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Article

Mixed diffusive-convective relaxation of a broad beam of energetic particles in cold plasma

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Abstract: We revisit the applications of quasi-linear theory as a paradigmatic model for weak plasma turbulence and the associated bump-on-tail problem. The work, presented here, is built around the idea that large-amplitude or strongly shaped beams do not relax through diffusion only and that there exists an intermediate time scale where the relaxations are *convective* (ballistic-like). We cast this novel idea in the rigorous form of a self-consistent nonlinear dynamical model, which generalizes the classic equations of the quasi-linear theory to "broad" beams with internal structure. We also present numerical simulation results of the relaxation of a broad beam of energetic particles in cold plasma. These generally demonstrate the mixed diffusive-convective features of supra-thermal particle transport; and essentially depend on nonlinear wave-particle interactions and phase-space structures. Taking into account modes of the stable linear spectrum is crucial for the self-consistent evolution of distribution function and the fluctuation intensity spectrum.

Keywords: Plasma kinetic equations; Particle beam interactions in plasmas; Nonlinear phenomena.

PACS classifications: 52.25.Dg; 52.40.Mj; 52.35.Mw

1. Introduction

The relaxation of supra-thermal particle beams in plasmas is a problem of fundamental significance. The problem of a broad particle beam interacting with a one-dimensional background plasma, in fact, underlies the classic "bump-on-tail" (BoT) problem addressed in the pioneering work by Bernstein-Greene-Kruskal (BGK) [1]. It also provides the paradigm for the original quasi-linear description of weak plasma turbulence [2,3], where the beam was assumed sufficiently tenuous to neglect wave-wave coupling; and sufficiently energetic that nonlinear wave-particle interactions did not affect thermal plasma particles. Other examples of the important influence of the classic BoT problem are Landau damping in a finite amplitude wave [4,5] and nonlinear behavior of the beam-plasma instability due to wave-particle interactions [6,7] (see also [8–13]).

The relevance of the BoT problem for fusion plasmas research was revived in the 90's by Berk and Breizman [14–16], who proposed it as paradigm to investigate and understand nonlinear interaction of supra-thermal particles with Alfvénic fluctuations, provided the system is sufficiently close to marginal stability [16–19]. In fact, nonlinear interplay of energetic particles (EPs) with Alfvénic fluctuations, such as Alfvén eigenmodes (AEs), EP modes and drift Alfvén waves [17,18,20,21], and their consequences for fluctuation induced cross-field transport constitute basic phenomena for thermonuclear plasmas [19, 21–31].

The fluctuation spectrum of AEs, EP driven modes and drift Alfvén waves in fusion plasmas covers disparate spatiotemporal scales and can have both "broad" features, such as those of typical plasma turbulence, as well as an almost "coherent" ("narrow") nearly periodic component [20,29]. A line-broadened quasi-linear approach has been proposed in Refs. [32,33] for computing EP transport by means of a diffusion equation, which could address not only overlapping resonant Alfvénic fluctuations but also the broadening of the resonant spectrum for isolated instabilities in the case of multiple AEs. This model has been extended and compared with experimental observations [34] and with numerical solutions of the BoT paradigm [35]. In general, however, understanding the complex features of EP transport in fusion plasmas requires going beyond the local description of fluctuation induced fluxes and extending the diffusive transport paradigm [20,29]. Accounting for modes of the linear stable spectrum is also crucial [36,37]. Thus, posing these issues for the BoT problem becomes an interesting and relevant research topic, in the light of its possible implications as paradigm for Alfvénic fluctuation-induced supra-thermal particle transport in fusion plasmas near marginal stability.

In this work, we revisit the classic BoT problem beyond the Brownian random-walk paradigm (cf. Sec. 2) and show the crucial role played wave-particle nonlinearity in determining the non-diffusive feature of supra-thermal particle transport, which self-consistently evolves with the fluctuation intensity spectrum. Phase-space structures and, thereby, nonlinear wave-particle interactions are known to be important, *e.g.*, in enhancing velocity space diffusion in longitudinal plasma turbulence [38–40]. In this work, in particular, it is further demonstrated that they are essential processes in the mixed diffusion-convection relaxation of a broad supra-thermal particle beam in a cold one-dimensional plasma in the presence of weak Langmuir turbulence.

We assume the Chirikov parameter to be sufficiently larger than one. That is, that the fluctuation spectrum be sufficiently dense that Cherenkov resonances overlap. Following Ref. [41], we also consider a "broad" beam to be composed of a large number of overlapping single beams, each having a characteristic nonlinear width Δv^{NL}, and the spacing between the beams represented by Δv^{SEP}. For sufficiently large nonlinearity parameter $K = \Delta v^{NL}/\Delta v^{SEP}$, we show in Sec. 3 that convective relaxation of limited duration occurs on meso-scales, and is always followed by a long-time, diffusive relaxation of the quasi-linear type. These processes are described by a self-consistent system of coupled nonlinear differential equations, which generalize their quasi-linear counterparts by directly taking into account wave-particle nonlinear interactions.

We demonstrate mixed diffusion-convection relaxation by numerical simulation of the classic BoT problem with varying Chirikov parameter and wave-number spectrum width (cf. Sec 4). In particular, we demonstrate the well-known and important role of the spectral width for controlling the time-asymptotic diffusion of quasi-linear type; and emphasize the necessity of including modes of the linear stable spectrum for a proper description of supra-thermal particle transport and fluctuation intensity spectrum evolution. Numerical simulation results, furthermore, show that convective relaxation on meso-scales also occurs for small nonlinearity parameter, due to the self-consistent evolution of fluctuation intensity on the same time scale of particle transport. This interpretation is demonstrated by an analytical toy-model, derived for understanding numerical simulation results.

Finally, Sec. 5 is devoted to conclusions and discussions.

2. The classic "bump-on-tail" problem and the Brownian random-walk paradigm

We will assume that the reader is familiar with the classic "BoT" BGK problem [1] of a distribution function that excites electrostatic waves. The basic insight is that the velocity space gradient drives (damps) the instabilities via the Cherenkov resonance $v = v_{res} = \omega_{k_j}/k_j$ with the plasma species (customarily associated with the energetic electrons). Here v is the particle velocity and ω_{k_j} and k_j are respectively the frequency and the wavevector of the resonant mode. Instability occurs when the gradient is positive ($\partial f/\partial v > 0$) and is damped when it is negative ($\partial f/\partial v < 0$). The latter phenomenon is known as the Landau damping and in many ways is the inversion of the BoT problem.

In the quasi-linear theory [2,3,42] of weak plasma turbulence, one assumes that the electron distribution function contains a small "bump" in the parameter range of large energies (much larger than the characteristic thermal energy) and that the dispersion of the electron velocities is also large compared with the thermal velocity. The bump being *small* implies that the level of the excited electrostatic noises is so low that the different unstable modes do not interact, hence one neglects any nonlinearities in the wave-field. Then the only nonlinearity one indeed takes into account is feedback effect of the waves onto the averaged velocity distribution function (hence the name of this theory: *quasi-linear*). The linear instability growth rate, which is frequency and wavevector dependent, is given by

$$\gamma_{k_j} = \frac{\pi\omega_p^3}{2k_j^2 n_p}\frac{\partial f}{\partial v}|_{v_{res}} \ , \tag{1}$$

where ω_p is the unperturbed Langmuir frequency (*i.e.*, $\omega_p \equiv \sqrt{4\pi n_p e^2/m_e}$, with n_p the thermal plasma density; and m_e and e denoting the mass and charge of electrons, respectively) and the gradient in the velocity space is taken at the Cherenkov resonance exactly. The large velocity dispersion within the bump is at the basis of another important assumption of the quasi-linear theory, namely, that the number of the excited modes is so large that their phases are represented by the random functions. Then the kicks received by the plasma particles via the resonant interactions with the waves will be also random in the limit $t \to \infty$ (here t is the time). It is generally believed that this quasi-linear effect of the many excited waves onto the plasma is contained in a Brownian random walk of the energetic particles toward the thermal core and the formation of a characteristic "plateau" ($\partial f / \partial v = 0$) in the equilibrium velocity distribution function [2,3].

The conclusion that the transport in the velocity space is diffusive or Brownian random walk-like is not at all trivial, though, and must be addressed. It occurs as a consequence of the idea that the resonant particles are not caught by one single resonance on time scales longer than a certain characteristic time (thought as the typical bouncing time in the potential well of the wave). Instead these particles will hop in a random fashion between the many overlapping resonances, hence their motion is statistical, rather than deterministic. This statistical approach to the particle random motion in the velocity space is in many ways due to the Novosibirsk school of nonlinear science (*e.g.*, books [43,44] and reviews [45,46]).

Consider the exact equations of motion of a charged particle in the potential electric field of a wave packet

$$\ddot{x} = \frac{e}{m} E(x,t) = \frac{e}{m} \sum_j \left[E_{k_j} e^{i(k_j x - \omega_{k_j} t)} + c.c. \right] , \tag{2}$$

where $\dot{x} = v$ is the particle velocity, and E_{k_j} is the Fourier amplitude of a mode with wavevector k_j. The electric field is assumed to be "small" in the sense of quasi-linear theory; and here it is represented as the sum of a large number of Fourier harmonics with frequencies ω_{k_j} and wavevectors k_j. Hence, it is shown, following Zaslavsky [43], that barely trapped particles $-$ those staying close to separatrices in phase space $(x, \dot{x})$ $-$ may occasionally become detrapped by jumping onto an open trajectory. These phenomena of occasional trapping-detrapping occur because the integrals of the motion are essentially destroyed by the perturbation $E(x,t)$ within a small layer surrounding the separatrix (known as the ergodic layer) [43,44]. For not too small perturbations, the width of the ergodic layer with respect to frequency is of the order of

$$\Delta\omega_{k_j} = \sqrt{\frac{ek|E_{k_j}|}{m}} . \tag{3}$$

The diffusion behavior occurs, when the width $\Delta\omega_{k_j}$ exceeds by a large margin the distance between the resonances $\delta\omega_{k_j}$ (for a statistically significant number of k's), implying that the resonances strongly overlap within a certain interval of wavevectors k_j. This is usually quantified by saying that the Chirikov parameter is large, *i.e.,*

$$\mathcal{S} = \Delta\omega_{k_j} / \delta\omega_{k_j} \gg 1 . \tag{4}$$

One sees that the Chirikov parameter $\mathcal{S} \propto \sqrt{|E_{k_j}|}$ being much larger than 1 implies that the level of the excited electrostatic waves in turn cannot be as small as one likes. It might not be taken for granted that this level just lies within the assumptions of the quasi-linear theory discussed above, even though we know *by experience* that the conflict does not normally occur in the classic BoT setting.

In what follows, we revise the applications of quasi-linear theory as a paradigmatic model for weak plasma turbulence and the associated BGK problem. *The main idea here is that large-amplitude or* *strongly shaped beams do not relax through diffusion only* and that there exists an intermediate time scale where the relaxations are *convective* (ballistic-like). We cast this novel idea in the rigorous form of a self-consistent nonlinear dynamical model, *i.e.,* mixed diffusion-convection model, which generalizes the classic equations of the quasi-linear theory to "broad" beams with internal structure.

3. Revising the classic "bump-on-tail" problem: Diffusion-convection model

We follow Ref. [41] and consider a "broad" beam as composed of a large number of overlapping single beams, each having a characteristic nonlinear width Δv^{NL}, and the spacing between the beams represented by Δv^{SEP}. The value of Δv^{NL} is determined by the kinematics of nonlinear resonance broadening in the coupled system of electrostatic waves and beam-plasma particles (*e.g.*, Ref.[44,47]). The dynamics crucially depend upon the value of the nonlinearity parameter

$$K = \Delta v^{NL}/\Delta v^{SEP} \ . \tag{5}$$

We note that the nonlinearity parameter above and the Chirikov parameter, introduced in Eq. (4), are intrinsically connected for the self-consistent beam-plasma problem, as emphasized in Ref. [41]. In fact, each individual beam, which the "broad" beam is thought to be composed of, can drive, when considered independent of others, a plasma wave that naturally saturates when $\Delta v^{NL} \simeq |\Delta\omega_{k_j}/k_j|$. Thus, K and $\mathcal{S}$ are interlinked for the case we consider in the present work (cf. also Sec. 4.5), while, in general, the two parameters may be controlled independently of each other; *e.g.*, when the fluctuation spectrum is externally imposed, rather than being self-consistently determined by the broad beam distribution, as evolved from the linearly unstable modes.

For $K \ll 1$, the comprising beams are well separated in velocity space, and the transport problem for the beam particles is reduced to a superposition of the respective BGK problems for each beam. We are not interested in this regime in this section.

The behavior changes drastically, if $K \gg 1$. In this parameter range, the beams strongly overlap as a consequence of the strong nonlinearity, and are not separable in the BGK sense. The main effect the overlap between the beams has on the instability growth rate is amplification of the number density of the resonant particles. Summing across all overlapping beams (and dropping the j-index in this whole Section for the sake of simplicity), we have, instead of Eq.(1),

$$\Gamma_k \simeq \frac{\pi\omega_p^3}{2k^2 n_p}\left(\frac{\partial f}{\partial v}|_{v_{res}} + \frac{\Delta f}{\Delta v^{NL}}\right) \ , \tag{6}$$

where Δf is calculated in vicinity of the resonant velocity $v_{res} = \omega_k/k$ within the velocity spread Δv^{NL}. Eq.(6) suggests that the instability growth rate in the broad-beam problem is incremented by

$$\Delta\gamma_k = \Gamma_k - \gamma_k \simeq \frac{\pi\omega_p^3}{2k^2 n_p}\frac{\Delta f}{\Delta v^{NL}} \ , \tag{7}$$

as compared to the classic BoT problem, with the growth rate γ_k. *A priori*, one may expect that the amplification Δf is proportional to the local number density, hence to the velocity distribution itself; that is, $\Delta f \simeq \chi f$, where the coefficient χ quantifies the coupling properties among the beams. Following Refs.[48,49], we write this coefficient as the Boltzmann factor $\chi = \exp[-\Delta v^{SEP}/\Delta v^{NL}]$.

The implication is that Δf is limited to the transition probabilities of resonance particles on a grid, with the spacing Δv^{SEP}, and the effective "temperature" of nonlinear interaction Δv^{NL}. Using the K parameter, we may also write $\chi = \exp[-1/K]$, which interpolates between the classic BoT problem ($K \ll 1$; $\chi \simeq 0$) and the regime of strong nonlinearity of interest here ($K \gg 1$; $\chi \simeq 1$). We note in passing that the functional dependence in the χ value is non-perturbative in that it goes as exponential of $-1/K$ and not as $1/K$, this being a small parameter of the model. Given that the Chirikov parameter is large, *i.e.,* that the condition in Eq.(4) holds, the relaxation current in velocity space J_k (*i.e.,* flux of the energetic particles in the direction of the thermal core) will be proportional to the instability growth rate Γ_k. We have $J_k = -\zeta\Gamma_k$, where ζ is a numerical normalization parameter and will be obtained below. The minus sign indicates that the flux goes against the v axis. With the aid of Eq.(6) one obtains

$$J_k = -\zeta \frac{\pi\omega_p^3}{2k^2 n_p}\left(\frac{\partial f}{\partial v}|_{v_{res}} + \chi \frac{f}{\Delta v_{\rm NL}}\right) . \tag{8}$$

Note that the resonance part of J_k, *i.e.,* first term on the right-hand-side of Eq.(8), has the sense of Fick's second law in velocity space.[1] The continuity of the flux-function implies

$$\frac{\partial f}{\partial t} + \nabla_v \cdot J_k = 0, \quad \nabla_v \equiv \partial/\partial v . \tag{9}$$

Combining with Eq.(8), one is led to a Fokker-Planck equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v}\Lambda_v \delta(v - \omega_k/k)\frac{\partial f}{\partial v} + \frac{\partial}{\partial v}\chi\Lambda_v \frac{f}{\Delta v^{NL}} , \tag{10}$$

where we have denoted for simplicity $\Lambda_v = \zeta(\pi\omega_p^3/2k^2 n_p)$. Clearly, the first term on the right-hand-side of Eq.(10) represents the well-known quasi-linear diffusion in the limit $t \to +\infty$. Following the analysis of the previous Section, in a basic theory of weak plasma turbulence one writes the quasi-linear diffusion coefficient as [42]

$$D_{QL} = \frac{\pi e^2}{m_e^2}\sum_k |E_k|^2 \delta(\omega_k - kv) = \frac{\pi e^2}{k m_e^2}\sum_k |E_k|^2 \delta(v - \omega_k/k) . \tag{11}$$

Comparing with Eq.(10), and remembering the expression of Λ_v, one arrives at

$$\zeta \frac{\pi\omega_p^3}{2k^2 n_p}\delta(v - \omega_k/k) \simeq \frac{\pi e^2}{k m_e^2}\sum_k |E_k|^2 \delta(v - \omega_k/k), \tag{12}$$

from which the ζ value can be inferred via an asymptotic matching procedure. At this point, no free parameters have remained in the Fokker-Planck model in Eq.(10).

In the discussion above, we have implicitly assumed that the distribution function f does not involve any coordinate dependence in real space. Relaxing this, we should obtain, instead of Eq.(10),

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = \frac{\partial}{\partial v}\Lambda_v \delta(v - \omega_k/k)\frac{\partial f}{\partial v} + \frac{\partial}{\partial v}\chi\Lambda_v \frac{f}{\Delta v^{NL}} , \tag{13}$$

[1] The Fick paradigm states that internal fluxes are driven by point-wise gradients with local coefficients: diffusivities and conductivities. Models based on these assumptions are referred to as local transport models.

where x is the position variable in real space. The implication is that the time derivative $\partial/\partial t$ is replaced by the Stokes operator $\partial/\partial t + v\partial/\partial x$ as soon as eventual x-dependencies come into play. Eq.(13) is a partial case of the Klein-Kramers equation away from external force fields. The closure of the Klein-Kramers equation is obtained through

$$\frac{\partial |E_k|^2}{\partial t} + v_g \frac{\partial |E_k|^2}{\partial x} = 2\Gamma_k |E_k|^2 \ , \tag{14}$$

which generalizes the well-known quasi-linear equation [42] for spectral energy density in that it uses the amplified increment Γ_k in place of γ_k. In the above, $v_g = \partial\omega_k/\partial k$ is group velocity. Note that the instability growth rate on the right-hand-side of Eq.(14) steps in with the front factor 2; this is because the square of the wave amplitude decays twice faster than the amplitude itself. Eqs.(13) and (14) represent the basic system of dynamical equations for our model. These equations describe the beam relaxation and the evolution of the wave spectrum as coupled processes and extend via the Γ_k value the known equations of weak plasma turbulence to the broad-beam problem.

3.1. Towards multi-scale dynamics: Comparing the relaxation times

An essentially new element of our model is the case of *convective relaxation* $-$ contained in the last term on the right-hand-side of Eq.(13). Indeed the convection term enters on an equal footing with the classic diffusion term via the generalization of the instability growth rate in Eq.(6). The characteristic relaxation time in the convection domain is easily seen to be given by

$$\tau_{\rm conv} \simeq (\Delta v^{NL})^2/\chi\Lambda_v \ . \tag{15}$$

This should be compared with the characteristic relaxation time via the quasi-linear diffusion, which we assess as $\tau_{\rm diff} \simeq \Delta v_b^2/\Lambda_v$, where Δv_b is the broad beam width in velocity space. In general, we can assume $\Delta v_b \gg \Delta v^{NL}$, suggesting that the convective relaxation is a meso-scale process. We interpret this result as follows.

3.2. Mixed diffusive-convective behavior and the asymptotic character of the diffusion

For $K \gg 1$ (*i.e.,* $\chi \simeq 1$, strong-overlap limit), the relaxation dynamics are completely dominated by the nonlinear amplification of the instabilities via the share in the resonance particle population. The coupling processes among the beams are such as to increase the density of the resonant particles where the second derivative of the distribution function is positive, *i.e.,* $\partial^2 f/\partial v^2 > 0$, and, via the conservation of the total number of the particles, act to decrease, at the same time, the resonant density where the second derivative is negative ($\partial^2 f/\partial v^2 < 0$). This generates an unstable propagating front in velocity space directed to the thermal core of the distribution function comprising the beams. Moreover the front is self-reinforcing: its strength, *i.e.,* the density pedestal, initially grows with time as more resonance particles are trapped in it. The process is analogous to a snow avalanche or self-amplifying chain reaction.[2]

[2] The idea that the avalanching dynamics occur via an amplification of instabilities in the parameter range of large nonlinearity is indeed very general and have been discussed for "strong" electrostatic turbulence in Refs.[50,51].

As the instability amplifies itself, the velocity gradient on its edge becomes persistently steeper. This feeds back the diffusive relaxation via the increasing $\partial f/\partial v$ term. Above some level of the front steepening and amplification, the convective and the diffusive relaxations start to operate on essentially an equal footing. The process ends with sharp (shock-wavelike) instability front and the relaxations rapidly switching to diffusion. It is in this sense that we have termed the convective relaxation to be meso-, respectively, beam size scale on which the amplifications occur. The asymptotic ($t \to +\infty$) relaxation process will be *always* diffusive quasi-linear.

One sees that the relaxation process acquires multi-scale features in the broad-beam problem: It begins as initially a convective process, with time scale in Eq.(15), followed by an asymptotic diffusion process on very long time scales. Note that the convective relaxation occurs despite the condition that the Chirikov parameter is large, *i.e.,* $\mathcal{S} \gg 1$, and is attributed to the fact that the nonlinearity of the interaction, contained in the ordering $K \gg 1$, is also large in its turn, giving rise to a Fokker-Planck generalization of the velocity-space diffusion equation. We should stress that by "generalizing" the quasi-linear theory we mean in fact the inclusion of the meso-scale relaxation of the convective type (via the coupling term in the respective Klein-Kramers equation), without touching on the asymptotic, diffusive behavior and the formation of the plateau.

3.3. Boltzmann's H-theorem and the entropy growth rates

Although obvious, it should be emphasized that the Chirikov parameter being much larger than 1 implies that the dynamics are random (chaotic-like) on micro-, respectively, wave-particle interactions, scales. It is convenient to characterize this implication of the chaotic dynamics in terms of Boltzmann's H-theorem and assess the corresponding entropy growth rates through respectively the diffusion and the convective relaxation processes.

For weakly interacting classical systems, the entropy $S = S(t)$ is related to the probability density function through

$$S(t) = -\int_{-\infty}^{+\infty} f(t,v)\log f(t,v)dv \; . \tag{16}$$

We shall assume for simplicity, without lacking generality, that the distribution function f represents solely the beam particles, so it is identically zero outside the bump region. So we differentiate the functional dependence in $S(t)$ over the time, then substitute the time derivative $\partial f(t,v)/\partial t$ with the sum of the diffusion and the convection terms using the Fokker-Planck equation (10), and integrate by parts over the velocity variable under the assumption that the velocity gradients vanish on both sides of the integration domain. Making use of the resonance condition, after a simple algebra one obtains

$$\frac{d}{dt}S(t) = \Lambda_v f^{-1}\left(\frac{\partial f}{\partial v}\right)^2 |_{v_{res}} \geq 0 \; . \tag{17}$$

One sees that the time derivative of the entropy is always non-negative, *i.e.,* $dS/dt \geq 0$, and is moreover restricted to the relaxation process of the diffusive type, as it should. Indeed the convective relaxation does *not* as a matter of fact contribute to the entropy growth rate and in this case can be considered *adiabatic*.

4. Numerical simulations of broad beam relaxation in cold plasma

In this Section, in support of the theoretical framework introduced above, we discuss numerical simulation results of the mixed diffusive-convective relaxation of a tenuous broad beam of energetic particles, whose density n_B is much smaller than that of the 1D background plasma (n_p), which is considered as a cold linear dielectric medium. In particular, we adopt the Hamiltonian formulation of the problem described in Ref.[41], where the broad supra-thermal particle beam is discretized as superposition of $n \gg 1$ cold beams self-consistently evolving in the presence of $m \geq n$ modes nearly degenerate with Langmuir waves, *i.e.,* with frequency $\omega_{k_j} \simeq \omega_p$ for $j = 1, ..., m$. This ensures that the dielectric function of the cold background plasma is nearly vanishing and to cast the Poisson equation for each plasma oscillation into the form of a simple evolution equation, while particles trajectories are solved from the equations of motion (2) [6].

In the following, we first briefly summarize the Hamiltonian formulation of the broad beam relaxation in cold plasma derived in Ref.[41] (Sec.4.1). Then, we introduce the dimensionless parameters that are used to characterize the different nonlinear dynamic regimes (Sec.4.2), which will be studied numerically with four different set-ups of initial conditions (Sec.4.3). Numerical simulation results are then presented and discussed, adopting also a test particle transport analysis to illustrate the mixed diffusive-convective nature of beam relaxation (Sec.4.4). Finally, we discuss a simple toy-model of diffusion-convection relaxation to illustrate the important role played by the self-consistent evolution of fluctuation intensity on the same time scale of beam transport (Sec.4.5).

4.1. The multi beam approach: BoT Hamiltonian formulation

This model is described in Ref. [41], which we refer to for details and discussion of its relationship to the vast literature on Hamiltonian formulation of the BoT problem. The model consists in the description of n cold beams self-consistently evolving in the presence of $m \geq n$ modes with $\omega_{k_j} \simeq \omega_p$. Among these waves, we select n modes which are resonantly driven by the beams (linearly unstable), *i.e.,* $k_\alpha = \omega_p/v_\alpha$ (where v_α are the initial beam velocities and $\alpha = 1, ..., n$) and $m - n$ linearly stable ones. The nonlinear evolution of this system, as shown in Ref.[41], is equivalent to a broad beam and m Langmuir waves for properly chosen initial conditions [52]. More specifically, the n cold beams can be considered as an initial discretization of the broad beam, when particles are conveniently distributed.

The dynamics of beam electrons is determined by the Newton's law, while Poisson's equation give the the self-consistent mode evolution. The 1D cold plasma equilibrium is taken as a periodic slab of length L, and the position along the x direction for each beam is labeled by x_α. Following Ref.[6], we assume that single beams consist of N_α particles located at $x_{\alpha i}$ (*i.e.,* x_{1i}, x_{2i}, ..., x_{ni}; $i = 1, ..., N_\alpha$), so that $\sigma_\alpha n_B$ is the particle density of the "α"-beam, with $\sigma_\alpha = N_\alpha/N$ and $N = N_1 + .. + N_n$ the total particle number. For the sake of convenience, beam particle positions are represented as $x_{\alpha i} = \bar{v}t + \xi_{\alpha i}(t)$, with $\bar{v} = (v_1 + .. + v_n)/n$ the initial average beam speed. Meanwhile, the Langmuir wave scalar potential $\varphi(x, t)$ is expressed in terms of the Fourier components $\varphi_j(k_j, t)$ (with $j = 1, ..., m$), yielding the corresponding electric fields $E_{k_j}(k_j, t) = -ik_j\varphi_j(k_j, t)$, consistent with Eq.(2). Finally, consistent with Ref.[6], the following scaled variables are introduced: $\ell_j = (2\pi/L)^{-1}k_j$, $\bar{\xi}_{\alpha i} = 2\pi\xi_{\alpha i}/L$, $\bar{\eta} = (n_B/2n_p)^{1/3}$, $\tau = t\omega_p\bar{\eta}$, $\phi_j = (\varphi_j\, ek_j^2)/(m\bar{\eta}^2\omega_p^2)$, $\beta_j = \big(k_j\, \bar{v}/\omega_p - 1\big)/\bar{\eta}$. Noting that the plasma dielectric constant, $\epsilon_p(\omega_k)$, is nearly vanishing, and adopting time scale separation between τ evolution and ω_p^{-1} [6],

$$\epsilon_p(\omega_{k_j}) = 1 - \frac{\omega_p^2}{\omega_{k_j}^2} \simeq 2i\bar{\eta}\frac{\partial}{\partial\tau} \; . \tag{18}$$

Noting Eqs.(2) and (18), the system of equation governing the interaction between m Langmuir modes and n beams finally reads as [41]

$$\bar{\xi}''_{\alpha i} = \sum_{j=1}^{m} \left(i\, \ell_j^{-1}\, \phi_j\, e^{i\ell_j \bar{\xi}_{\alpha i} + i\beta_j \tau} + c.c. \right) \; , \tag{19a}$$

$$\phi'_j = i e^{-i\beta_j\tau} \sum_{\alpha=1}^{n} \frac{\sigma_\alpha}{N_\alpha} \sum_{i=1}^{N_\alpha} e^{-i\ell_j \bar{\xi}_{\alpha i}} \; , \tag{19b}$$

where the prime denotes the derivative with respect to τ. Momentum and energy conservation properties of Eqs.(19) are discussed in Ref.[41].

In the simulations, each beam has an initial velocity expressed as $v_\alpha = \Theta_\alpha v_1$; and we get $\bar{v} = \Theta v_1/n$, with $\Theta = \Theta_1 + ... + \Theta_n$. As we assume that the first n modes are resonantly driven by the beams ($k_\alpha = \omega_p/v_\alpha$), one obtains $\Theta_\alpha = \ell_1/\ell_\alpha$. The initial conditions for the beam nonlinear shifts $\bar{\xi}_{\alpha i}$ are given random for each beam between 0 and 2π, while the initial velocities are $\bar{\xi}'_{\alpha i}(0) = (\Theta_\alpha - \Theta/n)/(\ell_1\bar{\eta})$. Furthermore, the resonance mismatch terms read $\beta_{j=1,\ldots,m} = (\Theta\ell_j/n\ell_1 \; - 1)/\bar{\eta}$. Finally, we note that the broad beam distribution function can be described by a suitable choice of the n beam partial densities σ_α. In the following, we solve Eqs.(19) via a Runge-Kutta (fourth order) algorithm and $N = 10^5$ total particles. For the considered time scales and for an integration step $h = 0.01$, both the total momentum and energy (for the explicit expressions, see [41]) are conserved with relative fluctuations of about 1.4×10^{-5}.

4.2. Characterization of the nonlinear dynamic regime

Numerical simulations, discussed here, illustrate different nonlinear dynamic behaviors, which can be characterized introducing proper dimensionless parameters. In particular, we can introduce the Chirikov parameter [53] that characterizes the resonance overlap of the fluctuation spectrum as in Eq. (4). For each of the n cold beams, we can evaluate the corresponding Chirikov parameter $\mathcal{S}_\alpha$ for the initial conditions discussed in the previous subsection and consistent with Eq. (4). Assuming $\ell_{\alpha+1} = \ell_\alpha + 1 \gg 1$, we get the following scaled expression of the Chirikov parameters

$$\mathcal{S}_\alpha =\simeq \ell_\alpha \bar{\eta}\sqrt{|\phi_\alpha^{sat}|} \; . \tag{20}$$

As already stated in the previous Section, for $\mathcal{S}_\alpha < 1$, the beams can be treated as independent, while resonance overlap occurs for $\mathcal{S}_\alpha > 1$. These two different situations correspond to different transport processes in the velocity space. The case with $\mathcal{S}_\alpha \lesssim 1$ was studied in Ref.[41] with the multi beam approach adopted here. In the present work, we focus on the $\mathcal{S}_\alpha > 1$ case, when many resonances are overlapping in the sense that the nonlinear velocity broadening in comparable or greater that the resonance separation. Our aim, in fact, is to assess the different role of diffusion and convection in the relaxation of a broad beam, interacting with multiple Langmuir fluctuations in a cold background plasma. In particular, one of the main goals of our numerical simulations, presented below, is to clarify how linear stable modes affect particle transport in the BoT problem and how it depends on the Chirikov parameter.

As already discussed in Secs. 2 and 3, we remind that large $\mathcal{S}_\alpha > 1$, intuitively, corresponds to diffusive beam relaxation in velocity space [38,39] and to the typical condition of applicability of quasi-linear theory [2,3]. However, such a case also requires the fluctuation spectrum be sufficiently broad; *i.e.,* characterized by a sufficiently large $\Delta k/k$ and a correspondingly large $\Delta v/v$, with $\Delta k = |k_m - k_1|$ and $\Delta v = |v_m - v_1|$. This is because the characteristic scattering time of beam particles with fluctuations (in the present $(\bar{\eta}\omega_p)^{-1}$ units)

$$(\bar{\eta}\omega_p)^{-1}\tau_s \simeq \frac{2\pi}{v\Delta k} \simeq \frac{2\pi}{k\Delta v} \ , \tag{21}$$

must be sufficiently short compared to the characteristic time, τ^{NL}, for fluctuation induced changes of the beam distribution function to occur. Using quasi-linear theory [2,3], we can estimate τ^{NL} with the quasi-linear diffusion time that, by means of Eq. (11), can be cast as

$$\begin{aligned}\bar{\eta}\omega_p \tau_{\rm diff}^{NL\,-1} \simeq \frac{D_{QL}}{(\Delta v)^2} \simeq \frac{\bar{\eta}\omega_p}{4\pi^2}\sum_\alpha \omega_{B\alpha}^4 \tau_{s\alpha}^3 \simeq \frac{\bar{\eta}\omega_p}{4\pi^2}\sum_\alpha \left(ek_\alpha^2\varphi_\alpha^{sat}/m\right)^2 \tau_{s\alpha}^3 \\ \simeq 2\pi\bar{\eta}\omega_p \sum_\alpha |\phi_\alpha^{sat}|^2 \bar{\eta}^3 \ell_\alpha^3 (\Delta\ell)^{-3} \ .\end{aligned} \tag{22}$$

Here, we have denoted the normalized wave-particle trapping frequency, estimated from Eq. (3) for an isolated resonant mode at saturation, as $\omega_B = (\bar{\eta}\omega_p)^{-1}\left(ek^2\varphi^{sat}/m\right)^{1/2}$. Thus, noting that for n resonant modes

$$\left(\tau_s/\tau_{\rm diff}^{NL}\right) \simeq 4\pi^2 n (\tau_s/\tau_B)^4 \ll 1 \ , \tag{23}$$

it follows that $\tau_s \ll \tau_B = 2\pi\omega_B^{-1}$ is a necessary but not sufficient condition in order for quasi-linear theory to be valid and beam relaxation to be diffusive. Meanwhile, adopting the theoretical framework of Secs. 2 and 3, $\tau_{\rm conv}^{NL}$ in the convection domain can be estimated from Eqs. (11) and (15) as

$$\bar{\eta}\omega_p \tau_{\rm conv}^{NL\,-1} \simeq \frac{D_{QL}}{(\Delta v^{NL})^2} \simeq \bar{\eta}\omega_p \sum_\alpha \omega_{B\alpha}^2 \tau_{s\alpha} \simeq 2\pi\bar{\eta}\omega_p \sum_\alpha |\phi_\alpha^{sat}| \bar{\eta}\ell_\alpha (\Delta\ell)^{-1} \ . \tag{24}$$

Introducing a typical mode number ℓ_α and the parameter Q

$$Q \equiv \tau_B/\tau_s \simeq \ell_\alpha^{-1}\bar{\eta}^{-1}|\phi_\alpha^{sat}|^{-1/2}\Delta\ell \simeq (\Delta\ell/\mathcal{S}_\alpha) \gg 1 \ , \tag{25}$$

whose reciprocal physically represents the fraction of a trapping time over which a particle is accelerated by any of the fluctuating fields, we see that larger $\mathcal{S}_\alpha > 1$ corresponds to smaller Q for fixed $\Delta\ell$. Note that

$$\left(\tau_s/\tau_{\rm diff}^{NL}\right) \simeq 4\pi^2 n/Q^4 \ll 1 \ , \tag{26}$$

for quasi-linear theory to be valid. Furthermore,

$$\left(\tau_s/\tau_{\rm conv}^{NL}\right) \simeq 4\pi^2 n/Q^2 \ll 1 \ , \tag{27}$$

and $\tau_{\text{diff}}^{NL}/\tau_{\text{conv}}^{NL} \sim Q^2$. Clearly, Eq.(26) implies Eqs. (25) and (27), but not vice-versa. In particular, we note that if Q is not sufficiently large, the nonlinear behavior may be significantly coherent as consequence of phase space structures. Meanwhile, characteristic time scales of convective and diffusive relaxation of the particle beam become comparable. In the following, we will discuss four different cases with varying combinations of Q and $\mathcal{S}$ (with $\mathcal{S}$ a representative value of $\mathcal{S}_\alpha$), illustrating the mixed diffusive-convective relaxation of a broad beam.

4.3. Numerical simulation results

We report, here, numerical simulation results with $n = 50$ beams for 4 distinct cases, which differ for the number of modes m and for the magnitude of the relevant Chirikov parameters: *(i)* $\mathcal{S} \simeq 2$ and $m = n = 50$ (only resonant modes); *(ii)* $\mathcal{S} \simeq 2$ and $m = 120$ (50 resonant modes and 70 linearly stable modes); *(iii)* $\mathcal{S} \simeq 15$ and $m = n = 50$; *(iv)* $\mathcal{S} \simeq 15$ and $m = 120$. The initial values of the modes are set with random phases ψ_j and with random amplitude of $\mathcal{O}(10^{-2})$ for the resonant (linearly unstable) n modes, while the random amplitude of non-resonant (linearly stable) modes is of $\mathcal{O}(10^{-3})$. The initial densities of the beams, *i.e.,* the σ_α values ($\alpha = 1, ..., n$), are assumed Gaussian distributed in the velocity space as depicted in FIG.1 (left-hand panel), where we also plot the initial phase space for the cases *(i)*, *(ii)* (central panel) and *(iii)*, *(vi)* (right-hand panel). Cases *(i)*, *(ii)* are chosen to illustrate the

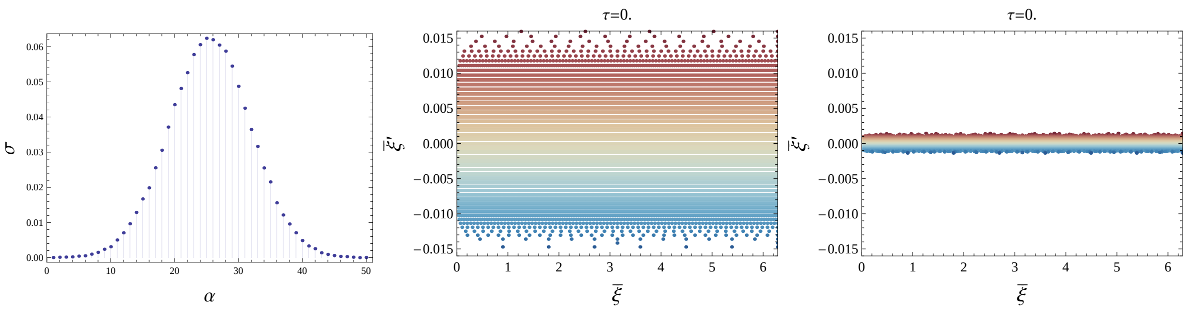

Figure 1. Left-hand panel: densities of the beams (σ_α) as a function of the beam index $\alpha = 1, ..., n$. Initial ($\tau = 0$) phase space ($\bar{\xi}, \bar{\xi}'$) for the cases *(i)*, *(ii)* (central panel) and *(iii)*, *(vi)* (right-hand panel): each point corresponds to a single charge sheet and different colors denote different beams.

mixed diffusive-convective character of a broad beam under the effect of a broad fluctuation spectrum. In fact, case *(i)* has $Q \simeq 30$; thus, we expect that the addition of non-resonant modes in case *(i)* may change the beam relaxation quantitatively but not qualitatively. Meanwhile, cases *(iii)*, *(iv)* are selected to show the possible crucial qualitative effect of including the stable part of the fluctuation spectrum. More precisely, $Q \simeq 3.6$ in case *(iii)* and, thus, significant coherent behavior can be expected; while in case *(iv)*, the inclusion of stable modes more than doubles the spectrum width, restoring the time asymptotic diffusive behavior. Let us now analyze separately these 4 distinct cases, discussing their features and the different underlying physics processes.

Case (i) : $\mathcal{S} \simeq 2$ and $376 \leq \ell_{res} \leq 425$. In this case, the mode evolution and the corresponding particle velocity distribution function, for sufficiently large time, are consistent with the quasi-linear model prediction. In the left panel of FIG.2, we can observe the saturation of each spectral component, while the other two panels show the intensity spectrum at different times; with a peak structure shifting towards large ℓ, consistent with total energy conservation. This suggests, at least in the initial phase

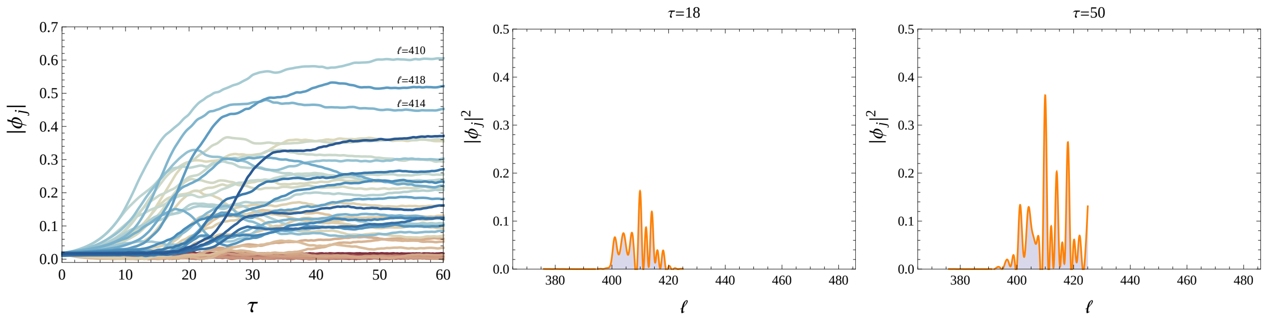

Figure 2. Case *(i)*. Left-hand panel: temporal evolution of the mode amplitude $|\phi_j|$; colors are fixed from blue (largest ℓ) to red (smallest ℓ), and time-asymptotic dominant wave-numbers are indicated. Central and right-hand panels: intensity spectrum at different times.

of the evolution, the presence of a convection (drag) in velocity space, reflecting the effect of coherent structures. Time asymptotically, FIG.3 shows how the velocity distribution function evolves towards a plateau, consistent with quasi-linear diffusion. In other words, the dominant diffusion process is

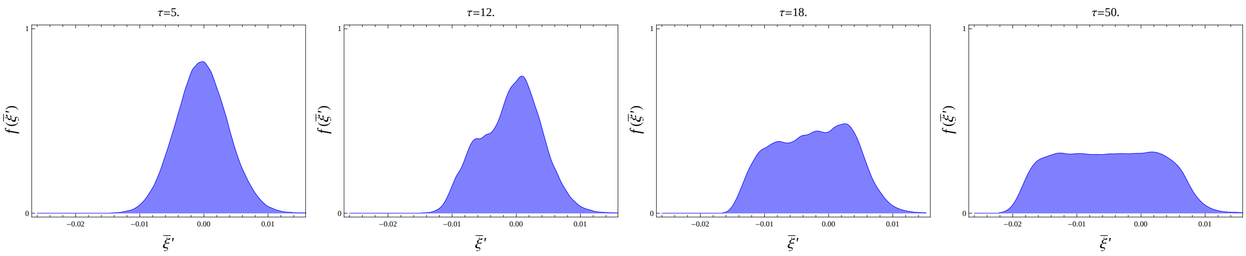

Figure 3. Case *(i)*. Evolution of the velocity distribution function $f(\bar{\xi}')$ (in arbitrary units), shown at different times.

due to particles interacting with multiple modes with random phases. It is also worth observing how the coarse-grained features of this distribution function are consistent with the results obtained in [52]. Moreover, in that work, the energy spectrum is shown to have irregularities associated with significant jumps in the particle velocities, characterized by Gaussian distribution. Similar irregularities can also be observed in our energy spectrum and, therefore, we can conclude that they do not depend on the morphology of the initial fields phase (in [52], the initial phases are not taken as random).

In this case, we get the following estimations: $\tau_s = 0.5$, $\tau_{\text{diff}}^{NL} \simeq 113$ and $Q \simeq 30$ ($\tau_B \simeq 16$). This is consistent with the broad character of the considered spectrum and condition (26) is properly satisfied. Thus, we can conclude that case *(i)*, time asymptotically, can be described by the well known quasi-linear approximation.

Case (ii) : $\mathcal{S} \simeq 2$ and $366 < \ell < 376 \leq \ell_{res} \leq 425 < \ell < 485$. In this case, which is the same as case *(i)* but with linear stable modes included, we see in FIG.4 how the fluctuation spectrum is effectively broadened and the behavior of the saturated modes is significantly different from that of isolated resonant modes. A significant convection (drag) in velocity space is associated with the energy transfer from fast particles to the initially linear stable modes. The presence of well-marked peaks in

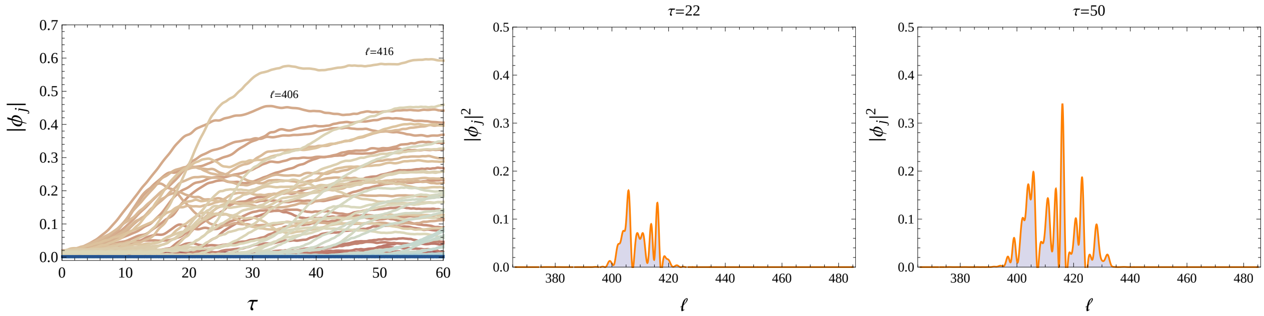

Figure 4. Case *(ii)*. Left-hand panel: temporal evolution of $|\phi_j|$. Central and right-hand panels: intensity spectrum at different times.

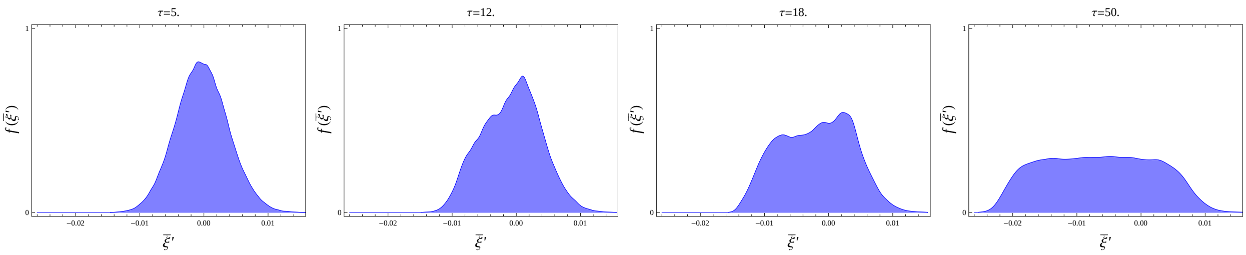

Figure 5. Case *(ii)*. Evolution of the velocity distribution function.

the intensity spectrum is limited to a maximum ℓ-value, beyond which energy transfer to new modes is almost suppressed and the saturation level of the excited modes is decreasing. The reason for such behavior must is due to a progressive decrease of resonant particle population. The velocity distribution function (see FIG.5) behaves similarly to the previous case, with density flattening followed by a uniform plateau formation. However, with respect to case *(i)*, the velocity broadening of the distribution function is more extended as a consequence of stable modes.

Case (iii) : $\mathcal{S} \simeq 15$ and $476 \leq \ell_{res} \leq 525$. This case, when the Chirikov parameter is significantly greater then one and without the inclusion of linear stable modes, is characterized by $Q \simeq 3.6$ ($\tau_s = 4.5$, $\tau_B \simeq 16$). The spectrum, thus, is not broad and it is expected that the quasi-linear paradigm is not appropriate in this case. In fact, in FIG.6 we can observe a morphology of mode saturation typical of isolated resonances and the intensity spectrum is characterized by a single dominant peak, which is progressively shifted up to the largest ℓ-value in the simulation domain. Such behavior of the fluctuation spectrum is clearly reflected by the velocity distribution function in FIG.7: after initial density flattening, the particle distribution function shows coherent behavior related to structures due to wave-particle trapping. In this scenario, a uniform plateau is never formed and the asymptotic evolution resembles that of an isolated resonance more than the typical morphology of the quasi-linear paradigm. The explanation for such behavior, again, is due to the fact that we are not dealing, in the present case, with a broad spectrum morphology. In fact, the ratio between the Chirikov parameter and the number

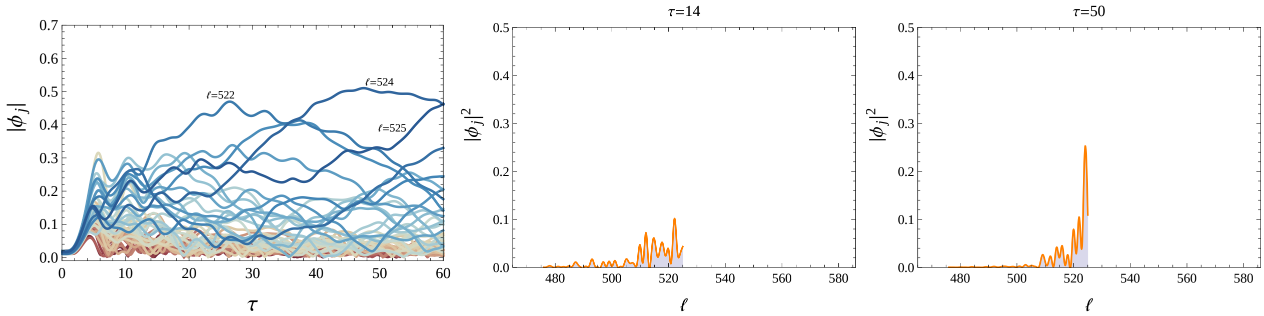

Figure 6. Case *(iii)*. Left-hand panel: temporal evolution of $|\phi_j|$. Central and right-hand panels: intensity spectrum at different times.

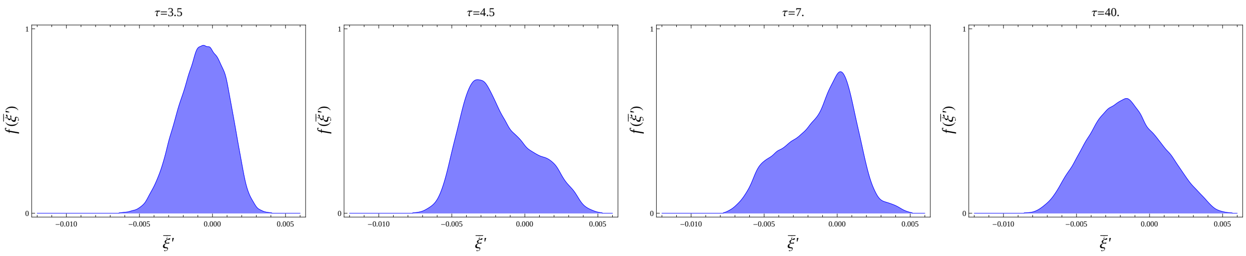

Figure 7. Case *(iii)*. Evolution of the velocity distribution function.

of overlapping resonances is of order unity. Thus, a given particle feels the effect of many modes acting as a nearly periodic spectrum. Furthermore, considering the contribution of all particles near a given resonance condition, modes instantaneously transferring their energy to particles are damped, while those receiving energy are amplified. Thus, there exists a mechanism able to create and enforce wave-particle phase synchronization. This is the reason why, in this case, the formation of a plateau in the velocity distribution function is suppressed.

Case (iv) : $\mathcal{S} \simeq 15$ and $466 < \ell < 476 \leq \ell_{res} \leq 525 < \ell < 585$. The conjecture described for case *(iii)* is clearly confirmed in this case, in which the ratio between the Chirikov parameter and the resonant beams is still of order unity, but now the presence of linearly stable modes is included in the evolution. As main effect, see FIG.8, a broadening of the intensity spectrum is clearly observable and, overall, the plateau in the particle distribution function is restored at sufficiently large times (see FIG.9). Indeed, the linearly stable modes, once excited by the nonlinear velocity spread (due to both convective and diffusive transport), reproduce a broad spectrum, significantly enhancing the effective Q value in the system. This analysis illuminates the relevance that linear stable modes have in determining the evolution of intensity spectrum and particle distribution function.

4.4. *Test particle transport*

To illustrate the mixed diffusive-convective character of the broad beam relaxation process, we have carried out test particle transport analyses of the simulation results discussed in the previous section. In particular, we analyze the time behavior of velocity fluctuations. For a given simulation, we extract the self-consistent potential fields and we monitor the evolution of a set of test particles (1000) evolving under the influence of those fields. The tracers are initialized to represent one single cold beam, among

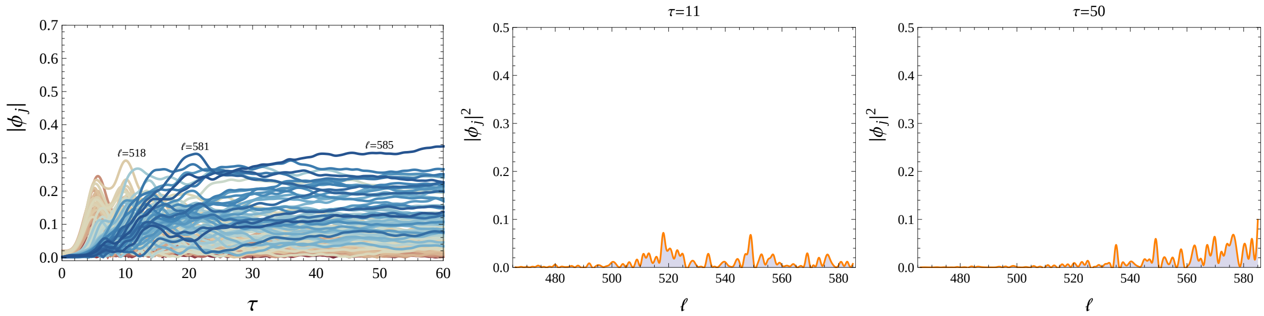

Figure 8. Case *(iv)*. Left-hand panel: temporal evolution of $|\phi_j|$. Central and right-hand panels: intensity spectrum at different times.

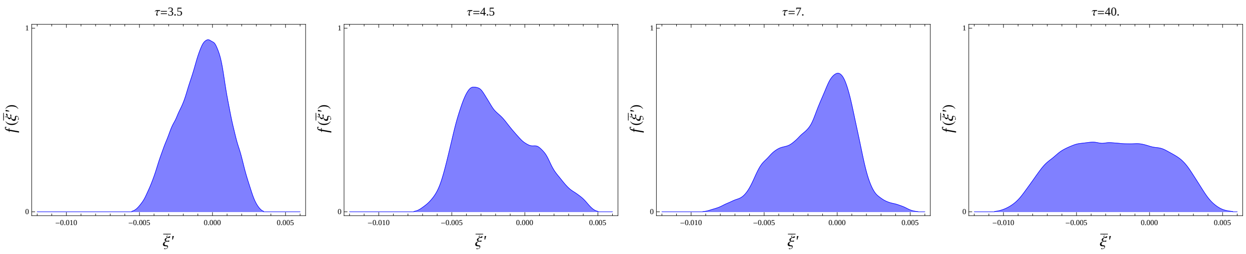

Figure 9. Case *(iv)*. Evolution of the velocity distribution function.

those of which the broad beam initially consists. For a single initial condition (*i.e.,* by setting all the test-particle initial velocities equal to $\bar{\xi}'_{\alpha i}(0)$), we plot the mean square path as function of time:

$$\langle\delta\bar{\xi}'^2_\alpha\rangle \equiv \langle[\bar{\xi}'(\tau) - \langle\bar{\xi}'(\tau)\rangle]^2\rangle \ , \tag{28}$$

where the average $\langle\ldots\rangle$ is taken over the tracers. When $\langle\delta\bar{\xi}'^2_\alpha\rangle$ shows a linear behavior in τ, we can speak of diffusive transport. This analysis is, in general, non trivial, because $\langle\delta\bar{\xi}'^2\rangle$ is characterized by complicated nonlinear time evolution when tracers are significantly displaced in velocity space (for an in-depth analysis of this problem, see Ref. [52]).

Nonetheless, limiting our attention to specific class of tracers and appropriate time intervals, in cases *(i)* and *(ii)* we can clearly identify a diffusion phase of the evolution, see FIG.10. There, we show the behavior of $\langle\delta\bar{\xi}'^2_\alpha\rangle$ for tracers of the beam $\alpha = 33$ (left panel refers to case *(i)*, while right panel to case *(ii)*). The existence of a well traced linear behavior of $\langle\delta\bar{\xi}'^2_\alpha\rangle$, *i.e.,* a net diffusion feature, is clearly recognizable at least for intermediate times. It is worth noting that the presence of linear stable modes (case *(ii)*) does not significantly affect the diffusion process since the corresponding coefficients are very close in the two cases. In this respect, it is interesting to compare the present study with the analysis in Ref.[40], where it is shown that the self-consistency of the model is broken when the plateau is sufficiently broad. Such a study demonstrates that the ratio between the obtained diffusion coefficient to the quasi-linear one can take a wide range of values, depending on the spectral morphology and initial beam profile. However, this ratio tends to unity when the random phase approximation is assumed. This is consistent with our findings in FIG.10, showing a slight decreases diffusion coefficient for the broader fluctuation spectrum case, including non-resonant modes.

In order to demonstrate convection in the velocity space in addition to diffusion, in FIG.11 we plot the quantities $\sqrt{\langle\delta\bar{\xi}'^2_{33}\rangle}$ (blue line) and $\langle\bar{\xi}'_{33} - \bar{\xi}'_{33}(0)\rangle$ (red line) for the same set of tracers considered above (left panel refers to case *(i)*, while right panel to case *(ii)*). The two processes are of the same order of

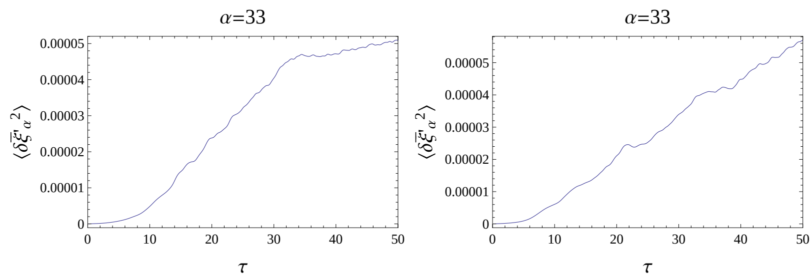

Figure 10. Temporal evolution of $\langle\delta\bar{\xi}'^2_{33}\rangle(\tau)$ for test particles initialized at $\bar{\xi}'_{33i}(0)$ (near the central beam): in the left-hand panel for the case *(i)* and in the right-hand panel for case *(ii)*. The linear (diffusive) time behavior is evident.

magnitude; and we stress how convection is enhanced in the presence of linear stable modes. This can be interpreted as a more efficient drag in the velocity space, due to the positive power exchange between linear stable modes and particles. In other words, particles can decrease their energy by exciting new modes and, thus, explore (on average) a wider velocity range.

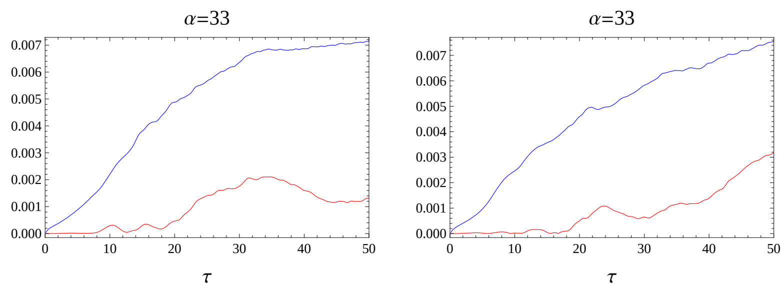

Figure 11. Overlap of the temporal evolution of $\sqrt{\langle\delta\bar{\xi}'^2_{33}\rangle}$ (blue line) and $\langle\bar{\xi}'_{33}-\bar{\xi}'_{33}(0)\rangle$ (red line), for test particles initialized at $\bar{\xi}'_{33i}(0)$. Case *(i)* in the left-hand panel and case *(ii)* in the right-hand panel.

Let us now repeat this analysis for cases *(iii)* and *(iv)*, corresponding to a "nearly periodic" spectrum in the presence or absence of linear stable modes, respectively. For these cases, we consider tracers representing beam $\alpha = 50$, because it is the directly influenced by the stable spectrum. We see in FIG.12 how the case *(iii)* (left-hand panel) does not suggest a diffusive process. This is clearly due to the coherent structures even in the late time evolution of the system, which introduce a strong dependence of transport on velocity and time. As soon as linear stable modes are accounted for (case *(iv)*, right-hand panel), an almost linear behavior of $\langle\delta\bar{\xi}'^2_{50}\rangle$ is recovered in the early evolution of the system ($10 \lesssim \tau \lesssim 20$). The physics underling these two cases is better elucidated when analyzing the behaviors in FIG.13. In case *(iii)* (left-hand panel), the existence of a significant number of trapped tracers strongly reduces the convection phenomenon in the velocity space. The latter is instead restored to the order of magnitude of $\sqrt{\langle\delta\bar{\xi}'^2_{50}\rangle}$ when stable mode are accounted for. Consistent with the analysis of the previous subsection, all these considerations suggest that the stable part of the spectrum is crucial for diffusion and convection phenomena, in contrast to the coherent structure morphology of case *(iii)*.

4.5. A toy-model of diffusion-convection relaxation

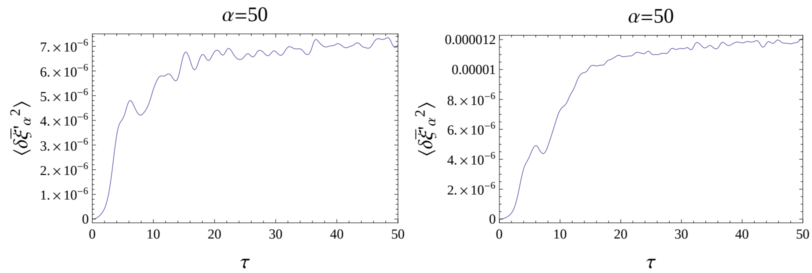

Figure 12. Time evolution of of $\langle\delta\bar{\xi}'^2_{50}\rangle(\tau)$ for tracers initialized at $\bar{\xi}'_{50i}(0)$: in the left-hand panel for the case *(iii)* and in the right-hand panel for case *(iv)*.

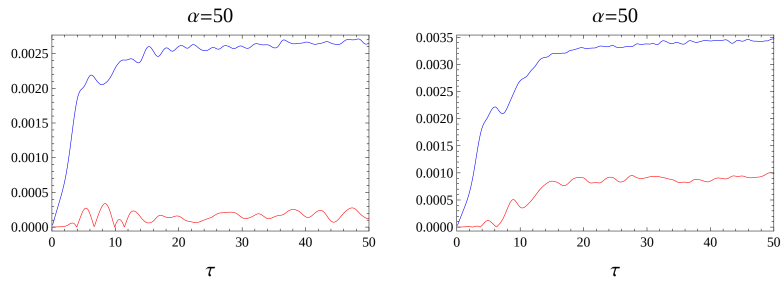

Figure 13. Overlap of the evolution of $\sqrt{\langle\delta\bar{\xi}'^2_{50}\rangle}$ (blue line) and $\langle\bar{\xi}' - \bar{\xi}'_{50}(0)\rangle$ (red line), for test particles initialized at $\bar{\xi}'_{50i}(0)$. Case *(iii)* in the left-hand panel and case *(iv)* in the right-hand panel.

Theoretical analyses of Secs. 2 and 3, and numerical simulation results of Sec.4 show that, when $\mathcal{S} > 1$, the relaxation of a broad beam in cold plasma can be due to both diffusion and convection processes. The value of Q, meanwhile, controlled by the width of the fluctuation spectrum, can determine whether coherent periodic behaviors are to be expected, as reflection of wave-particle trapping in a narrow (nearly periodic) spectrum. As the nonlinearity parameter $K \simeq \mathcal{S}/Q \simeq \mathcal{S}^2/\Delta\ell$, from Eqs. (4) and (5), one may conclude that the convective relaxation discussed in Secs. 2 and 3 is relatively unimportant in case *(i)* and/or *(ii)*, where $K < 1$. Here, we show that, even in this "standard" weak-turbulence limit, significant non-diffusive behavior can be expected, consistent with our numerical simulation results. In order to illustrate this, we derive a toy-model of diffusion-convection under the assumption that the spectrum be sufficiently broad, *i.e.,* that $Q \gg 1$ as well as $\mathcal{S} > 1$ (the case *(i)* and/or *(ii)* previously treated).

The goal is to capture the self-consistent evolution of fluctuation spectrum and beam distribution function. Thus, the first model equation should render Eq.(1), that is the destabilization of Langmuir waves by the beam particles; while the second one must reflect the nonlinear beam relaxation due to "weak" turbulence as described in Eq.(11). Thus, we can write

$$\begin{aligned}\partial_\tau \mathcal{E} &= \mathcal{E}\partial_{\bar{\xi}'} G_0 \ ,\\ \partial_\tau G_0 &= \partial_{\bar{\xi}'}\left(\mathcal{E}\partial_{\bar{\xi}'} G_0\right) \ ,\end{aligned} \tag{29}$$

where the dimensionless function $\mathcal{E}$ is connected with the fluctuation intensity and G_0 with the $k = 0$ Fourier component of the beam distribution function, f_0. This model can be properly derived from the

Vlasov-Poisson system, as represented in the Fourier space under the assumptions above. By inspection of Eqs.(29) and comparison with Eqs.(1) and (11), it is possible to show

$$\mathcal{E}(\bar{\xi}',\tau) \;=\; 2\pi\bar{\eta}\frac{m}{\ell_0\Delta\ell}\left|\phi(k,\tau)\right|^2\Big|_{k=(2\pi\ell_0/L)(1-\bar{\eta}\ell_0\bar{\xi}')}\;, \tag{30}$$

$$G_0(\bar{\xi}',\tau) \;=\; \frac{\bar{\eta}}{\ell_0^2}\omega_p L\,\frac{f_0(v,\tau)}{n_B}\Big|_{v=(\omega_p L/2\pi\ell_0)(1+\bar{\eta}\ell_0\bar{\xi}')}\;, \tag{31}$$

where we recall that m is the number of modes in the Langmuir fluctuation spectrum and ℓ_0 the reference (central) mode number. By direct substitution, it is readily verified that Eqs.(29) admit the solution

$$G_0(\bar{\xi}',\tau) = \bar{G}_0(\bar{\xi}') + \partial_{\bar{\xi}'}\mathcal{E}(\bar{\xi}',\tau)\;, \tag{32}$$

with $\mathcal{E}(\bar{\xi}',\tau)$ satisfying

$$\partial_\tau\mathcal{E} = \mathcal{E}\partial_{\bar{\xi}'}\bar{G}_0 + \mathcal{E}\partial^2_{\bar{\xi}'}\mathcal{E}\;. \tag{33}$$

Here, $\bar{G}_0(\bar{\xi}')$ represents the initial beam distribution function and the source of instability.

To illustrate the ability of Eqs.(29) to capture essential features of the broad beam relaxation for $Q \gg 1$ as well as $\mathcal{S} > 1$, we solve Eq.(33) numerically and compare $\mathcal{E}(\bar{\xi}',\tau)$ and $G_0(\bar{\xi}',\tau)$, obtained from Eq. (32), with the numerical simulation results for case *(i)* discussed in FIG.2 and FIG.3. The initial

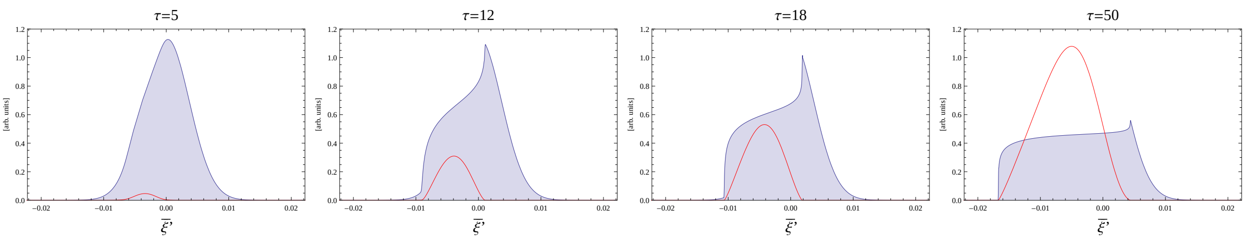

Figure 14. Evolution of the velocity distribution function G_0 (blue filled line) and the spectral density $\mathcal{E}$ (red line) as a function of the normalized velocity $\bar{\xi}'$ at different stages of the evolution of FIG.3.

conditions for $\bar{G}_0(\bar{\xi}')$ are given in order to properly represent the Gaussian distribution of the simulation (see FIG.1) and, for convenience, we also assign a Gaussian $\mathcal{E}(\bar{\xi}',0)$ profile resembling the initial mode amplitude. The behavior of $\mathcal{E}$ (red line) and G_0 (blue filled line) for different time steps is shown in FIG.14, well reproducing both the early stages of evolution and the plateau formation at later τ.

Note that $\mathcal{E}(\bar{\xi}',\tau)$ represents the intensity spectrum evolution discussed in Sec. 4.3 and plays the role of an "effective diffusion coefficient". In this respect, we can compare, for fixed times, the behavior of $\mathcal{E}$ and the normalized spectrum evaluated from the right-hand side of Eq.(30) with the simulation results for $|\phi(k,\tau)|$. This is shown in FIG.15 for $\tau = 50$, comparing the numerical solution of Eq.(33) and the value of $\mathcal{E}$ from Eq.(30) evaluated by substituting a smoothed profile of the simulated spectrum of case *(i)*. There is a good agreement between numerical simulation results and the toy-model proposed here.

One interesting implication of Eqs.(29) is to illustrate how non-diffusive behavior may arise in such a system, and how this is a consequence of the self-consistent evolution of fluctuation intensity on the same time scale of particle transport. The crucial role of fastest growing modes was already pointed out

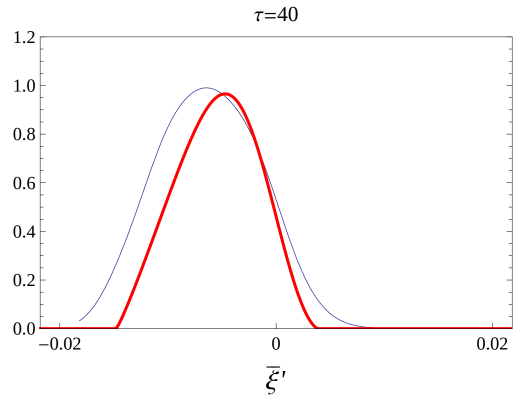

Figure 15. Plot of $\mathcal{E}$ numerically estimated from Eq.(30) (blue line) and the theoretical values of $\mathcal{E}$ calculated from Eq.(33) (red line), versus the normalized velocity $\bar{\xi}'$ at a fixed time $\tau = 40$.

is Sec. 3, since such fluctuations play a key role in the nonlinear evolution. Fastest growing modes are resonant near an inflection point of G_0, in the neighborhood of which Eqs.(29) can be cast as

$$\partial_\tau G_0 \simeq \partial_{\bar{\xi}'} \mathcal{E} \partial_{\bar{\xi}'} G_0 \simeq \left(G_0 - \bar{G}_0 \right) \partial_{\bar{\xi}'} G_0 \ . \tag{34}$$

After sufficiently long time that the inflection point of G_0 has evolved to a region where the initial beam distribution $\bar{G}_0(\bar{\xi}')$ is sufficiently small, Eq.(34) can be cast as the inviscid Burgers equation [54]. This has the formal solution $G_0 = \mathcal{G}_0\left(\Xi(\bar{\xi}', \tau)\right)$, with $\Xi(\bar{\xi}', \tau)$ obtained from

$$\bar{\xi}' = \Xi - \tau \mathcal{G}_0(\Xi) \ ; \tag{35}$$

and it is known to generate a propagating shock solution, which is visible in the plateau formation at lower speed in the rightmost panel of FIG.14. It is also worthwhile noting that, away from the $\bar{G}_0(\bar{\xi}')$ localization region, Eq. (33) for $\mathcal{E} \equiv \exp \mathcal{W}$ reduces to the heat equation with exponential nonlinearity [55]

$$\partial_\tau \mathcal{W} = \partial_{\bar{\xi}'} \bar{G}_0 + \partial_{\bar{\xi}'} \left(e^{\mathcal{W}} \partial_{\bar{\xi}'} \mathcal{W} \right) \ . \tag{36}$$

5. Conclusions and Discussions

In this work, we analyzed the relaxation of a broad supra-thermal particle beam in a cold one-dimensional collisionless plasma, modeled as a linear dielectric medium. In particular, we show that such a relaxation process must be characterized by convection and diffusion in the velocity space, when the Chirikov parameter is larger than one. We adopt the Hamiltonian formulation of the problem described in Ref. [41], where the broad supra-thermal particle beam is discretized as superposition of $n \gg 1$ cold beams self-consistently evolving in the presence of $m \geq n$ modes nearly degenerate with Langmuir waves. Essential element of this analysis is the crucial role played by wave-particle nonlinearity in determining the non-diffusive feature of supra-thermal particle transport, which self-consistently evolves with the fluctuation intensity spectrum. Thus, parameters other than Chirikov play important roles, such as the ratio of wave-particle trapping time to the scattering time (which is related with the ratio of wave-number spectrum width to the Chirikov parameter), and the nonlinearity parameter, straightforwardly related to the former two.

A detailed theoretical analysis for sufficiently large nonlinearity parameter has shown how the convective relaxation of large-amplitude or strongly shaped beams is meso-scale; in that it has limited duration in time and is always followed by a long-time, diffusive relaxation of the quasi-linear type. The quasi-linear relaxation is, therefore, asymptotic ($t \rightarrow +\infty$), while the convective relaxation is not. It represents, nevertheless, the initial, "violent" relaxation phase for a "broad" beam in a plasma and is driven by excessively large free-energy reservoir associated with this type relaxation problem. Mathematically, the relaxation problem for broad beams is described by a self-consistent system of coupled nonlinear differential equations, Eqs.(13) and (14), which generalize their quasi-linear relatives by directly taking into account the amplification of the resonance domain through the nonlinear interaction and the communication among the beams.

We carried out numerical simulations of broad supra-thermal particle beam relaxation changing parameters controlling particle transport and the nonlinear dynamic regime. In particular, we have chosen two cases, respectively *(i)* and *(iii)*, characterized by a "broad" and a "nearly periodic" ("narrow") fluctuation spectrum consisting of linearly unstable modes only; and the two corresponding cases, respectively *(ii)* and *(iv)*, where modes of the linear stable spectrum have been accounted for as well. These show show the crucial role played by the linear stable spectrum for transport phenomena, which are characterized by both diffusive and convective processes, as demonstrated by test particle transport analyses.

Numerical simulation results for the "narrow" spectrum exhibit the coherent behavior of spectrum intensity and particle distribution function typical of persistent phase space structures due to wave particle trapping, which suppress convective transport. The plateau in the particle distribution function, in this case, is not formed, unless modes of the linear stable spectrum are accounted for, which not only enhance diffusion but convection as well, because of enhanced detrapping rate and drag in velocity space.

Numerical simulation results for the "broad" spectrum cases, meanwhile, show that a plateau in the distribution function is always formed time asymptotically. Controlling the wave-number spectrum width by including or not the modes of the linear stable spectrum yields behaviors of the diffusion coefficient that are consistent with earlier analyses by Escande et al. [38–40]. Further to this, convective transport is observed even for small nonlinearity parameter. This persistent mixed diffusion-convection relaxation is due to the self-consistent evolution of fluctuation intensity on the same time scale of particle transport, as demonstrated by an analytical toy-model, whose solution is in good qualitative and quantitative agreement with numerical simulations. This toy-model provides a valuable tool, allowing the identification of respective roles of convection and diffusion in the velocity space relaxation; in particular, elucidating how convection processes are not negligible during the relaxation of a bump-on-tail initial profile, especially during the meso-time-scales.

In conclusion, the present study provides understanding and insights into the mixed diffusion-convection relaxation of a broad beam in a cold one-dimensional plasma in the presence of weak Langmuir turbulence. In particular, crucial roles are played by wave-particle nonlinearity and the self-consistent evolution of particle distribution function with the fluctuation intensity spectrum. These results are of general interest as well as practical importance, in the light of their possible implications as paradigm for Alfvénic fluctuation-induced supra-thermal particle transport in fusion plasmas near marginal stability.

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Author Contributions

N. Carlevaro, A. V. Milovanov, G. Montani and F. Zonca conceived this work, its motivation, scope and overall structure. A. V. Milovanov conceived the generalization of the quasi-linear theory based on the idea of convective amplification of the resonant domain, using the notion of a propagating instability and the formalism of Klein-Kramers equation. A. V. Milovanov, G. Montani, M. V. Falessi and F. Zonca developed the theoretical framework for the description and interpretation of the dynamics of the system. N. Carlevaro wrote the code and performed numerical simulations. N. Carlevaro, D. Terzani and M. V. Falessi analyzed numerical simulation results. D. Terzani, N. Carlevaro and M. V. Falessi analyzed numerically the toy-model. All authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

References

1. Bernstein, I.; Greene, J.; Kruskal, M. Exact nonlinear plasma oscillations. *Phys. Rev.* **1957**, *108*, 546.
2. Vedenov, A.A.; Velikhov, E.P.; Sagdeev, R.Z. *Nucl. Fusion* **1961**, *1*, 82.
3. Drummond, W.E.; Pines, D. *Nucl. Fusion Suppl. Pt.* **1962**, *3*, 1049.
4. O’Neil, T.M. Collisionless Damping of Nonlinear Plasma Oscillations. *Phys. Fluids* **1965**, *8*, 2255.
5. Mazitov, R.K. *Zh. Prikl. Mekh. Fiz.* **1965**, *1*, 27.
6. O’Neil, T.; Winfrey, J.; Malmberg, J. Nonlinear Interaction of a Small Cold Beam and a Plasma. *Phys. Fluids* **1971**, *14*, 1204–1212.
7. O’Neil, T.; Malmberg, J. Transition of the Dispersion Roots from Beam-Type to Landau-Type Solutions. *Phys. Fluids* **1968**, *11*, 1754–1760.
8. Shapiro, V. Nonlinear theory of the interaction of a monoenergetic beam with a plasma. *Sov. Phys. JEPT* **1963**, *17*, 416–423.
9. Onishchenko, I.; Linetskii, A.; Matsiborko, N.; Shapiro, V.; Shevchenko, V. Contribution to the Nonlinear Theory of Excitation of a Monochromatic Plasma Wave by an Electron Beam. *JETP Letters* **1970**, *12*, 281.

10. Shapiro, V.; Shevchenko, V. Contribution to the Nonlinear Theory of Relaxation of a "Monoenergetic" Beam in a Plasma. *Sov. Phys. JEPT* **1971**, *33*, 555–561.
11. Thompson, J. Nonlinear Evolution of Collisionless Electron Beam-Plasma Systems. *Phys. Fluids* **1971**, *14*, 1532–1541.
12. Levin, M.; Lyubarskiǐ, M.; Onishchenko, I.; Shapiro, V.; Shevchenko, V. Contribution to the Nonlinear Theory of Kinetic Instability of an Electron Beam in Plasma. *Sov. Phys. JEPT* **1972**, *35*, 898–901.
13. Matsiborko, N.; Onishchenko, I.; Shapiro, V.; Shevchenko, V. On non-linear theory of instability of a mono-energetic electron beam in plasma. *Plasma Phys.* **1972**, *14*, 591–600.
14. Berk, H.; Breizman, B. Saturation of a single mode driven by an energetic injected beam. I. Plasma wave problem. *Phys. Fluids B* **1990**, *2*, 2226–2234.
15. Berk, H.; Breizman, B. Saturation of a single mode driven by an energetic injected beam. II. Electrostatic "universal" destabilization mechanism. *Phys. Fluids B* **1990**, *2*, 2235–2245.
16. Berk, H.; Breizman, B. Saturation of a single mode driven by an energetic injected beam. III. Alfvén wave problem. *Phys. Fluids B* **1990**, *2*, 2246–2252.
17. Chen, L.; Zonca, F. Theory of Alfvén waves and energetic particle physics in burning plasmas. *Nucl. Fusion* **2007**, *47*, S727–S734.
18. Chen, L.; Zonca, F. On nonlinear physics of shear Alfvén waves. *Phys. Plasmas* **2013**, *20*, 055402.
19. Breizman, B.; Sharapov, S. Major minority: energetic particles in fusion plasmas. *Plasma Phys. Contr. Fusion* **2011**, *53*, 054001.
20. Chen, L.; Zonca, F. Physics of Alfvén waves and energetic particles in burning plasmas. *to be published in Rev. Mod. Phys.* **2016**.
21. Fasoli, A.; Gormenzano, C.; Berk, H.; Breizman, B.; Briguglio, S.; Darrow, D.; Gorelenkov, N.; Heidbrink, W.; Jaun, A.; Konovalov, S.; Nazikian, R.; Noterdaeme, J.M.; Sharapov, S.; Shinohara, K.; Testa, D.; Tobita, K.; Todo, Y.; Vlad, G.; Zonca, F. Chapter 5: Physics of energetic ions. *Nucl. Fusion* **2007**, *47*, S264–S284.
22. Heidbrink, W. Basic physics of Alfvén instabilities driven by energetic particles in toroidally confined plasmas. *Phys. Plasmas* **2008**, *15*, 055501.
23. Lauber, P. Super-thermal particles in hot plasmas - Kinetic models, numerical solution strategies, and comparison to tokamak experiments. *Phys. Rept.* **2013**, *533*, 33–68.
24. Vlad, G.; Briguglio, S.; Fogaccia, G.; Zonca, F.; Fusco, V.; Wang, X. Electron fishbone simulations in tokamak equilibria using XHMGC. *Nucl. Fusion* **2013**, *53*, 083008.
25. Wang, X.; Briguglio, S.; Chen, L.; Di Troia, C.; Fogaccia, G.; Vlad, G.; Zonca, F. An extended hybrid magnetohydrodynamics gyrokinetic model for numerical simulation of shear Alfvén waves in burning plasmas. *Phys. Plasmas* **2011**, *18*, 052504.
26. Schneller, M.; Lauber, P.; Bilato, R.; García-Muñoz, M.; Brüdgam, M.; Günter, S.; the ASDEX Upgrade Team. Multi-mode Alfvénic fast particle transport and losses: numerical versus experimental observation. *Nucl. Fusion* **2013**, *53*, 123003.
27. Schneller, M.; Lauber, P.; Brüdgam, M.; Pinches, S.; Günter, S. Double-resonant fast particle-wave interaction. *Nucl. Fusion* **2012**, *52*, 103019.

28. Gorelenkov, N.; Pinches, S.; Toi, K. Energetic particle physics in fusion research in preparation for burning plasma experiments. *Nucl. Fusion* **2014**, *54*, 125001.
29. Zonca, F.; Chen, L.; Briguglio, S.; Fogaccia, G.; Milovanov, A.; Qiu, Z.; Vlad, G.; Wang, X. Energetic particles and multi-scale dynamics in fusion plasmas. *Plasma Phys. Contr. Fusion* **2015**, *57*, 014024.
30. Zonca, F.; Chen, L.; Briguglio, S.; Fogaccia, G.; Vlad, G.; Wang, X. Nonlinear dynamics of phase space zonal structures and energetic particle physics in fusion plasmas. *New J. Phys.* **2015**, *17*, 013052.
31. Pinches, S.; Chapman, I.; Lauber, P.; Oliver, H.; Sharapov, S.; Shinohara, K.; Tani, K. Energetic ions in ITER plasmas. *Phys. Plasmas* **2015**, *22*, 021807.
32. Berk, H.; Breizman, B.; Fitzpatrick, J.; Wong, H. Line broadened quasi-linear burst model [fusion plasma]. *Nucl. Fusion* **1995**, *35*, 1661–1668.
33. Berk, H.; Breizman, B.; Fitzpatrick, J.; Pekker, M.; Wong, H.; Wong, K. Nonlinear response of driven systems in weak turbulence theory. *Phys. Plasmas* **1996**, *3*, 1827–1838.
34. Ghantous, K.; Gorelenkov, N.; Berk, H.; Heidbrink, W.; Van Zeeland, M. 1.5D quasilinear model and its application on beams interacting with Alfvén eigenmodes in DIII-D. *Phys. Plasmas* **2012**, *19*, 092511.
35. Ghantous, K.; Berk, H.; Gorelenkov, N. Comparing the line broadened quasilinear model to Vlasov code. *Phys. Plasmas* **2014**, *21*, 032119.
36. Lauber, Ph. Local and global kinetic stability analysis of Alfvén eigenmodes in the 15MA ITER scenario. *Plasma Phys. Control. Fusion* **2015**, *57*, 054011.
37. Schneller, M.; Lauber, P.; Briguglio, S. Nonlinear energetic particle transport in the presence of multiple Alfvénic waves in ITER. *Plasma Phys. Control. Fusion* **2016**, *58*, 014019.
38. Cary, J.; Doxas, I.; Escande, D.; Verga, A. Enhancement of the velocity diffusion in longitudinal plasma turbulence. *Phys. Fluids B* **1992**, *4*, 2062.
39. Elskens, Y.; Escande, D. *Microscopic Dynamics of Plasmas Chaos*; Taylor Francis Ltd, 2003.
40. Escande, D.; Elskens, Y. Self-consistency vanishes in the plateau regime of the bump-on-tail instability. *arXiv:0807.1839* **2008**.
41. Carlevaro, N.; Falessi, M.; Montani, G.; Zonca, F. Nonlinear physics and energetic particle transport features of the beam-plasma instability. *J. Plasma Phys.* **2015**, *81*, 495810515.
42. Kadomtsev, B. *Collective phenomena in plasma. 2*; Nauka, 1988.
43. Zaslavsky, G. Statistical irreversibility in nonlinear systems, 1970.
44. Sagdeev, R.; Zaslavsky, G. Introduction to the Nonlinear Physics. *Harwood* **1988**.
45. Zaslavsky, G.; Chirikov, B. Stochastic instability of non-linear oscillations. *Phys. Uspekhi* **1972**, *14*, 549–568.
46. Zaslavsky, G. Chaos, fractional kinetics, and anomalous transport. *Phys. Repts.* **2002**, *371*, 461–580.
47. Shapiro, V. *Sov. Phys. JETP* **1963**, *17*, 416.
48. Milovanov, A.; Iomin, A. Localization-delocalization transition on a separatrix system of nonlinear Schrödinger equation with disorder. *Eur. Phys. Lett.* **2012**, *100*, 10006.

49. Milovanov, A.; Iomin, A. Topological approximation of the nonlinear Anderson model. *Phys. Rev. E* **2014**, *89*, 062921.
50. Milovanov, A.; Rasmussen, J. A mixed SOC-turbulence model for nonlocal transport and Lévy-fractional Fokker–Planck equation. *Phys. Lett. A* **2014**, *378*, 1492–1500.
51. Milovanov, A.; Rasmussen, J. Self-organized criticality revisited: non-local transport by turbulent amplification. *J. Plasma Phys.* **2015**, *81*, 495810606.
52. Volokitin, A.; Krafft, C. Velocity diffusion in plasma waves excited by electron beams. *Plasma Phys. Control. Fusion* **2012**, *54*, 085002.
53. Chirikov, B. A universal instability of many-dimensional oscillator systems. *Phys. Rep.* **1979**, *52*, 263–379.
54. Burgers, J. *Adv. Appl. Mech.* **1948**, *1*, 171.
55. Ovsiannikov, L. Group properties of nonlinear heat equations [in Russian]. *Doklady AN SSSR* **1959**, *125*, 492.