infty} d_n e^{in\alpha}. \quad \text{(C.6)}$$

Recall that c_1, c_2 and $\tilde{c}_2$ have been defined in (78), (79), (81). Since the indefinite integrals of $\ln x$ are finite everywhere, the source term on the right side of (C.4) does not introduce singularities in f_n and we conclude that g_n is finite for any value of x; in particular, it is finite at $x=0$.

Now, we turn to study the solutions of (C.1) at $x=\infty$. Actually, it is enough to analyze the homogeneous equation (C.2) because the right side of (C.1) is negligible except in a neighborhood of λ_c. We can employ the standard technique to find local solutions of homogeneous linear equations around irregular singular points described in reference [25]. We start by introducing a formal small parameter γ in (C.1),

$$\partial_x^2 g_n + in\gamma^{-2}\frac{a_1}{\nu_\lambda \xi}\ln(\tilde{a}_2 x)\, g_n = 0. \tag{C.7}$$

At the end of the calculation, we will set $\gamma = 1$. Now, we write

$$g_n = e^{\gamma^{-1}S_n^{(0)} + S_n^{(1)}}, \tag{C.8}$$

and plug (C.8) into (C.1), finding

$$\gamma^{-1}\partial_x^2 S_0 + \partial_x^2 S_1 + (\gamma^{-1}\partial_x S_0 + S_1)^2 + \gamma^{-2} in\frac{a_1}{\nu_\lambda \xi}\ln(\tilde{a}_2 x) = 0. \tag{C.9}$$

Solving the orders γ^{-2} and γ^{-1} of this equation and finally setting $\gamma = 1$, one gets the asymptotic expressions

$$g_n \sim \frac{1}{(\ln(\tilde{a}_2 x))^{1/4}}\exp\left(\pm\frac{1}{\sqrt{2}}(i-1)\sqrt{\frac{na_1}{\nu_\lambda \xi}}\int^x \sqrt{\ln(\tilde{a}_2 y)}\,\mathrm{d}y\right) \tag{C.10}$$

for $n > 0$ and

$$g_n \sim \frac{1}{(\ln(\tilde{a}_2 x))^{1/4}}\exp\left(\pm\frac{1}{\sqrt{2}}(i+1)\sqrt{\frac{-na_1}{\nu_\lambda \xi}}\int^x \sqrt{\ln(\tilde{a}_2 y)}\,\mathrm{d}y\right) \tag{C.11}$$

for $n < 0$. In particular, for every $n \neq 0$ there exist solutions to (C.1) that vanish when $x \to \infty$.

In summary, we have shown that (C.1) can be solved with the boundary conditions given by (88) and the vanishing of g_n when $x \to \infty$.

Appendix D. Drift-kinetic equation in coordinates u and μ

It is pedagogical to derive the drift-kinetic equation (52) by starting with the drift-kinetic equation in coordinates u and μ, where u is the parallel velocity. Furthermore, this derivation helps understand why (52) is not valid when the collisionality is small enough, as pointed out in Section 6.

We have derived the drift-kinetic equation to $O(\rho_i^{*2})$ in coordinates u and μ in reference [11]. One of the original results of [11] is the explicit calculation of all the $O(\rho_i^{*2})$ terms, given in equation (79) of that reference. From equation (79) of [11] we only have to keep terms that become large when $\nu_{i*} \ll 1$; that is, terms that contain either $\widehat{F}_{i1}$ or φ_1. Here, we are using the notation

$$\widehat{F}_i(\mathbf{r}, u, \mu) = F_{i0}(\mathbf{r}, u, \mu) + \widehat{F}_{i1}(\mathbf{r}, u, \mu) + O(\rho_{i*}^2 F_{i0}), \tag{D.1}$$

where

$$F_{i0}(\mathbf{r}, u, \mu) = n_i(\psi(\mathbf{r}))\left(\frac{m_i}{2\pi T_i(\psi(\mathbf{r}))}\right)^{3/2}\exp\left(-\frac{m_i(u^2/2 + \mu B(\mathbf{r}))}{T_i(\psi(\mathbf{r}))}\right). \tag{D.2}$$

Retaining only the relevant terms $O(\rho_i^{*2})$ and adding the standard $O(\rho_{i*})$ terms, we have

$$\begin{aligned}
&(\mathbf{v}_{M,i} + \mathbf{v}_{E,0}) \cdot \nabla \widehat{F}_{i1} \\
&\quad + \left[u\boldsymbol{\kappa} \cdot (\mathbf{v}_{\nabla B,i} + \mathbf{v}_{E,0}) - \frac{Z_i e}{m_i} \hat{\mathbf{b}} \cdot \nabla \varphi_1 \right] \partial_u \widehat{F}_{i1} \\
&\quad + \mathbf{v}_{E,1} \cdot \nabla F_{s0} + u\boldsymbol{\kappa} \cdot \mathbf{v}_{E,1} \partial_u F_{i0} \\
&\quad + (u\hat{\mathbf{b}} \cdot \nabla - \mu \hat{\mathbf{b}} \cdot \nabla B \partial_u) \widehat{F}_{s1} + \frac{Z_i e}{T_i} u \hat{\mathbf{b}} \cdot \nabla \varphi_1 F_{i0} \\
&\quad + \Upsilon_i \mathbf{v}_{M,i} \cdot \nabla \psi F_{i0} = C_{ii}^{\ell}[\widehat{F}_{i1}],
\end{aligned} \tag{D.3}$$

where

$$\Upsilon_i = \frac{n_i'}{n_i} + \frac{T_i'}{T_i} \left(\frac{m_i u^2/2 + \mu B}{T_i} - \frac{3}{2} \right) + \frac{Z_i e \varphi_0'}{T_i}, \tag{D.4}$$

$$\mathbf{v}_{M,i} = \frac{1}{\Omega_i} \hat{\mathbf{b}} \times \left(u^2 \boldsymbol{\kappa} + \mu \nabla B \right), \tag{D.5}$$

$$\mathbf{v}_{\nabla B,i} = \frac{1}{\Omega_i} \hat{\mathbf{b}} \times \mu \nabla B, \tag{D.6}$$

$$\mathbf{v}_{E,0} = \frac{c}{B} \hat{\mathbf{b}} \times \nabla \varphi_0, \tag{D.7}$$

$$\mathbf{v}_{E,1} = \frac{c}{B} \hat{\mathbf{b}} \times \nabla \varphi_1 \tag{D.8}$$

and

$$F_{i0}(\mathbf{r}, u, \mu) = n_i(\psi(\mathbf{r})) \left(\frac{m_i}{2\pi T_i(\psi(\mathbf{r}))} \right)^{3/2} \exp \left(-\frac{m_i (u^2/2 + \mu B(\mathbf{r}))}{T_i(\psi(\mathbf{r}))} \right). \tag{D.9}$$

The last two lines in (D.3) contain the terms of the standard drift kinetic equation. The third line in (D.3) can be rewritten as

$$\begin{aligned}
&\mathbf{v}_{E,1} \cdot \nabla F_{s0} + u\boldsymbol{\kappa} \cdot \mathbf{v}_{E,1} \partial_u F_{i0} = \\
&\quad \frac{Z_i e}{T_i} \mathbf{v}_{M,i} \cdot \nabla \varphi_1 F_{i0} + \mathbf{v}_{E,1} \cdot \nabla \psi \left(\frac{n_i'}{n_i} + \frac{T_i'}{T_i} \left(\frac{m_i u^2/2 + \mu B}{T_i} - \frac{3}{2} \right) \right) F_{i0}.
\end{aligned} \tag{D.10}$$

These terms are not very surprising. However, the first two lines in (D.3) look a bit awkward at first sight. The awkwardness disappears if one employs $\{\mathbf{r}, \mathcal{E}, \mu, \sigma\}$ as independent coordinates, where $\mathcal{E}$ is the total energy per mass unit and σ is the sign of the parallel velocity. The coordinate $\mathcal{E}$ expressed as a function of the coordinates $\{\mathbf{r}, u, \mu\}$ reads

$$\mathcal{E}(\mathbf{r}, u, \mu) = \frac{1}{2} u^2 + \mu B(\mathbf{r}) + \frac{Z_i e}{m_i} (\varphi_0(\psi(\mathbf{r})) + \varphi_1(\mathbf{r})). \tag{D.11}$$

We define $\tilde{F}_{i1}(\mathbf{r},\mathcal{E},\mu)$ by $\widehat{F}_{i1}(\mathbf{r},u,\mu)=\tilde{F}_{i1}(\mathbf{r},\mathcal{E}(\mathbf{r},u,\mu),\mu)$, Then, we can write

$$(\mathbf{v}_{M,i}+\mathbf{v}_{E,0})\cdot\nabla\widehat{F}_{i1}$$
$$+\Big[u\boldsymbol{\kappa}\cdot(\mathbf{v}_{\nabla B,i}+\mathbf{v}_{E,0})\Big]\partial_u\widehat{F}_{i1}=$$
$$(\mathbf{v}_{M,i}+\mathbf{v}_{E,0})\cdot\nabla\tilde{F}_{i1}+\frac{Z_ie}{m_i}\left(\mathbf{v}_{M,i}+\mathbf{v}_{E,0}\right)\cdot\nabla\varphi_1\partial_{\mathcal{E}}\tilde{F}_{i,1}. \tag{D.12}$$

Finally, the last term in the second line of (D.3) combines with the parallel streaming terms in the last line of (D.3) to give

$$-\frac{Z_ie}{m_i}\hat{\mathbf{b}}\cdot\nabla\varphi_1\partial_u\widehat{F}_{i1}+(u\hat{\mathbf{b}}\cdot\nabla-\mu\hat{\mathbf{b}}\cdot\nabla B\partial_u)\widehat{F}_{s1}=v_{||}\hat{\mathbf{b}}\cdot\nabla\tilde{F}_{i1}, \tag{D.13}$$

where

$$v_{||}(\mathbf{r},\mathcal{E},\mu,\sigma)=\sigma\sqrt{2\left(\mathcal{E}-\mu B(\mathbf{r})-\frac{Z_ie}{m_i}\left(\varphi_0(\psi(\mathbf{r}))+\varphi_1(\mathbf{r})\right)\right)}\,. \tag{D.14}$$

Hence, (D.3) is recast as

$$(\mathbf{v}_{M,i}+\mathbf{v}_{E,0})\cdot\nabla\tilde{F}_{i1}$$
$$+\frac{Z_ie}{m_i}\left(\mathbf{v}_{M,i}+\mathbf{v}_{E,0}\right)\cdot\nabla\varphi_1\partial_{\mathcal{E}}\tilde{F}_{i,1}+v_{||}\hat{\mathbf{b}}\cdot\nabla\tilde{F}_{i1}+\frac{Z_ie}{T_i}v_{||}\hat{\mathbf{b}}\cdot\nabla\varphi_1F_{i0}$$
$$+\frac{Z_ie}{T_i}\mathbf{v}_{M,i}\cdot\nabla\varphi_1F_{i0}+\mathbf{v}_{E,1}\cdot\nabla\psi\left(\frac{n_i'}{n_i}+\frac{T_i'}{T_i}\left(\frac{m_iu^2/2+\mu B}{T_i}-\frac{3}{2}\right)\right)F_{i0}$$
$$+\Upsilon_i\mathbf{v}_{M,i}\cdot\nabla\psi F_{i0}=C_{ii}^{\mathcal{E},\ell}[\tilde{F}_{i1}] \tag{D.15}$$

and, after a slight rearrangement, as

$$(\mathbf{v}_{M,i}+\mathbf{v}_{E,0})\cdot\left(\nabla\tilde{F}_{i1}+\frac{Z_ie}{T_i}\nabla\varphi_1F_{i0}\right)$$
$$+\frac{Z_ie}{m_i}\left(\mathbf{v}_{M,i}+\mathbf{v}_{E,0}\right)\cdot\nabla\varphi_1\partial_{\mathcal{E}}\tilde{F}_{i,1}+v_{||}\hat{\mathbf{b}}\cdot\left(\nabla\tilde{F}_{i1}+\frac{Z_ie}{T_i}\nabla\varphi_1F_{i0}\right)$$
$$+\Upsilon_i(\mathbf{v}_{M,i}+\mathbf{v}_{E,1})\cdot\nabla\psi F_{i0}=C_{ii}^{\mathcal{E},\ell}[\tilde{F}_{i1}]. \tag{D.16}$$

With the notation $C_{ii}^{\mathcal{E},\ell}$ we emphasize that the kernel of the collision operator has a different expression in coordinates $\{\mathbf{r},u,\mu\}$ and $\{\mathbf{r},\mathcal{E},\mu\}$. Note that the first term in the second line of the previous equation is very small. It is a nominally $O(\rho_{i*}^3)$ term, so we drop it and arrive at

$$(\mathbf{v}_{M,i}+\mathbf{v}_{E,0})\cdot\left(\nabla\tilde{F}_{i1}+\frac{Z_ie}{T_i}\nabla\varphi_1F_{i0}\right)$$
$$+v_{||}\hat{\mathbf{b}}\cdot\left(\nabla\tilde{F}_{i1}+\frac{Z_ie}{T_i}\nabla\varphi_1F_{i0}\right)$$
$$+\Upsilon_i(\mathbf{v}_{M,i}+\mathbf{v}_{E,1})\cdot\nabla\psi F_{i0}=C_{ii}^{\mathcal{E},\ell}[\tilde{F}_{i1}]. \tag{D.17}$$

Observe that

$$v_{||}\hat{\mathbf{b}}\cdot\left(\nabla\tilde{F}_{i1}+\frac{Z_ie}{T_i}\nabla\varphi_1 F_{i0}\right)=$$

$$v_{||}\hat{\mathbf{b}}\cdot\nabla\left(\tilde{F}_{i1}+F_{i0}^{\mathcal{E}}\exp\left(\frac{Z_ie\varphi_1}{T_i}\right)\right), \tag{D.18}$$

where

$$F_{i0}^{\mathcal{E}}(\psi,\mathcal{E})=n_i(\psi)\left(\frac{m_i}{2\pi T_i(\psi)}\right)^{3/2}\exp\left(-\frac{m_i\mathcal{E}-Z_ie\varphi_0}{T_i(\psi)}\right). \tag{D.19}$$

is constant over trajectories with constant total energy.

In the first line of (D.17), we can replace F_{i0} by $F_{i0}^{\mathcal{E}}$ because their difference gives a nominally $O(\rho_i^{*3})$ contribution to the equation. We can do this in the last term of the left side of (D.17) as well. However, in principle, we cannot do it in the rest of terms of that equation and we have to be careful to keep all terms that are nominally $O(\rho_i^{*2})$. Then, to the required accuracy, (D.17) can be recast as

$$\begin{aligned}&(\mathbf{v}_{M,i}+\mathbf{v}_{E,0})\cdot\nabla\tilde{G}_{i1}-\varphi_1(\mathbf{v}_{M,i}+\mathbf{v}_{E,0})\cdot\nabla\left(\frac{Z_ie}{T_i}F_{i0}^{\mathcal{E}}\right)\\&\quad+v_{||}\hat{\mathbf{b}}\cdot\nabla\left(\frac{1}{2}\left(\frac{Z_ie\varphi_1}{T_i}\right)^2F_{i0}^{\mathcal{E}}\right)\\&\quad+\Upsilon_i\mathbf{v}_{M,i}\cdot\nabla\psi\frac{Z_ie\varphi_1}{T_i}F_{i0}^{\mathcal{E}}\\&\quad+v_{||}\hat{\mathbf{b}}\cdot\nabla\tilde{G}_{i1}+\Upsilon_i(\mathbf{v}_{M,i}+\mathbf{v}_{E,1})\cdot\nabla\psi F_{i0}^{\mathcal{E}}=C_{ii}^{\mathcal{E},\ell}[\tilde{G}_{i1}],\end{aligned} \tag{D.20}$$

where

$$\tilde{G}_{i1}:=\tilde{F}_{i1}+\frac{Z_ie}{T_i}\varphi_1F_{i0}^{\mathcal{E}}. \tag{D.21}$$

Noting that φ_0, $F_{i0}^{\mathcal{E}}$ and T_i are flux functions, one has

$$\begin{aligned}&-\varphi_1(\mathbf{v}_{M,i}+\mathbf{v}_{E,0})\cdot\nabla\left(\frac{Z_ie}{T_i}F_{i0}^{\mathcal{E}}\right)\\&\quad+\Upsilon_i\mathbf{v}_{M,i}\cdot\nabla\psi\frac{Z_ie\varphi_1}{T_i}F_{i0}^{\mathcal{E}}=\\&\quad\frac{Z_ieT_i'}{T_i^2}\mathbf{v}_{M,i}\cdot\nabla\psi\,\varphi_1F_{i0}^{\mathcal{E}}\end{aligned} \tag{D.22}$$

and (D.20) becomes

$$\begin{aligned}&(\mathbf{v}_{M,i}+\mathbf{v}_{E,0})\cdot\nabla\tilde{G}_{i1}+\frac{Z_ieT_i'}{T_i^2}\mathbf{v}_{M,i}\cdot\nabla\psi\,\varphi_1F_{i0}^{\mathcal{E}}\\&\quad+v_{||}\hat{\mathbf{b}}\cdot\nabla\left(\frac{1}{2}\left(\frac{Z_ie\varphi_1}{T_i}\right)^2F_{i0}^{\mathcal{E}}\right)\\&\quad+v_{||}\hat{\mathbf{b}}\cdot\nabla\tilde{G}_{i1}+\Upsilon_i(\mathbf{v}_{M,i}+\mathbf{v}_{E,1})\cdot\nabla\psi F_{i0}^{\mathcal{E}}=C_{ii}^{\mathcal{E},\ell}[\tilde{G}_{i1}].\end{aligned} \tag{D.23}$$

The last term on the first line combines with the last term on the left side, so (neglecting again terms $O(\rho_{i*}^3)$),

$$(\mathbf{v}_{M,i} + \mathbf{v}_{E,0}) \cdot \nabla \tilde{G}_{i1}$$
$$+ v_{||} \hat{\mathbf{b}} \cdot \nabla \left(\frac{1}{2} \left(\frac{Z_i e \varphi_1}{T_i} \right)^2 F_{i0}^{\mathcal{E}} \right)$$
$$+ v_{||} \hat{\mathbf{b}} \cdot \nabla \tilde{G}_{i1} + \Upsilon_i^{\mathcal{E}} (\mathbf{v}_{M,i} + \mathbf{v}_{E,1}) \cdot \nabla \psi F_{i0}^{\mathcal{E}} = C_{ii}^{\mathcal{E},\ell} [\tilde{G}_{i1}], \tag{D.24}$$

where

$$\Upsilon_i^{\mathcal{E}} = \frac{n_i'}{n_i} + \frac{T_i'}{T_i} \left(\frac{m_i \mathcal{E} - Z_i e \varphi_0}{T_i} - \frac{3}{2} \right) + \frac{Z_i e \varphi_0'}{T_i}. \tag{D.25}$$

Finally, we define

$$F_{i1} = \tilde{G}_{i1} + \frac{1}{2} \left(\frac{Z_i e \varphi_1}{T_i} \right)^2 F_{i0}^{\mathcal{E}}. \tag{D.26}$$

Dropping some small terms again, (D.24) can be rewritten as

$$(\mathbf{v}_{M,i} + \mathbf{v}_{E,0}) \cdot \nabla F_{i1}$$
$$+ v_{||} \hat{\mathbf{b}} \cdot \nabla F_{i1} + \Upsilon_i^{\mathcal{E}} (\mathbf{v}_{M,i} + \mathbf{v}_{E,1}) \cdot \nabla \psi F_{i0}^{\mathcal{E}} = C_{ii}^{\mathcal{E},\ell} [F_{i1}], \tag{D.27}$$

Clearly, this equation coincides with (52) when expansions in $\nu_{i*} \ll 1$ and $\delta \ll 1$ are taken.

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