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ASYMPTOTIC TRANSITION FROM KINETIC TO ADIABATIC ELECTRONS ALONG MAGNETIC FIELD LINES

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Abstract. Plasma dynamics is known to involve several time and space scales, fact which renders its study particularly challenging, from an analytical as well as numerical point of view. In this work we focus on the electron dynamics, studied on the time scale of the ion thermal motion, leading to a quasi-adiabatic response of the electrons in this regime. Starting from a bi-fluid kinetic model with intra- and inter-particle collision operators, we establish via a scaling procedure the non-dimensional equations, in which the ion Mach number is assumed of order one and the electron-ion mass ratio as well as the Knudsen number are embodied by a single scaling parameter $\varepsilon \ll 1$. Our studies are done in one space- and velocity dimensions, modeling electron/ion motions along (straight) magnetic field lines (no cyclotron motion or drifts). We introduce two numerical schemes for solving the electron evolution equation along the transition $\varepsilon \to 0$ from the kinetic regime to the adiabatic regime (Boltzmann relation), in a uniformly stable way with respect to ε. Our approach is based on micro-macro techniques, separating the microscopic kinetic distribution part from the macroscopic part.

Keywords: Plasma modelling, kinetic equations, mass disparate particles, multi-scale problem, Boltzmann adiabatic relation, Asymptotic-Preserving numerical method.

1. Introduction

A kinetic treatment of a tokamak plasma, composed of ions and electrons, is very precise and appropriate for detailed studies of phenomena like temperature-gradient driven instabilities (ITG, ETG), internal kink modes or reconnection processes, to mention only some examples. The difficulty with a fully kinetic treatment comes among others from the small mass ratio $\delta := m_e/m_i \approx 10^{-4}$ of the particles, inducing, for a typical tokamak plasma with similar electron and ion temperatures, faster electron dynamics than ion dynamics. In particular, the ratio of the thermal velocities is given by $v_{th,e}/v_{th,i} = 1/\sqrt{\delta} \approx 10^2$. This fact poses for a standard discretization of the bi-kinetic system rather restrictive time-step constraints related to the fast electron motion, *i.e.* numerical stability requires $v_{th,e}\Delta t \leq \Delta x$.

However, in many cases one is interested in phenomena evolving on the ion time-scales which describe the macroscopic evolution of the plasma, as for example the time-scale of

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the ITG instabilities and of the plasma drifts. The primary goal is hence to construct a numerical scheme being able to describe accurately the needed (ion) physics, however without being forced to follow the fast electron motion. This was achieved in past works by hybrid strategies [4, 7, 31], describing the heavy, slow ions via a kinetic equation (or a hydrodynamic system if accurate enough) and the light and fast electrons via a massless or inertial-less fluid system (so-called Boltzmann response or adiabatic electrons). This Boltzmann relation is obtained by assuming zero electron inertia ($m_e \to 0$) and zero viscosity in the "parallel" electron equation of motion ("parallel" with respect to the magnetic field lines), leading to the relation

$$\nabla_{||} p_e = -q\, n_e\, \mathbf{E}_{||}\,, \qquad \mathbf{E} = -\nabla\phi\,. \tag{1}$$

This relation signifies that the pressure-gradient and electrostatic forces acting on the electrons (parallel to the magnetic field) are in balance. Moreover, rapid parallel thermal conduction assures that $\nabla_{||} T_e \sim 0$, such that with the thermodynamic equation of state $p_e = n_e\, k_B\, T_e$ one gets

$$n_e(t,\mathbf{x}) = c(t,\mathbf{x}_\perp)\, \exp\left(\frac{q\,\phi(t,\mathbf{x})}{k_B\, T_e(t,\mathbf{x}_\perp)}\right)\,, \quad \mathbf{x} = (\mathbf{x}_\perp, \mathbf{x}_{||}) \in \mathbb{R}^3\,,\ \ t \in \mathbb{R}^+\,. \tag{2}$$

This is the so-called Boltzmann relation or adiabatic response, relating the electron density with the electric potential. Here, $c(t,\mathbf{x}_\perp)$ and $T_e(t,\mathbf{x}_\perp)$ are functions to be determined from the remaining transport equations as well as initial and boundary conditions; they do not depend on the parallel coordinate $\mathbf{x}_{||}$. Once c and T_e are known the relation (2) can be inserted into the Poisson equation for the electrostatic potential, which can then be coupled to a model for the ion dynamics (kinetic or fluid). Such a procedure is common in plasma simulations [4, 7, 31], because it leads to large reductions in computational cost.

The smallness of the electron-ion mass ratio makes (2) a good approximation for typical fusion plasmas. But even under these circumstances, there are situations where the Boltzmann electron approximation is not adapted, as for example near to the boundaries (in the so-called sheath and pre-sheath regions of a tokamak) or when describing trapped electrons [4, 19]. It is therefore more reasonable to use in these regions a standard electron kinetic model, whereas in the rest of the domain the adiabatic electron response would give sufficiently accurate results. Considering this new difficulty, the aim of this work is to design a numerical scheme capable to describe both electron regimes, *i.e.* the non-adiabatic kinetic regime as well as the adiabatic Boltzmann regime, as an alternative to strategies such as domain decomposition. We shall make use of Asymptotic-Preserving (AP) techniques [12, 27, 35], which preserve at the discrete level the asymptotic passage from the kinetic to the adiabatic regime, as a small parameter ε, responsible for this asymptotics, goes to zero. AP techniques enable ε-independent time and spatial steps Δt resp. Δx, adapted to the physical phenomena one wants to describe, and are hence very interesting

from a computational point of view.

The model we are starting from is a 1D1V two-species kinetic model, describing a non-magnetized, isothermal plasma. One can imagine that we are investigating a gas of charged particles evolving along the magnetic field lines, and being additionally in a thermal bath of given temperature. This model, though rather simplified, still keeps all the numerical difficulties one can encounter in a more physical 3D3V, strongly magnetized model, with varying temperatures (the 3D adiabatic limit has been studied in [36]). The aim of the present simplification was to understand the difficult points in the construction of an AP-scheme in the adiabatic scaling.

The structure of this paper is the following. In Section 2 we are introducing the two-species kinetic model and its scaling. Section 3 reviews the obtention of some macroscopic ion/electron models. The electron adiabatic limit differs from hydrodynamic and drift-diffusion limits in that frictional terms between ions and electrons are neglected. Section 4 contains a reformulation of the original electron kinetic equation via the micro-macro approach. In Section 5 we introduce two AP-procedures for the resolution of the electron evolution in the kinetic as well as the adiabatic regime. Numerical results are finally presented in Section 6.

2. The fully kinetic model and its scaling

Starting point of our study is the following one-dimensional Boltzmann system for the two species (ions, electrons) of charged particles

$$\begin{cases} \partial_t f_i + v\,\partial_x f_i + \dfrac{q}{m_i}\,E\,\partial_v f_i = Q_{ii}(f_i) + Q_{ie}(f_i, f_e) \\ \partial_t f_e + v\,\partial_x f_e - \dfrac{q}{m_e}\,E\,\partial_v f_e = Q_{ee}(f_e) + Q_{ei}(f_e, f_i)\,, \end{cases} \tag{3}$$

coupled to the Poisson equation for the computation of the electrostatic potential

$$-\partial_{xx}\phi = \frac{q}{\epsilon_0}(n_i - n_e)\,, \quad E = -\partial_x\phi\,, \tag{4}$$

where $f_{i,e}$ are the particle density functions, q is the elementary charge, ϵ_0 the vacuum permittivity, $m_{e,i}$ the electron respectively ion mass and $n_{e,i}$ the electron respectively ion density, defined as

$$n_{e,i}(t,x) := \int_{\mathbb{R}} f_{e,i}(t,x,v)\,dv. \tag{5}$$

The particle fluxes (flow velocities) are defined via

$$(n_{e,i}\,u_{e,i})(t,x) := \int_{\mathbb{R}} v\,f_{e,i}(t,x,v)\,dv. \tag{6}$$

At this point we do not need to precise the specific form of the collision operators. It suffices to list some of their important properties and give their particular form later on. In particular, the self- as well as inter-species collision operators are supposed to conserve mass and momentum, *i.e.* we assume

$$\int_{\mathbb{R}} Q_{kl}(v)\, dv = 0\,, \quad \forall k,l \in \{i,e\}\,, \tag{7}$$

$$\int_{\mathbb{R}} Q_{ee}(v)\, m_e\, v\, dv = \int_{\mathbb{R}} Q_{ii}(v)\, m_i\, v\, dv = 0\,, \qquad \mathcal{S}_{ei} + \mathcal{S}_{ie} = 0\,, \tag{8}$$

with the friction terms defined as

$$\mathcal{S}_{ei} := \int_{\mathbb{R}} Q_{ei}(v)\, m_e\, v\, dv\,, \qquad \mathcal{S}_{ie} := \int_{\mathbb{R}} Q_{ie}(v)\, m_i\, v\, dv\,. \tag{9}$$

To simplify the present study, we shall suppose in this paper, that the particles evolve in a thermal bath of given temperature T ($T_e = T_i = T$), such that no energy conservation is demanded from the collision operators. The more general $3D3V$ case, with variable temperatures and strong magnetic field, will be treated in a forthcoming work.

2.1. **Regime/scaling.** Let us now introduce the dimensionless form of the kinetic model. This procedure shall permit to identify relevant parameters, describing different asymptotic regimes of the plasma. With this ambition in mind, the characteristic scales of our problem are summarized here:

- Disparate masses (Parameter: δ):

$$\delta := \frac{m_e}{m_i}. \tag{10}$$

- Microscopic (thermal) velocity scales:

$$\bar{v}_e := v_{th,e} = \sqrt{\frac{k_B T}{m_e}}\,, \qquad \bar{v}_i := v_{th,i} = \sqrt{\frac{k_B T}{m_i}} = \sqrt{\delta}\, \bar{v}_e. \tag{11}$$

- Microscopic time and length scale:

$$\tau_c := \tau_{ii} \quad \text{(elapsed time between 2 ionic collisions)}\,, \tag{12a}$$

$$l_c := \bar{v}_i\, \tau_c \quad \text{(mean free path between 2 ionic collisions)}\,. \tag{12b}$$

- Macroscopic velocity scale (parameter: ionic Mach number M):

$$\bar{u}_i = \bar{u}_e = \bar{u} \quad \text{(characteristic mean velocities)}\,, \tag{13a}$$

$$\frac{\bar{u}}{\bar{v}_i} =: M \quad \Rightarrow \quad \frac{\bar{u}}{\bar{v}_e} = \sqrt{\delta}\, M\,. \tag{13b}$$

- Macroscopic length scale (parameter: ionic Knudsen number κ):

$$\bar{x} = L \quad \text{(distance of interest)}\,, \qquad \frac{l_c}{\bar{x}} =: \kappa\,. \tag{14}$$

- Macroscopic time scale:

$$\bar{t} := \frac{\bar{x}}{\bar{u}} = \frac{1}{\kappa M}\,\tau_c \quad \text{(observation time)}\,. \tag{15}$$

- Electric force scale:

$$\bar{E} = \frac{k_B\,T}{\bar{x}\,q} = \frac{\bar{v}_i^2\,m_i}{q\bar{x}} \qquad \Longrightarrow \qquad \bar{\phi} = \frac{k_B\,T}{q}\,. \tag{16}$$

- Collision operators, distribution functions:

$$\bar{n}_i = \bar{n}_e = \bar{n}\,, \qquad \bar{f}_e := \frac{\bar{n}}{\bar{v}_e}\,, \quad \bar{f}_i := \frac{\bar{n}}{\bar{v}_i}\,, \tag{17}$$

$$\bar{Q}_{ee} = \bar{\nu}_{ee}\bar{f}_e\,, \qquad \bar{Q}_{ii} = \bar{\nu}_{ii}\bar{f}_i\,, \qquad \bar{Q}_{ei} = \bar{\nu}_{ei}\bar{f}_e\,, \qquad \bar{Q}_{ie} = \bar{\nu}_{ie}\bar{f}_i\,. \tag{18}$$

- Collisional frequencies $\bar{\nu}_{kl}$ [22] and corresponding collisional periods $\tau_{kl} = 1/\bar{\nu}_{kl}$:

$$\bar{\nu}_{ii} = \sqrt{\delta}\,\bar{\nu}_{ee}\,, \quad \bar{\nu}_{ie} = \delta\,\bar{\nu}_{ei}\,, \quad \bar{\nu}_{ie} = \sqrt{\delta}\bar{\nu}_{ii}\,, \quad \bar{\nu}_{ee} = \bar{\nu}_{ei}\,, \tag{19}$$

$$\tau_{ee} = \tau_{ei} = \sqrt{\delta}\,\tau_c\,, \quad \tau_{ie} = \frac{1}{\sqrt{\delta}}\,\tau_c\,. \tag{20}$$

- Debye length (parameter λ):

$$\lambda_D := \sqrt{\frac{\epsilon_0 k_B T}{\bar{n}\,q^2}} = \frac{v_{th,i}}{\omega_p}\,, \qquad \omega_p := \sqrt{\frac{\bar{n}\,q^2}{\epsilon_0\,m_i}} \quad \text{(plasma frequency)}\,, \tag{21}$$

$$\lambda := \frac{\lambda_D}{\bar{x}} = \frac{v_{th,i}}{\bar{x}}\frac{1}{\omega_p} = \kappa\,\frac{1}{\omega_p\,\tau_c}\,. \tag{22}$$

In order to rescale our system, we perform the following variable substitutions in (3)-(4):

$$t = \bar{t}\,t'\,, \qquad x = \bar{x}\,x'\,, \qquad v = \bar{v}_{e,i}\,v'\,, \qquad f_{e,i}(t,x,v) = \bar{f}_{e,i}\,f'_{e,i}(t',x',v')\,, \tag{23}$$

$$E(t,x) = \bar{E}\,E'(t',x')\,, \qquad Q_{kl}(f_k,f_l) = \bar{Q}_{kl}\,Q'_{kl}(f'_k,f'_l)\,, \tag{24}$$

and finally obtain the dimension-less kinetic model

$$\begin{cases} \partial_{t'} f'_i + \dfrac{1}{M}\,v'\partial_{x'} f'_i + \dfrac{1}{M}\,E'\,\partial_{v'} f'_i = \dfrac{1}{\kappa M}\,\left[Q'_{ii}(f'_i) + \sqrt{\delta}\,Q'_{ie}(f'_i,f'_e)\right] \\ \partial_{t'} f'_e + \dfrac{1}{\sqrt{\delta}M}\,v'\partial_{x'} f'_e - \dfrac{1}{\sqrt{\delta}M}\,E'\,\partial_{v'} f'_e = \dfrac{1}{\kappa\,\sqrt{\delta}M}\,\left[Q'_{ee}(f'_e) + Q'_{ei}(f'_e,f'_i)\right]\,, \end{cases} \tag{25}$$

coupled with the Poisson equation

$$-\lambda^2\,\partial_{xx}\phi' = n'_i - n'_e\,, \qquad E' = -\partial_x\phi'\,, \qquad n'_{e,i} = \int_{\mathbb{R}} f'_{e,i}\,dv'\,. \tag{26}$$

The rescaled collision operators satisfy now the rescaled conservation laws

$$\int_{\mathbb{R}} Q'_{kl}(v')\,dv' = 0\,, \quad \forall k,l \in \{i,e\}\,, \tag{27}$$

$$\int_{\mathbb{R}} Q'_{ee}(v')\, v'\, dv' = \int_{\mathbb{R}} Q'_{ii}(v')\, v'\, dv' = 0\,, \qquad \mathcal{S}'_{ei} + \sqrt{\delta}\, \mathcal{S}'_{ie} = 0\,, \tag{28}$$

with the friction terms given by

$$\mathcal{S}'_{ei} := \int_{\mathbb{R}} Q'_{ei}(v')\, v'\, dv'\,, \qquad \mathcal{S}'_{ie} := \int_{\mathbb{R}} Q'_{ie}(v')\, v'\, dv'\,. \tag{29}$$

It is worthwhile to remark also the scaled first moments of the distribution functions:

$$M\, n'_i u'_i = \int_{\mathbb{R}} v' f'_i(v')\, dv'\,, \qquad \sqrt{\delta} M\, n'_e u'_e = \int_{\mathbb{R}} v' f'_e(v')\, dv'\,. \tag{30}$$

Several time-scales are apparent in the non-dimensional system (25). The overall time-scale is obviously determined by the Mach number M. It becomes also clear that the electron dynamics is faster by a factor $1/\sqrt{\delta}$ compared to the ion dynamics, fact which justifies the adiabatic electron treatment when $\delta \to 0$ formally. The collisional time-scale is determined by the Knudsen number κ in our setting. Moreover, we remark that the different collision operators do not operate on the same time-scale. Firstly, due to the small mass ratio δ, the ion-electron collision term is negligible in the relaxation process of the ions towards their thermodynamic equilibrium. Secondly, both collision operators Q_{ee} and Q_{ei} act on the same time-scale and contribute together to the thermodynamic relaxation of the electrons. Finally, one also remarks that the ions relax much slower than the electrons towards their corresponding equilibrium, namely $\sqrt{\delta} = \sqrt{m_e/m_i}$-slower, such that the electron fluid equations are established in advance to the ion hydrodynamic ones.

Keeping now all parameters fixed but one leads to well-known asymptotic limits. In particular, $\lambda \to 0$ is the so-called quasi-neutrality limit [29, 30], $\kappa \to 0$ the hydrodynamic limit [1, 13, 23] and $M \to 0$ the long-time asymptotics [3, 18], which is a rather challenging limit. Finally $\delta \to 0$ denotes the zero-mass limit [24]. The limit we are interested to treat in this paper is the so-called adiabatic limit. We have seen in [36] that in order to get from the kinetic to the electron Boltzmann regime, one needs a collisional, low-mass or inertial-less situation. In order to mimic this situation we suppose

$$M \sim 1\,, \qquad \delta \sim \varepsilon\,, \qquad \kappa \sim \sqrt{\varepsilon}\,, \tag{31}$$

where $\varepsilon \in [0, 1]$ is the asymptotic parameter tending to zero[1] in the following studies. The main reasons for the particular study of (31) can be summarized as follows:

- **Collisionality:** Even if tokamak plasmas are low-collisional, our choice of a high collisionality comes from the fact that nowadays fluid models are still used for numerical plasma simulations, being less consuming (in time and memory) than

[1]The mass ratio δ being fixed in reality, the limit $\varepsilon \to 0$ has to be viewed as a mathematical trick, or approximation. The study of this limit permits a) to understand better the differences between the ion and electron motions and b) to construct efficient numerical schemes for regimes of $\varepsilon \ll 1$.

kinetic models. However, there are situations (or regions in the tokamak) where fluid models give erroneous or inaccurate results, such that one has to come back to the more precise kinetic models. Our aim was thus to construct a scheme being able to switch automatically between the corresponding kinetic and fluid regimes, and this via the action of the collisional parameter κ.

Remark also that, even for low-collisional plasmas, collisions can lead to important effects on long time scales, such as magnetic reconnection processes; this makes their inclusion in the kinetic equations necessary in order to describe observed physical phenomena.

- **Inertia-less electron regime:** In many cases the phenomena of interest occur at ionic time-scales. Hence, in order to be performant, one would like to follow accurately the ion-dynamics, restricted by the time-step $v_{th,i}\,\Delta t \leq \Delta x$, without having to resolve the rapid electron evolution, restricted by the more limited time-step $v_{th,e}\,\Delta t = v_{th,i}\,\Delta t/\sqrt{\delta} \leq \Delta x$. This is the reason why we artificially set $M \sim 1$ and let $\varepsilon \to 0$ (even if the mass-ratio δ is fixed in reality), having in mind a scheme that gives accurate results for Δt independent of ε. The adiabatic limit could also be attained by setting $M \sim \varepsilon$ and δ fixed; however, this limit would not differentiate between ions and electrons and is the topic of other works [17] regarding the low-Mach ion regime.

- **AP-property:** Regarding (25) with $M \sim 1$ and $\sqrt{\delta} \sim 10^{-2}$ could suggest that studying the limit $\delta = \varepsilon \to 0$ is rather academic, with little use in practice. This is not true since the Mach number can be small in some regions of the Tokamak, such that the product $\sqrt{\delta}M$ is small enough to cause numerical difficulties (bad conditioning of linear systems, slow convergence in iterative schemes, numerical singularities). The AP-schemes designed here are devoid of such problems and are capable to treat each regime equally efficient and accurate.

2.2. **Collision operators.** Let us now focus on the collision operators and their respective scaling. The choice of a collision operator is important for the correct description of the underlying collisional process and the relaxation towards the respective thermodynamic equilibrium. The conclusions of the previous subsection are independent on the particular form of the collision operators, provided they satisfy the demanded conservation rules. However, for the further study, in particular for the determination of the structure of the limiting system in the long-time asymptotics, it is important to know more about these collision operators, especially about the thermodynamic equilibria, property embodied within them.

Several collision operators have been introduced in literature, describing various collision processes and varying in complexity, as for example the Boltzmann operator, the Fokker-Planck operator or the Landau operator [5, 6, 39]. Given their rather considerable elaboration, it is tempting to make the choice of more simpler operators, which are approximations of the latter ones, and share the same conservation-properties, entropy inequalities and equilibria. The advantage of these more simpler models is their manageable numerical treatment. For these reasons, we shall concentrate in the present paper on BGK-type collision operators, which substitute the detailed particle interactions by relaxation processes towards the requested equilibria. These self-species and inter-species operators read [37]

$$Q_{ee,ii}(f_{e,i}) := \nu_{ee,ii}\,(\mathcal{M}^{e,i}_{n_{e,i},u_{e,i},T} - f_{e,i})\,, \tag{32a}$$

$$Q_{ei}(f_e, f_i) := \nu_{ei}\,(\mathcal{M}^{e}_{n_e,u_i,T} - f_e)\,, \quad Q_{ie}(f_i, f_e) := \nu_{ie}\,(\mathcal{M}^{i}_{n_i,u_e,T} - f_i)\,, \tag{32b}$$

where $\nu_{kl} = \bar{\nu}_{kl}\,\nu'_{kl}$ are the previously introduced relaxation frequencies of the distribution functions towards the Maxwellian equilibria, given by

$$\mathcal{M}^{s}_{n_{e,i},u_{e,i},T}(t,x,v) = n_{e,i}(t,x)\,\left(\frac{m_s}{2\pi k_B T}\right)^{1/2}\,\exp\left(-\,m_s\,\frac{|v-u_{e,i}(t,x)|^2}{2k_BT}\right). \tag{33}$$

Here k_B is the Boltzmann constant, the electrons and ions are supposed to be in a thermal bath of given temperature $T > 0$ and $n_{e,i}$ and $u_{e,i}$ are the macroscopic particle density and mean velocity defined in Eqs. (5) and (6), respectively. Remark here also the upper index $s \in \{e, i\}$ in the notation of the Maxwellian $\mathcal{M}^s$, permitting to clarify which mass to take, especially in the definition of the inter-species collision operators. To have the classical mass and momentum conservation properties of these collision operators, we have to suppose

$$m_i n_i \nu_{ie} = m_e n_e \nu_{ei}\,, \qquad \Longrightarrow \qquad n'_i\,\nu'_{ie} = n'_e\,\nu'_{ei}\,.$$

The scaled relation follows then from our assumptions (17) and (19) on the characteristic densities and collision frequencies. Scaling the collision operators (32) leads to $Q_{kl}(f_k, f_l) =$

$\bar\nu_{kl}\,\bar f_k\,Q'_{kl}(f'_k,f'_l)$, where

$$Q'_{ee}(f'_e) := \nu'_{ee}\,(\mathcal{M}_{n'_e,\sqrt\delta\,M\,u'_e} - f'_e) = \nu'_{ee}\left(\frac{n'_e}{\sqrt{2\pi}}\,\exp\left(-\frac{|v'-\sqrt\delta\,M\,u'_e|^2}{2}\right) - f'_e\right)\,, \quad (34a)$$

$$Q'_{ii}(f'_i) := \nu'_{ii}\,(\mathcal{M}_{n'_i,M\,u'_i} - f'_i) = \nu'_{ii}\left(\frac{n'_i}{\sqrt{2\pi}}\,\exp\left(-\frac{|v'-M\,u'_i|^2}{2}\right) - f'_i\right)\,, \quad (34b)$$

$$Q'_{ei}(f'_e,f'_i) := \nu'_{ei}\,(\mathcal{M}_{n'_e,\sqrt\delta\,M\,u'_i} - f'_e) = \nu'_{ei}\left(\frac{n'_e}{\sqrt{2\pi}}\,\exp\left(-\frac{|v'-\sqrt\delta\,M\,u'_i|^2}{2}\right) - f'_e\right)\,, \quad (34c)$$

$$Q'_{ie}(f'_i,f'_e) := \nu'_{ie}\,(\mathcal{M}_{n'_i,M\,u'_e} - f'_i) = \nu'_{ie}\left(\frac{n'_i}{\sqrt{2\pi}}\,\exp\left(-\frac{|v'-M\,u'_e|^2}{2}\right) - f'_i\right)\,. \quad (34d)$$

The rescaled moments $n'_{e,i}$ and $u'_{e,i}$ of the distribution functions are given in (26) and (30).

3. Macroscopic models in the mass-disparate regime

3.1. **Drift-diffusion regime of electrons.** Before concentrating on the adiabatic electron limit, let us review an ion/electron macroscopic model issued from the scaled kinetic system (25), with our particular choice (31) for the occurring parameters, *i.e.*

$$\begin{cases} \partial_t f_i^\varepsilon + v\,\partial_x f_i^\varepsilon + E^\varepsilon\,\partial_v f_i^\varepsilon = \dfrac{1}{\sqrt\varepsilon}\,\left[Q_{ii}(f_i^\varepsilon) + \sqrt\varepsilon Q_{ie}(f_i^\varepsilon,f_e^\varepsilon)\right]\,, \\ \partial_t f_e^\varepsilon + \dfrac{1}{\sqrt\varepsilon}\,v\,\partial_x f_e^\varepsilon - \dfrac{1}{\sqrt\varepsilon}\,E^\varepsilon\,\partial_v f_e^\varepsilon = \dfrac{1}{\varepsilon}\,\left[Q_{ee}^{(\varepsilon)}(f_e^\varepsilon) + Q_{ei}^{(\varepsilon)}(f_e^\varepsilon,f_i^\varepsilon)\right]\,, \end{cases} \quad (35)$$

coupled with the Poisson equation

$$-\lambda^2\,\partial_{xx}\phi^\varepsilon = n_i^\varepsilon - n_e^\varepsilon\,, \qquad E^\varepsilon = -\partial_x\phi^\varepsilon\,. \quad (36)$$

Our aim is to identify the ion/electron asymptotic limit system as the small perturbation parameter ε tends towards zero. We term this limit a "macroscopic" limit since $\sqrt\varepsilon$ embodies the Knudsen number (ε expresses also the electron-ion mass-ratio). Remark that the primes have been omitted for simplicity reasons and that the collision operators in dimension-less form read

$$Q_{ii} = \nu_{ii}\left(\frac{n_i^\varepsilon}{\sqrt{2\pi}}\exp\left(-\frac{(v-u_i^\varepsilon)^2}{2}\right) - f_i^\varepsilon\right)\,, \quad Q_{ee}^{(\varepsilon)} = \nu_{ee}\left(\frac{n_e^\varepsilon}{\sqrt{2\pi}}\exp\left(-\frac{(v-\sqrt\varepsilon u_e^\varepsilon)^2}{2}\right) - f_e^\varepsilon\right)\,, \quad (37a)$$

$$Q_{ie} = \nu_{ie}\left(\frac{n_i^\varepsilon}{\sqrt{2\pi}}\exp\left(-\frac{(v-u_e^\varepsilon)^2}{2}\right) - f_i^\varepsilon\right)\,, \quad Q_{ei}^{(\varepsilon)} = \nu_{ei}\left(\frac{n_e^\varepsilon}{\sqrt{2\pi}}\exp\left(-\frac{(v-\sqrt\varepsilon u_i^\varepsilon)^2}{2}\right) - f_e^\varepsilon\right)\,, \quad (37b)$$

The electron collision operators depend explicitly on ε in the exponential, fact which stems from the scale assumptions (13) and which is highlighted in their notation. In many practical applications the rescaled Debye length λ will tend to zero as $\varepsilon \to 0$, since $\lambda = \kappa \frac{1}{\omega_p \tau_c}$ with ω_p the ion plasma frequency; we shall not investigate this quasi-neutrality limit in the present paper, in order to concentrate on the new adiabatic limit.

The $\varepsilon \to 0$ limit of the ion kinetic equation is a standard hydrodynamic limit. Indeed, f_i^ε tends towards a function belonging to the kernel of the dominant operator Q_{ii}, *i.e.* $f_i^0 = \mathcal{M}_{n_i^0, u_i^0}$, the moments of this Maxwellian being solution of the hydrodynamic model

$$\text{(ion-}HD\text{)} \quad \begin{cases} \partial_t n_i^0 + \partial_x (n_i^0 \, u_i^0) = 0 \,, \\ \partial_t (n_i^0 \, u_i^0) + \partial_x (n_i^0 \, (u_i^0)^2) + \partial_x \, p_i^0 - E^0 \, n_i^0 = \mathcal{S}_{ie}^0 \,, \end{cases} \tag{38}$$

with the friction term $\mathcal{S}_{ie}^0 = \nu_{ie} n_i^0 (u_e^0 - u_i^0)$ and the pressure defined as

$$p_i^\varepsilon(t,x) := \int_{\mathbb{R}} (v - u_i^\varepsilon)^2 \, f_i^\varepsilon(t,x,v) \, dv \,,$$

leading for $\varepsilon \to 0$ to the equation of state $p_i^0 = n_i^0$. The limit problem (38) is a completely macroscopic system, coupled to the electron system via the mean velocity u_e^0 in $\mathcal{S}_{ie}^0$ as well as via the electric field E^0, computed through Poisson. This hydrodynamic limit $\varepsilon \to 0$ has been largely investigated in literature; for more details we refer the interested reader to the non-exhaustive list [1, 9, 13, 20, 21, 23] and references therein.

To identify the $\varepsilon \to 0$ limit of the electron kinetic equation in (35), one observes from (37) that, for given ion-distribution function f_i, we have

$$\ker \Big\{ Q_{ee}^{(0)}(f_e) + Q_{ei}^{(0)}(f_e, f_i) \Big\} = \{ \mathcal{M}_{n_e^0, 0} \ , \ \ n_e^0 \in \mathbb{R} \} \,,$$

meaning that the $\varepsilon \to 0$ limit of the electron distribution function f_e^ε is a Maxwellian with zero mean velocity, $f_e^0 = \mathcal{M}_{n_e^0, 0}$, where the evolution of the density $n_e^0(t,x)$ has still to be determined. For this, taking the moments of the electron kinetic equation in (35) and using (30) leads to

$$\begin{cases} \partial_t n_e^\varepsilon + \partial_x (n_e^\varepsilon \, u_e^\varepsilon) = 0 \,, \\ \partial_t (n_e^\varepsilon \, u_e^\varepsilon) + \partial_x (n_e^\varepsilon \, (u_e^\varepsilon)^2) + \dfrac{1}{\varepsilon} \, \partial_x \, p_e^\varepsilon + \dfrac{1}{\varepsilon} \, E^\varepsilon \, n_e^\varepsilon = \dfrac{1}{\varepsilon} \mathcal{S}_{ei}^\varepsilon \,, \end{cases} \tag{39}$$

with the friction term $\mathcal{S}_{ei}^\varepsilon = \nu_{ei} n_e^\varepsilon (u_i^\varepsilon - u_e^\varepsilon)$ and the pressure given by the formula

$$p_e^\varepsilon(t,x) := \int_{\mathbb{R}} (v - \sqrt{\varepsilon} \, u_e^\varepsilon)^2 \, f_e^\varepsilon(t,x,v) \, dv \,. \tag{40}$$

It follows from this definition that $p_e^0 = n_e^0$; thus the limit regime $\varepsilon \to 0$ of system (39) is simply

$$(\text{electron-}DD) \quad \begin{cases} \partial_t n_e^0 + \partial_x(n_e^0\, u_e^0) = 0\,, \\ \partial_x n_e^0 + E^0\, n_e^0 = \nu_{ei} n_e^0\,(u_i^0 - u_e^0)\,, \end{cases} \tag{41}$$

which is a drift-diffusion model coupled to the ion dynamics through the mean velocity u_i^0 as well as via the electric field E^0. Kinetic equations in a drift-diffusive scaling and their asymptotic analysis have been also the context of several works, as for example [28, 34, 38]. Note here that, as expected, the electrons have reached in the limit a more macroscopic equilibrium (DD-model) than the ions (HD-model), due to their smaller mass or equivalently their higher thermal velocities.

3.2. **Adiabatic regime of electrons.** In contrast to the drift-diffusion electron regime, the adiabatic regime is attained when the friction term $\mathcal{S}_{ei}^\varepsilon$ in (39) is small. This could occur when for example the current is small, *i.e.* $u_i^\varepsilon - u_e^\varepsilon \sim \mathcal{O}(\sqrt{\varepsilon})$, such that the friction term $\mathcal{S}_{ie}^\varepsilon$ will disappear at leading order in (39). Passing then to the limit yields

$$\begin{cases} \partial_t\, n_e^0 + \partial_x(n_e^0\, u_e^0) = 0 \\ \partial_x n_e^0 + E^0\, n_e^0 = 0\,. \end{cases} \tag{42}$$

Assuming periodic boundary conditions, which makes sense if we keep in mind the idea of a plasma evolving along closed field lines, this is an ill-posed system. Indeed, there is no manner to determine completely the velocity u_e^0. This difficulty is similar to the singular low-Mach limit.

It is worth noting that one is however able to get the Boltzmann relation from (42). Indeed, for given $E^0 = -\partial_x \phi^0$ the second equation of (42) yields the density-potential relation

$$n^0(t,x) = c(t)\, e^{\phi^0(t,x)}\,, \quad \forall (t,x) \in \mathbb{R}^+ \times [0,L]\,. \tag{43}$$

To determine the constant $c(t)$, one has only to integrate the first equation of (42) over the periodic space-variable x to get

$$\partial_t\, \overline{n^0} = 0\,, \qquad \overline{n^0}(t) := \frac{1}{L}\int_0^L n^0(t,x)\, dx\,,$$

implying thus

$$\overline{n^0}(t) = \overline{n^0(0,\cdot)} = \overline{n_0^0}\,,$$

which permits to compute $c(t)$ for all $t \in \mathbb{R}^+$ from (43). The fact however that u^0 is not completely determined from the limit model (42) will lead to an ill-conditioned linear system to be solved if standard methods are used for the discretization of the electron kinetic equation in (35), schemes which will break down as $\varepsilon \to 0$.

The remainder of this paper is dedicated to the study of this adiabatic asymptotics, in particular its asymptotic-preserving formulation.

4. Passage to the electron Boltzmann regime

To investigate the adiabatic asymptotics in more details, let us neglect the ions and start in this section from the following rescaled electron kinetic equation

$$(K)_\varepsilon \quad \partial_t f_e^\varepsilon + \frac{1}{\sqrt{\varepsilon}}\, v\, \partial_x f_e^\varepsilon - \frac{1}{\sqrt{\varepsilon}}\, E\, \partial_v f_e^\varepsilon = \frac{1}{\varepsilon}\, Q_{ee}(f_e^\varepsilon) = \frac{1}{\varepsilon}\, [\mathcal{M}_{n_e^\varepsilon, \sqrt{\varepsilon} u_e^\varepsilon} - f_e^\varepsilon]. \tag{44}$$

The electrostatic potential $\phi(t,x)$ is assumed to be given and $E = -\partial_x \phi$. Firstly, our aim will be to identify formally the limiting regime of (44) as $\varepsilon \to 0$ and secondly to construct a numerical scheme for the resolution of this kinetic equation working uniformly accurate in all ε-regimes. For the sake of clarity, we shall omit in the following the 'e'-index of the distribution function and of the macroscopic quantities and shall keep in mind that we are dealing with the electron dynamics. Furthermore, to simplify, we set $\nu_{ee} = 1$.

To complete the electron kinetic equation (44) we have to specify the boundary and initial conditions. The time variable t belongs to $\mathbb{R}^+$, the spatial variable x to the interval $[0, L]$ with $L > 0$, and we shall consider in the following a periodic space-situation, meaning

$$\partial_x^k f(t,0,v) = \partial_x^k f(t,L,v)\,, \quad \forall t \in \mathbb{R}^+\,,\ \ \forall v \in \mathbb{R}\,, \quad k \in \mathbb{N}\,. \tag{45a}$$

The velocity variable v is considered in the whole domain $\mathbb{R}$, with the condition that

$$\lim_{v \to \pm\infty} f(t,x,v) = 0\,, \quad \forall t \in \mathbb{R}^+\,,\ \ \forall x \in [0,L]. \tag{45b}$$

In the numerical simulations one has to truncate the velocity space to $[v_{min}, v_{max}]$ supposing then that the test case and the final simulation time $T > 0$ are such that one can consider for fixed $(t,x) \in [0,T] \times [0,L]$ that $f(t,x,v) = 0$ outside the velocity space.

Concerning the initial condition, we shall start with a situation corresponding to the scaling of Eq. (44), i.e. adapted to the physical regime we consider. Thus, initially, our electron distribution function is given by the expression

$$f^\varepsilon(0,x,v) = \mathcal{M}_{n_0^\varepsilon, \sqrt{\varepsilon}\, u_0^\varepsilon} + \varepsilon\, g_0^\varepsilon\,, \tag{46}$$

with some given, bounded functions n_0^ε, u_0^ε, g_0^ε. This particular initial condition, which signifies that we start with a perturbation of the equilibrium Maxwellian, permits to avoid the creation of boundary layers near $t \approx 0$, which can be difficult to treat. Note that sometimes, an AP-scheme based on such well-prepared initial conditions is called "weakly Asymptotic Preserving".

4.1. The Micro-Macro decomposition.

To identify the asymptotic limit of (44), we start with a Chapman-Enskog Ansatz for the distribution function,

$$f^\varepsilon = \mathcal{M}(f^\varepsilon) + \varepsilon\, g^\varepsilon\,, \qquad \mathcal{M}(f^\varepsilon) = \mathcal{M}_{n^\varepsilon, \sqrt{\varepsilon} u^\varepsilon} := \frac{n^\varepsilon}{\sqrt{2\pi}}\, \exp\left(-\frac{|v - \sqrt{\varepsilon}\, u^\varepsilon|^2}{2} \right)\,, \tag{47}$$

and recall that the moments are given by

$$n^\varepsilon(t,x) = \int_{\mathbb{R}} f^\varepsilon(t,x,v)\,dv\,, \quad \sqrt{\varepsilon}\,n^\varepsilon\,u^\varepsilon(t,x) = \int_{\mathbb{R}} v\,f^\varepsilon(t,x,v)\,dv\,, \tag{48}$$

such that one has the properties

$$\int_{\mathbb{R}} g^\varepsilon(t,x,v)\,dv = \int_{\mathbb{R}} v\,g^\varepsilon(t,x,v)\,dv = 0\,. \tag{49}$$

Inserting this decomposition into (44), one obtains

$$\partial_t \mathcal{M}(f^\varepsilon) + \varepsilon\,\partial_t g^\varepsilon + \frac{1}{\sqrt{\varepsilon}}\,v\,\partial_x \mathcal{M}(f^\varepsilon) + \sqrt{\varepsilon} v\,\partial_x g^\varepsilon - \frac{1}{\sqrt{\varepsilon}}\,E^\varepsilon\,\partial_v \mathcal{M}(f^\varepsilon) - \sqrt{\varepsilon} E^\varepsilon\,\partial_v g^\varepsilon = -g^\varepsilon\,. \tag{50}$$

In order to separate the macroscopic and microscopic parts in this equation, we shall use a projection technique employed also in previous works, see for example [2, 10, 11]. Suppose, for the moment, that f^ε and thus the moments $(n^\varepsilon,\ \sqrt{\varepsilon}\,u^\varepsilon)$ as well as the associated Maxwellian $\mathcal{M}_{n^\varepsilon,\sqrt{\varepsilon}u^\varepsilon}$ are fixed. For notational reasons, we shall designate by $\mathcal{M}$ this Maxwellian, there where no confusion is possible. Let us moreover denote by $\Pi_{\mathcal{M}}$ the $L^2(\mathcal{M}^{-1}\,dv)$-orthogonal projection onto the space

$$\mathcal{N}(\mathcal{L}_{\mathcal{M}}) := \text{Span}\,\{\mathcal{M}\,, v\,\mathcal{M}\}\,, \tag{51}$$

which is the kernel of the linearization $(\mathcal{L}_{\mathcal{M}})$ of the BGK-collision operator Q_{ee} around the fixed Maxwellian $\mathcal{M}_{n^\varepsilon,\sqrt{\varepsilon}u^\varepsilon}$. This orthogonal projection operator on $\mathcal{N}(\mathcal{L}_{\mathcal{M}})$ has the form

$$\Pi_{\mathcal{M}}(h) = \Big[\,\frac{n_\star}{n^\varepsilon} + \sqrt{\varepsilon}\,u_\star\,(v - \sqrt{\varepsilon}u^\varepsilon)\,\Big]\,\mathcal{M}(f^\varepsilon)\,, \tag{52}$$

where $(n^\varepsilon,\ \sqrt{\varepsilon}\,u^\varepsilon)$ are the moments corresponding to the given, fixed distribution function f^ε and $(n_\star,\ \sqrt{\varepsilon}\,u_\star)$ correspond to the function h and are defined as

$$n_\star := \int_{\mathbb{R}} h\,dv\,, \quad \sqrt{\varepsilon}\,n^\varepsilon\,u_\star := \int_{\mathbb{R}} (v - \sqrt{\varepsilon}u^\varepsilon)\,h\,dv\,. \tag{53}$$

Having defined the projection operator, we successively apply $\Pi_{\mathcal{M}}$ and then $\mathbb{I} - \Pi_{\mathcal{M}}$ to (50), in order to separate the macroscopic and microscopic parts of the distribution function f^ε. This gives rise to the coupled system

$$\begin{cases} \varepsilon\partial_t g^\varepsilon + \dfrac{1}{\sqrt{\varepsilon}}(\mathbb{I} - \Pi_{\mathcal{M}})\,(v\,\partial_x \mathcal{M}(f^\varepsilon)) + \sqrt{\varepsilon}\,(\mathbb{I} - \Pi_{\mathcal{M}})\,(v\,\partial_x g^\varepsilon) - \sqrt{\varepsilon}\,E^\varepsilon\,\partial_v\,g^\varepsilon = -g^\varepsilon\,, \\ \partial_t\,\mathcal{M}(f^\varepsilon) + \dfrac{1}{\sqrt{\varepsilon}}\,\Pi_{\mathcal{M}}(v\,\partial_x \mathcal{M}(f^\varepsilon)\,) + \sqrt{\varepsilon}\Pi_{\mathcal{M}}(v\,\partial_x g^\varepsilon) - \dfrac{1}{\sqrt{\varepsilon}}E^\varepsilon\,\partial_v \mathcal{M}(f^\varepsilon) = 0\,. \end{cases} \tag{54}$$

Remark that for the obtention of this system, we used the following properties of the projection operator

$$\Pi_{\mathcal{M}}(\partial_t \mathcal{M}) = \partial_t \mathcal{M}\,, \qquad \Pi_{\mathcal{M}}(\partial_t g) = 0\,, \tag{55}$$

$$\Pi_{\mathcal{M}}(E^\varepsilon\, \partial_v \mathcal{M}) = E^\varepsilon\, \partial_v \mathcal{M}\,, \qquad \Pi_{\mathcal{M}}(E^\varepsilon\, \partial_v g) = 0\,. \tag{56}$$

Furthermore, one has

$$(\mathbb{I} - \Pi_{\mathcal{M}^\varepsilon})\,(v\,\partial_x \mathcal{M}^\varepsilon) = \sqrt{\varepsilon}\partial_x u^\varepsilon\, \left[(v - \sqrt{\varepsilon}u^\varepsilon)^2 - 1\right] \mathcal{M}^\varepsilon\,, \tag{57}$$

$$\Pi_{\mathcal{M}^\varepsilon}(v\,\partial_x g) = \frac{v - \sqrt{\varepsilon}u^\varepsilon}{n^\varepsilon}\,\partial_x \left\langle v^2\, g^\varepsilon \right\rangle \mathcal{M}^\varepsilon\,. \tag{58}$$

Taking now the moments of the second equation of (54), denoting the integration in v simply by $\langle\cdot\rangle := \int_{\mathbb{R}} \cdot\, dv$ and observing that the pressure defined in (40) satisfies $p^\varepsilon = n^\varepsilon + \varepsilon \left\langle v^2\, g^\varepsilon \right\rangle$, leads to the so-called Kinetic-Fluid Micro-Macro reformulation of (44),

$$(KF)_\varepsilon \quad \begin{cases} \varepsilon\,\partial_t g^\varepsilon + \sqrt{\varepsilon}\,v\,\partial_x g^\varepsilon - \sqrt{\varepsilon} E^\varepsilon \partial_v g^\varepsilon - \sqrt{\varepsilon}\,\dfrac{(v - \sqrt{\varepsilon}u^\varepsilon)}{n^\varepsilon}\,\partial_x \left\langle v^2\, g^\varepsilon \right\rangle \mathcal{M}^\varepsilon \\ \qquad\qquad + \partial_x u^\varepsilon\, \left[(v - \sqrt{\varepsilon}u^\varepsilon)^2 - 1\right] \mathcal{M}^\varepsilon = -g^\varepsilon\,, \\ \partial_t\, n^\varepsilon + \partial_x (n^\varepsilon\, u^\varepsilon) = 0\,, \\ \partial_t\, (n^\varepsilon\, u^\varepsilon) + \partial_x (n^\varepsilon (u^\varepsilon)^2) + \dfrac{1}{\varepsilon}\partial_x n^\varepsilon + \dfrac{1}{\varepsilon} E^\varepsilon\, n^\varepsilon + \partial_x \left\langle v^2\, g^\varepsilon \right\rangle = 0\,. \end{cases} \tag{59}$$

This corresponds to a coupled system, consisting of a microscopic kinetic equation for g^ε and the mass and momentum balance laws for the macroscopic quantities $(n^\varepsilon, u^\varepsilon)$. It is a completely equivalent model to the original kinetic equation (44) (for $\varepsilon > 0$), which will however behave better in the limit $\varepsilon \to 0$, due to the microscopic-macroscopic decomposition.

Note here also that if one considers $E^\varepsilon \equiv 0$ and $g^\varepsilon \equiv 0$, system (59) is nothing else than the low-Mach number isentropic Euler equations, treated from a numerical point of view in [8, 16, 25]. The present system provides thus a generalization of this low-Mach number model, including the kinetic effects (for large ε-values) as well as the effects coming from the electrostatic field. The here presented strategy for its efficient AP-resolution is based however on different techniques as those of the previously cited works.

4.2. **Identification of the Limit model and AP-reformulation.** The aim is now to avoid the singularity in the fluid equations of (59) as $\varepsilon \to 0$, c.f. Eqs. (42), by means of a projection / micro-macro technique. To do this, let us study the dominant operator in the momentum equation of system (59) and try to introduce a second Micro-Macro decomposition associated to this new dominant operator.

Let us denote in the sequel by $X_\sharp$ the subspace of the Banach-space X, consisting of the functions which are L-periodic in the space variable $x \in I := (0, L)$. The time-variable $t \in \mathbb{R}^+$ shall be considered as a parameter in the following arguments. With these definitions, we introduce for fixed electric field $E = -\partial_x \phi$, with $\phi \in W^{1,\infty}(I)$, the linear operator

$$\mathbb{L} : \mathcal{D}_\sharp \subset L^2_\sharp(I) \to L^2_\sharp(I)\,, \qquad \mathbb{L}(n) := \partial_x n + E\, n\,, \tag{60}$$

with definition domain

$$\mathcal{D}_\sharp := \{\xi \in L^2_\sharp(I) \ / \ \mathbb{L}(\xi) \in L^2_\sharp(I)\}\,.$$

This operator is the dominant operator in the momentum equation of (59) and has to be studied in more details in order to circumvent the singularity of this system as $\varepsilon \to 0$.

The kernel of $\mathbb{L}$ is given by the following one-dimensional space

$$\mathcal{G}_\mathbb{L} := \{\xi \in \mathcal{D}_\sharp \ / \ \mathbb{L}(\xi) = 0\} = \{c\, e^\phi\,,\ c \in \mathbb{R}\}\,. \tag{61}$$

Introducing now the following weighted scalar-product on $L^2_\sharp(I)$

$$\langle \xi, \zeta \rangle_\mathbb{L} := \frac{1}{L} \int_0^L \xi\, \zeta\, e^{-2\phi}\, dx\,, \qquad \forall \xi, \zeta \in L^2_\sharp(I)\,, \tag{62}$$

permits to decompose the Hilbert-space $L^2_\sharp(I)$ in a unique manner as follows

$$L^2_\sharp(I) = \mathcal{G}_\mathbb{L} \oplus^\perp \tilde{\mathcal{A}}_\mathbb{L}\,, \tag{63}$$

where

$$\tilde{\mathcal{A}}_\mathbb{L} := \{\xi \in L^2_\sharp(I) \ / \ \langle \xi, \zeta \rangle_\mathbb{L} = 0 \quad \forall \zeta \in \mathcal{G}_\mathbb{L}\} = \{\xi \in L^2_\sharp(I) \ / \ \frac{1}{L} \int_0^L \xi\, e^{-\phi}\, dx = 0\}\,, \tag{64}$$

which consists of functions with a zero weighted average over I. The decomposition (63) is associated with an orthogonal projection operator $\mathcal{P}_\mathbb{L}$ defined as

$$\mathcal{P}_\mathbb{L} : L^2_\sharp(I) \to \mathcal{G}_\mathbb{L}\,, \qquad \mathcal{P}_\mathbb{L}(\xi) := \frac{1}{L} \int_0^L \xi\, e^{-\phi}\, dx\ e^\phi\,. \tag{65}$$

The definition of this projection operator permits to rewrite the space $\tilde{\mathcal{A}}_\mathbb{L}$ as

$$\tilde{\mathcal{A}}_\mathbb{L} := \{\xi \in L^2_\sharp(I) \ / \ \mathcal{P}_\mathbb{L}(\xi) = 0\} = \mathcal{G}_\mathbb{L}^\perp\,. \tag{66}$$

By restriction, one can immediately show that we have also the following decomposition of the definition domain of the dominant operator $\mathbb{L}$

$$\mathcal{D}_\sharp = \mathcal{G}_\mathbb{L} \oplus^\perp \mathcal{A}_\mathbb{L}\,, \tag{67}$$

where this time

$$\mathcal{A}_\mathbb{L} := \{\xi \in \mathcal{D}_\sharp \ / \ \langle \xi, \zeta \rangle_\mathbb{L} = 0 \quad \forall \zeta \in \mathcal{G}_\mathbb{L}\} = \{\xi \in \mathcal{D}_\sharp \ / \ \mathcal{P}_\mathbb{L}(\xi) = 0\}\,. \tag{68}$$

With all these explanations, one can show finally that the dominant operator is a linear bijective mapping

$$\mathbb{L} : \mathcal{A}_{\mathbb{L}} \to \tilde{\mathcal{A}}_{\mathbb{L}} , \qquad \mathbb{L}(n) := \partial_x n + E\, n , \tag{69}$$

meaning that the problem

$$\begin{cases} \partial_x \xi + E\, \xi = \Psi , \quad \forall x \in (0, L) , \\ \mathcal{P}_{\mathbb{L}}(\xi) = 0 , \end{cases} \tag{70}$$

admits a unique solution $\xi \in \mathcal{A}_{\mathbb{L}}$ if and only if $\Psi \in \tilde{\mathcal{A}}_{\mathbb{L}}$.

Coming now back to our singularly-perturbed problem (59), let us now reformulate this problem using the just introduced projection-framework, in particular let us introduce the decomposition $n^\varepsilon = r^\varepsilon + \varepsilon s^\varepsilon$, with r^ε being the macroscopic part defined by $r^\varepsilon := \mathcal{P}_{\mathbb{L}}(n^\varepsilon)$. Hence, s^ε is the unique solution of (70) with the right hand side given by $\Theta = \frac{1}{\varepsilon}\left[\partial_x n^\varepsilon + E^\varepsilon\, n^\varepsilon\right] \in \tilde{\mathcal{A}}_{\mathbb{L}}$. Inserting this new decomposition into (59) yields the system

$$(AP)_\varepsilon \quad \begin{cases} \varepsilon\, \partial_t g^\varepsilon + \sqrt{\varepsilon}\, v\, \partial_x g^\varepsilon - \sqrt{\varepsilon} E^\varepsilon \partial_v g^\varepsilon - \sqrt{\varepsilon}\, \dfrac{(v - \sqrt{\varepsilon} u^\varepsilon)}{n^\varepsilon}\, \partial_x \left\langle v^2\, g^\varepsilon \right\rangle \mathcal{M}^\varepsilon \\ \qquad\qquad + \partial_x u^\varepsilon \, \left[(v - \sqrt{\varepsilon} u^\varepsilon)^2 - 1 \right] \mathcal{M}^\varepsilon = - g^\varepsilon , \\ \partial_t\, n^\varepsilon + \partial_x (n^\varepsilon\, u^\varepsilon) = 0 , \\ \partial_t \left(n^\varepsilon\, u^\varepsilon\right) + \partial_x (n^\varepsilon (u^\varepsilon)^2) + \partial_x s^\varepsilon + E^\varepsilon\, s^\varepsilon + \partial_x \left\langle v^2\, g^\varepsilon \right\rangle = 0 , \\ \partial_x n^\varepsilon + E^\varepsilon\, n^\varepsilon = \varepsilon \left(\partial_x s^\varepsilon + E^\varepsilon\, s^\varepsilon\right) , \qquad \mathcal{P}_{\mathbb{L}}(s^\varepsilon) = 0 . \end{cases} \tag{71}$$

This system is completely equivalent to the original kinetic system (44) as well as to the Kinetic-Fluid Micro-Macro reformulation (59) for all $\varepsilon > 0$. It has the essential advantage of capturing the well-posed limit regime, as $\varepsilon \to 0$. To determine this asymptotic limit of $(AP)_\varepsilon$, let us make the Hilbert-Ansatz $g^\varepsilon = g^0 + \sqrt{\varepsilon} g^1 + \cdots$ and equate in the kinetic equation of (71) the terms of the same order in ε. This yields immediately

$$g^0 = (1 - v^2)\, \partial_x u^0\, \mathcal{M}_{n^0, 0} \quad \Rightarrow \quad \left\langle v^2\, g^0 \right\rangle = -2\, n^0\, \partial_x u^0 . \tag{72}$$

Plugging this information in the macroscopic equations of (71), permits to get the limit model of $(AP)_\varepsilon$ as ε tends towards zero, namely

$$(L) \quad \begin{cases} \partial_t\, n^0 + \partial_x (n^0\, u^0) = 0 \\ \partial_t \left(n^0\, u^0\right) + \partial_x (n^0 (u^0)^2) + \partial_x s^0 + E^0\, s^0 - 2\, \partial_x \left(n^0\, \partial_x u^0\right) = 0 , \\ \partial_x n^0 + E^0\, n^0 = 0 , \qquad \mathcal{P}_{\mathbb{L}}(s^0) = 0 . \end{cases} \tag{73}$$

This limit system contains on one hand the adiabatic Boltzmann relation, hidden in the third equation (see (43)), and on the other hand it permits to compute the particle flux $n^0\,u^0$, which was not the case for the ill-posed system (42). Remark in particular the presence of the viscous term in the momentum conservation law, giving rise to regular solutions. Finally, the integral constraint $\mathcal{P}_{\mathbb{L}}(s^0) = 0$ guarantees the uniqueness of s_0, solution of a first-order ODE in (73).

To summarize, the distribution function f^ε, solution to the kinetic equation (44), converges as $\varepsilon \to 0$ towards a Maxwellian distribution function $f^0 = \mathcal{M}_{n^0,0}$, with density function given by the well-posed limit system (73). In this sense we shall call the problem (L) the limit model of the kinetic equation $(K)_\varepsilon$ and shall introduce in the next section a numerical scheme able to capture this limit model for vanishing ε.

5. AP-discretization of the electron kinetic model

The aim of this section is to suggest numerical schemes for the Vlasov-BGK system (44) that give the correct solution (up to discretization errors) when $\varepsilon = \mathcal{O}(1)$ and which yield the asymptotic solution as $\varepsilon \to 0$, i.e. the solution of system (73). In particular, it should be possible to choose the time step Δt independently of ε, thereby gaining an advantage over standard explicit schemes. With this ambition, we shall first present semi-discretizations in time of (59) and (71), respectively, which is sufficient to show the AP character of the schemes. In a second step we perform the space discretization using finite difference methods.

5.1. **Time-discretization of the scheme $(KF)_\varepsilon$.** For notational simplicity let us omit now the ε-indices on the unknowns. We fix the time step to $\Delta t > 0$ and denote by $n^k, u^k, g^k, \mathcal{M}^k$ the approximations of $n(t^k,\cdot)$, $u(t^k,\cdot)$, $g(t^k,\cdot,\cdot)$, $\mathcal{M}(f(t^k,\cdot,\cdot))$ at time $t^k = k\,\Delta t$, $k = 0, \cdots, K$ with $K \in \mathbb{N}$. Then a possible first order time-discretization of (59) reads

$$(KF)_{\varepsilon,\Delta t}\begin{cases} \varepsilon\,\dfrac{g^{k+1}-g^k}{\Delta t} + \sqrt{\varepsilon}\,v\,\partial_x g^k - \sqrt{\varepsilon}E^k\partial_v g^k - \sqrt{\varepsilon}\,\dfrac{(v-\sqrt{\varepsilon}u^k)}{n^k}\,\partial_x\left\langle v^2\,g^k\right\rangle\mathcal{M}^k \\ \qquad\qquad + \partial_x u^k\,\left[(v-\sqrt{\varepsilon}u^k)^2 - 1\right]\mathcal{M}^k = -g^{k+1}\,, \\ \dfrac{n^{k+1}-n^k}{\Delta t} + \partial_x(n^{k+1}u^{k+1}) = 0\,, \\ \dfrac{n^{k+1}\,u^{k+1} - n^k\,u^k}{\Delta t} + \partial_x\left(n^k(u^k)^2\right) + \dfrac{1}{\varepsilon}\partial_x n^{k+1} + \dfrac{1}{\varepsilon}E^k\,n^{k+1} + \partial_x\left\langle v^2\,g^{k+1}\right\rangle = 0\,. \end{cases} \tag{74}$$

The right-hand-side in the kinetic equation for g is implict, fact which permits the computation of g^{k+1} for all $\varepsilon \geq 0$. To see that this implicit-explicit choice in the g-equation leads

to an ε-independent CFL-condition, we detail in Appendix A the numerical stability study of a simplified but similar kinetic equation. The time-discretization of the macroscopic conservation laws is based on the following arguments:

a) the pressure-gradient and electrostatic force terms in the last equation are taken implicitly, as these terms are stiff,

b) the particle-flux term in the second equation is also taken implicitly, motivated by the fact that we want to recover in the limit $\varepsilon \to 0$ the electron Boltzmann relation, in turn obtained from the momentum equation. Hence, the particle conservation law will be employed to get some information about the velocity unknown u^{k+1}.

Remark here that one can improve this semi-discretization in order to treat better the regime $\varepsilon \sim 1$. We are referring here to schemes able to capture shocks if the perturbation parameter is of order unity and make reference to the work [25], where the stiff pressure term and the mass flux are splitted for hyperbolicity reasons. In the present work we shall concentrate more on the correct description of the adiabatic regime. In the following sections we present two AP-reformulations of system (74).

5.2. **First asymptotic-preserving scheme.** To avoid the $\varepsilon \to 0$ singularity in the momentum equation, we shall use in the semi-discretized system (74) the same decomposition for the density n^{k+1} as in the continuous case, based on the projection operator $\mathcal{P}_{\mathbb{L}}$. This yields the first semi-discrete AP-system

$$(AP)^1_{\varepsilon,\Delta t} \quad \begin{cases} \varepsilon\,\dfrac{g^{k+1}-g^k}{\Delta t} + \sqrt{\varepsilon}\,v\,\partial_x g^k - \sqrt{\varepsilon}E^k\partial_v g^k - \sqrt{\varepsilon}\,\dfrac{(v-\sqrt{\varepsilon}u^k)}{n^k}\,\partial_x\left\langle v^2\,g^k\right\rangle\mathcal{M}^k \\ \qquad\qquad + \partial_x u^k\,\left[(v-\sqrt{\varepsilon}u^k)^2-1\right]\mathcal{M}^k = -g^{k+1}\,, \\ \dfrac{n^{k+1}-n^k}{\Delta t} + \partial_x(n^{k+1}u^{k+1}) = 0\,, \\ \dfrac{n^{k+1}\,u^{k+1}-n^k\,u^k}{\Delta t} + \partial_x\left(n^k(u^k)^2\right) + \partial_x s^{k+1} + E^k\,s^{k+1} + \partial_x\left\langle v^2\,g^{k+1}\right\rangle = 0\,, \\ \partial_x n^{k+1} + E^k\,n^{k+1} = \varepsilon\left(\partial_x s^{k+1} + E^k\,s^{k+1}\right)\,, \qquad \mathcal{P}_{\mathbb{L}}(s^{k+1}) = 0\,. \end{cases} \tag{75}$$

It can be easily shown that in the limit $\varepsilon \to 0$ and for fixed time discretization parameter $\Delta t > 0$, this system leads to a semi-discrete version of (L). It is this particular property which is the main advantage of our AP-reformulation, as compared to standard time-discretizations of the kinetic equation (44). However, a weakness of system (75) is that the three fluid equations are fully coupled, leading to a large linear system to be solved. To avoid this new difficulty, a second AP-reformulation is proposed in the next section.

5.3. **Second asymptotic-preserving scheme.** In order to decouple somehow the system (75), we start from (74) and inject the density conservation law of (74) into the momentum conservation law, where for simplicity we denote the particle momentum by $q^{k+1} := n^{k+1}\, u^{k+1}$. This yields

$$\varepsilon q^{k+1} - (\Delta t)^2 \partial_{xx} q^{k+1} - (\Delta t)^2\, E^k\, \partial_x q^{k+1} = \varepsilon q^k - \varepsilon \Delta t\, \partial_x \left(\frac{(q^k)^2}{n^k} \right) - \varepsilon \Delta t\, \partial_x \langle v^2\, g^{k+1} \rangle \\ - \Delta t\, \partial_x n^k - \Delta t\, E^k\, n^k\,. \tag{76}$$

We thus transformed the momentum conservation law into an elliptic equation for q^{k+1}, which degenerates in the limit $\varepsilon \to 0$ (due to the periodic boundary conditions) and has hence to be treated with care. Such type of singularly-perturbed elliptic or diffusion equations have been the object of several works [14, 15] and were handled via various techniques. The duality-based approach we shall follow here is based on the micro-macro decomposition (67) applied to the quantity q^{k+1}, meaning

$$q^{k+1} = \eta^{k+1} + \xi^{k+1}\,, \qquad \eta^{k+1} := \mathcal{P}_{\mathbb{L}}(q^{k+1})\,, \qquad \mathbb{L}(\xi^{k+1}) = \mathbb{L}(q^{k+1})\,, \tag{77}$$

and consists in projecting (76) on the kernel $\mathcal{G}_{\mathbb{L}}$ to have an equation for the "macroscopic" quantity η^ε,

$$\varepsilon \eta^{k+1} = \varepsilon \eta^k - \varepsilon \Delta t\, \mathcal{P}_{\mathbb{L}} \left(\partial_x \left[\frac{(q^k)^2}{n^k} \right] \right) - \varepsilon \Delta t\, \mathcal{P}_{\mathbb{L}} (\partial_x \langle v^2\, g^{k+1} \rangle)\,, \tag{78}$$

where we used that $\mathcal{P}_{\mathbb{L}}(\mathbb{L}(\cdot)) = 0$. Furthermore, inserting the ansatz (77) into (76) yields an equation for the "microscopic" quantity ξ^ε,

$$\varepsilon \xi^{k+1} - (\Delta t)^2 \partial_{xx} \xi^{k+1} - (\Delta t)^2\, E^{k+1}\, \partial_x \xi^{k+1} = \varepsilon q^k - \varepsilon \Delta t\, \partial_x \left(\frac{(q^k)^2}{n^k} \right) - \varepsilon \Delta t\, \partial_x \langle v^2\, g^{k+1} \rangle \\ - \varepsilon \eta^{k+1} + (\Delta t)^2 \partial_{xx} \eta^{k+1} + (\Delta t)^2\, E^{k+1}\, \partial_x \eta^{k+1} - \Delta t\, \partial_x n^k - \Delta t\, E^k\, n^k\,. \tag{79}$$

We obtain thus a second (semi-discretized) AP-reformulation

$$
(AP)^2_{\varepsilon,\Delta t}\quad \begin{cases} \varepsilon\,\dfrac{g^{k+1}-g^k}{\Delta t}+\sqrt{\varepsilon}\,v\,\partial_x g^k-\sqrt{\varepsilon}E^k\partial_v g^k-\sqrt{\varepsilon}\,\dfrac{(v-\sqrt{\varepsilon}u^k)}{n^k}\,\partial_x\left\langle v^2\,g^k\right\rangle\mathcal{M}^k \\ \qquad\qquad +\partial_x u^k\,\left[(v-\sqrt{\varepsilon}u^k)^2-1\right]\mathcal{M}^k=-g^{k+1}\,, \\ \dfrac{n^{k+1}-n^k}{\Delta t}+\partial_x q^{k+1}=0\,,\qquad q^{k+1}=\eta^{k+1}+\xi^{k+1}\,, \\ \eta^{k+1}=\eta^k-\Delta t\,\mathcal{P}_{\mathbb{L}}\left(\partial_x\left[\dfrac{(q^k)^2}{n^k}\right]\right)-\Delta t\,\mathcal{P}_{\mathbb{L}}(\partial_x\langle v^2\,g^{k+1}\rangle)\,, \\ \varepsilon\xi^{k+1}-(\Delta t)^2\partial_{xx}\xi^{k+1}-(\Delta t)^2\,E^{k+1}\,\partial_x\xi^{k+1}=\varepsilon q^k-\varepsilon\Delta t\,\partial_x\left(\dfrac{(q^k)^2}{n^k}\right) \\ \qquad -\varepsilon\eta^{k+1}+(\Delta t)^2\partial_{xx}\eta^{k+1}+(\Delta t)^2\,E^{k+1}\,\partial_x\eta^{k+1} \\ \qquad -\Delta t\,\partial_x n^k-\Delta t\,E^k\,n^k-\varepsilon\Delta t\,\partial_x\langle v^2\,g^{k+1}\rangle\,,\qquad \mathcal{P}_{\mathbb{L}}(\xi^{k+1})=0\,. \end{cases} \tag{80}
$$

The advantage of the second semi-discretized AP-reformulation (80) as compared to the first one (75) is the fact that the macroscopic equations (conservation laws) are now fully decoupled and can be solved sequentially ($g^{k+1}\to\eta^{k+1}\to\xi^{k+1}\to n^{k+1}$), which permits a considerable gain in computational time.

5.4. **Space discretization of $(AP)^1_{\varepsilon,\Delta t}$.** The electron kinetic equation (44) is defined for $(t,x,v)\in\mathbb{R}^+\times[0,L]\times\mathbb{R}$ with the boundary conditions (45). The velocity space will be truncated far from the origin, thus reduced to $[v_{min},v_{max}]\subset\mathbb{R}$ where $v_{min}<0<v_{max}$. We define a Cartesian position-velocity grid via

$$
\begin{aligned} &\Delta x=\frac{L}{N_x-1}\,,\qquad x_i:=(i-1)\Delta x\,,\qquad i\in\{1\dots N_x\}\,,\qquad N_x\in\mathbb{N}\,, \\ &\Delta v=\frac{v_{max}-v_{min}}{N_v-1}\,,\qquad v_j:=v_{min}+(j-1)\Delta v\,,\qquad j\in\{1\dots N_v\}\,,\qquad N_v\in\mathbb{N}\,, \end{aligned} \tag{81}
$$

and denote by $f_{i,j}:=f(x_i,v_j)$, and in general $a_i:=a(x_i)$ and $b_j:=b(v_j)$. Dirichlet boundary conditions are imposed in velocity space, $f_{i,1}=f_{i,N_v}=0\ \forall i$. We first present a discretization of the scheme (75). The micro-part of the kinetic equation, i.e. the equation for g, is discretized with a simple upwind scheme. Hence we set

$$
\begin{aligned} &(v\partial_x g^k)_{i,j}\approx(v_-)_j\frac{g_{i+1,j}-g_{i,j}}{\Delta x}+(v_+)_j\frac{g_{i,j}-g_{i-1,j}}{\Delta x}\,, \\ &(v_-)_j=\min(0,v_j)\,,\qquad (v_+)_j=\max(0,v_j)\,, \end{aligned} \tag{82}
$$

and

$$
\begin{aligned}
(-E^k \partial_v g^k)_{i,j} &\approx (F_-)_i \frac{g_{i,j+1} - g_{i,j}}{\Delta v} + (F_+)_i \frac{g_{i,j} - g_{i,j-1}}{\Delta v}\,, \\
(F_-)_i &= \min(0, -E_i^k)\,, \qquad (F_+)_i = \max(0, -E_i^k)\,.
\end{aligned}
\tag{83}
$$

The second velocity moment of g^k is approximated via the trapezoidal rule,

$$
\Theta_i^k := \langle v^2 g^k \rangle_i \approx \frac{\Delta v}{2} \sum_{j=1}^{N_v - 1} v_j \, g_{i,j}^k + \frac{\Delta v}{2} \sum_{j=2}^{N_v} v_j \, g_{i,j}^k\,. \tag{84}
$$

Centered finite difference approximations are used for the remaining derivatives in the g-equation,

$$
(\partial_x \Theta^k)_i \approx \frac{\Theta_{i+1}^k - \Theta_{i-1}^k}{2\Delta x}\,, \qquad (\partial_x u^k)_i \approx \frac{u_{i+1}^k - u_{i-1}^k}{2\Delta x}\,. \tag{85}
$$

Moreover, we recall the electron Maxwellian,

$$
\mathcal{M}_{i,j}^k = \frac{n_i^k}{\sqrt{2\pi}} \exp\left(-\frac{(v_j - \sqrt{\varepsilon} u_i^k)^2}{2} \right)\,. \tag{86}
$$

The tricky part is the discretization of the macroscopic conservation laws in (75). First, the integral constraint $\mathcal{P}_{\mathbb{L}}(s^{k+1}) = 0$ is implementd via a Lagrange multiplier technique, i.e. we add an unknown constant $\lambda \in \mathbb{R}$ to the system,

$$
\partial_x n^{k+1} + E^k \, n^{k+1} = \varepsilon \left(\partial_x s^{k+1} + E^k \, s^{k+1} \right) + \lambda\,. \tag{87}
$$

The additional unknown allows us to add to the linear system the equation $\mathcal{P}_{\mathbb{L}}(s^{k+1}) = 0$, in the form of

$$
\Delta x \sum_{i=1}^{N_x - 1} s_i^{k+1} e^{-\phi_i^k} = 0\,. \tag{88}
$$

Applying the projection $\mathcal{P}_{\mathbb{L}}$ to equation (87) shows that $\lambda = 0$; this means that the introduction of the Lagrange multiplier does not change the solution of system (75). Secondly, let us write the particle flux as $q_i^k := n_i^k u_i^k$. A convenient way to approximate the first-order derivatives in the fluid equations is to use centered finite differences at the half-points $i + 1/2$. The macroscopic fluid part of (75) is thus approximated as

$$
\begin{cases}
\dfrac{n_{i+1}^{k+1} - n_{i+1}^k}{2\Delta t} + \dfrac{n_i^{k+1} - n_i^k}{2\Delta t} + \dfrac{(q_{i+1}^{k+1} - q_i^{k+1})}{\Delta x} + \dfrac{\mathcal{F}_{i+1}^k - \mathcal{F}_i^k}{\Delta x} = 0\,, \\[2ex]
\dfrac{q_{i+1}^{k+1} - q_{i+1}^k}{2\Delta t} + \dfrac{q_i^{k+1} - q_i^k}{2\Delta t} + \dfrac{s_{i+1}^{k+1} - s_i^{k+1}}{\Delta x} + \dfrac{E_{i+1}^k s_{i+1}^{k+1} + E_i^k \, s_i^{k+1}}{2} + \dfrac{\mathcal{G}_{i+1}^k - \mathcal{G}_i^k}{\Delta x} = 0\,, \\[2ex]
\dfrac{n_{i+1}^{k+1} - n_i^{k+1}}{\Delta x} + \dfrac{E_{i+1}^k n_{i+1}^{k+1} + E_i^k \, n_i^{k+1}}{2} = \varepsilon \left(\dfrac{s_{i+1}^{k+1} - s_i^{k+1}}{\Delta x} + \dfrac{E_{i+1}^k s_{i+1}^{k+1} + E_i^k \, s_i^{k+1}}{2} \right) + \lambda\,.
\end{cases}
\tag{89}
$$

Here, $\mathcal{F}_i^k$ and $\mathcal{G}_i^k$ stand for the particle and momentum fluxes with artificial viscosity of Rusanov type [33], given at the grid point i by

$$\begin{aligned}
\mathcal{F}_i^k &:= -\frac{a_i}{4}(n_{i+1}^k - n_{i-1}^k)\,, \\
\mathcal{G}_i^k &:= \frac{1}{4}\left[\Big(\frac{q^2}{n}\Big)_{i+1}^k + 2\Big(\frac{q^2}{n}\Big)_i^k + \Big(\frac{q^2}{n}\Big)_{i-1}^k\right] + \frac{1}{4}(\Theta_{i+1}^k + 2\Theta_i^k + \Theta_{i-1}^k) - \frac{a_i}{4}(q_{i+1}^k - q_{i-1}^k)\,,
\end{aligned} \tag{90}$$

where $a_i = \max(|\frac{q_i^k}{n_i^k} + 1|, |\frac{q_i^k}{n_i^k} - 1|)$. The scheme with (82)-(90) is stable if the following CFL-condition is satisfied (c.f. Appendix A):

$$\Delta t < CFL * \min(\Delta t_F, \Delta t_K)\,, \tag{91a}$$

where $CFL < 1$ and

$$\begin{aligned}
\Delta t_F &:= \frac{\Delta x}{\max_i(a_i)} \\
\Delta t_K &:= \begin{cases} \Delta t_F & \text{if} \quad \sqrt{\varepsilon}\,\gamma^k - 1 \le 0\ , \\ \dfrac{\varepsilon}{\sqrt{\varepsilon}\gamma^k - 1} & \text{if} \quad \sqrt{\varepsilon}\gamma^k - 1 > 0\,. \end{cases} \qquad \gamma^k := \sqrt{2\left(\frac{\max_j(v_j^2)}{(\Delta x)^2} + \frac{\max_i((E_i^k)^2)}{(\Delta v)^2}\right)}
\end{aligned} \tag{91b}$$

Clearly, for $\varepsilon \to 0$ and Δx and Δv fixed, the number $\sqrt{\varepsilon}\gamma^k - 1$ is negative, such that the CFL-condition is independent of ε in the adiabatic regime and corresponds to a fluid CFL condition.

5.5. **Space discretization of $(AP)^2_{\varepsilon,\Delta t}$.** The kinetic g-equation in the second AP-scheme (80) is discretized in exactly the same way as for the first AP-scheme, i.e. by means of Eqs. (82)-(86). In the macroscopic equations the space derivatives are this time standard centered finite differences at the mesh points, for example

$$(\partial_x \eta^k)_i \approx \frac{\eta_{i+1}^k - \eta_{i-1}^k}{2\Delta x}\,, \qquad (\partial_{xx}\eta^k)_i \approx \frac{\eta_{i+1}^k - 2\eta_i^k + \eta_{i-1}^k}{\Delta x^2}\,. \tag{92}$$

The projection $\mathcal{P}_{\mathbb{L}}(\cdot)$ is discretized as

$$[\mathcal{P}_{\mathbb{L}}(b)]_i = \frac{e^{\phi_i^k}\Delta x}{L}\sum_{l=1}^{N_x-1} b_l\, e^{-\phi_l^k}\,. \tag{93}$$

The integral constraint $\mathcal{P}_{\mathbb{L}}(\xi^{k+1}) = 0$ is again implementd via a Lagrange multiplier $\lambda \in \mathbb{R}$, c.f. eqs. (87)-(88), which is added in the equation for ξ^{k+1}. Artificial fluxes of Rusanov type are added in the particle conservation law and in the equation for ξ^{k+1} (the Rusanov flux being of order ε in the latter case). In contrast to (90) the fluxes are now defined at

the half points $i+1/2$, i.e.

$$\mathcal{F}_{i+1/2} := -\frac{a_{i+1/2}}{2}(n_{i+1}^k - n_i^k)\,, \tag{94}$$

$$\mathcal{G}_{i+1/2} := \frac{\varepsilon}{2}\left[\left(\frac{q^2}{n}\right)_{i+1}^k + \left(\frac{q^2}{n}\right)_i^k\right] + \frac{\varepsilon}{2}(\Theta_{i+1}^k + \Theta_i^k) - \frac{\varepsilon\, a_{i+1/2}}{2}(q_{i+1}^k - q_i^k)\,, \tag{95}$$

where $a_{i+1/2} = \max(a_i, a_{i+1})$. The time step is again restricted by the CFL-condition (91) as for the first AP-scheme.

6. Numerical results

The aim of this section is to study the efficiency of the proposed AP-schemes, in particular to demonstrate numerically their asymptotic-preserving property as $\varepsilon \to 0$. Numerical tests will be performed on a domain with $L = 1$, $v_{min} = -5$ and $v_{max} = 5$ in the fixed time interval $[0, T]$ with $T = 0.1$. We assume a given electrostatic potential, independent of time,

$$\phi(x) = \cos(2\pi x) \qquad \Longrightarrow \quad E^k(x) = -2\pi \sin(2\pi x) \quad \forall k \in \mathbb{N}\,. \tag{96}$$

The initial distribution function f_0 for all simulations is a Maxwellian with zero mean velocity,

$$f_0(x, v) = \frac{n_0(x)}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right)\,, \tag{97}$$

with

$$n_0(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-0.5)^2}{2\sigma^2}\right)\,, \tag{98}$$

and $\sigma = 0.05$. On the time scale ε, we expect the density to approach the Boltzmann relation

$$n(t/\varepsilon \to \infty, x) = c\, e^{\phi(x)}\,, \qquad c = \frac{\int_0^1 n_0\, dx}{\int_0^1 e^{\phi}\, dx}\,. \tag{99}$$

To get a basic understanding of the test case at hand, we plot in Figure 1 a solution obtained with the first AP-scheme (75) for $\varepsilon = 10^{-2}$. If not stated otherwise, we used $N_x = 51$ and $N_v = 251$ mesh points in simulations. From the middle column one clearly observes the convergence in time of the particle density towards the Boltzmann relation (99). The electric field is responsible for the asymmetric acceleration of the electrons, namely to the left for $x < 0$ and to the right for $x > 0$.

For the purpose of validating the new AP-schemes, we implement an explicit upwind scheme for the kinetic equation (44). Numerical solutions obtained with this scheme shall serve as a reference in the regime $\varepsilon \geq 10^{-6}$. For smaller ε-values the time step in the explicit scheme is heavily restricted and the scheme becomes inefficient. Figure 2 depicts the convergence over time towards the Boltzmann relation (99) for solutions obtained with

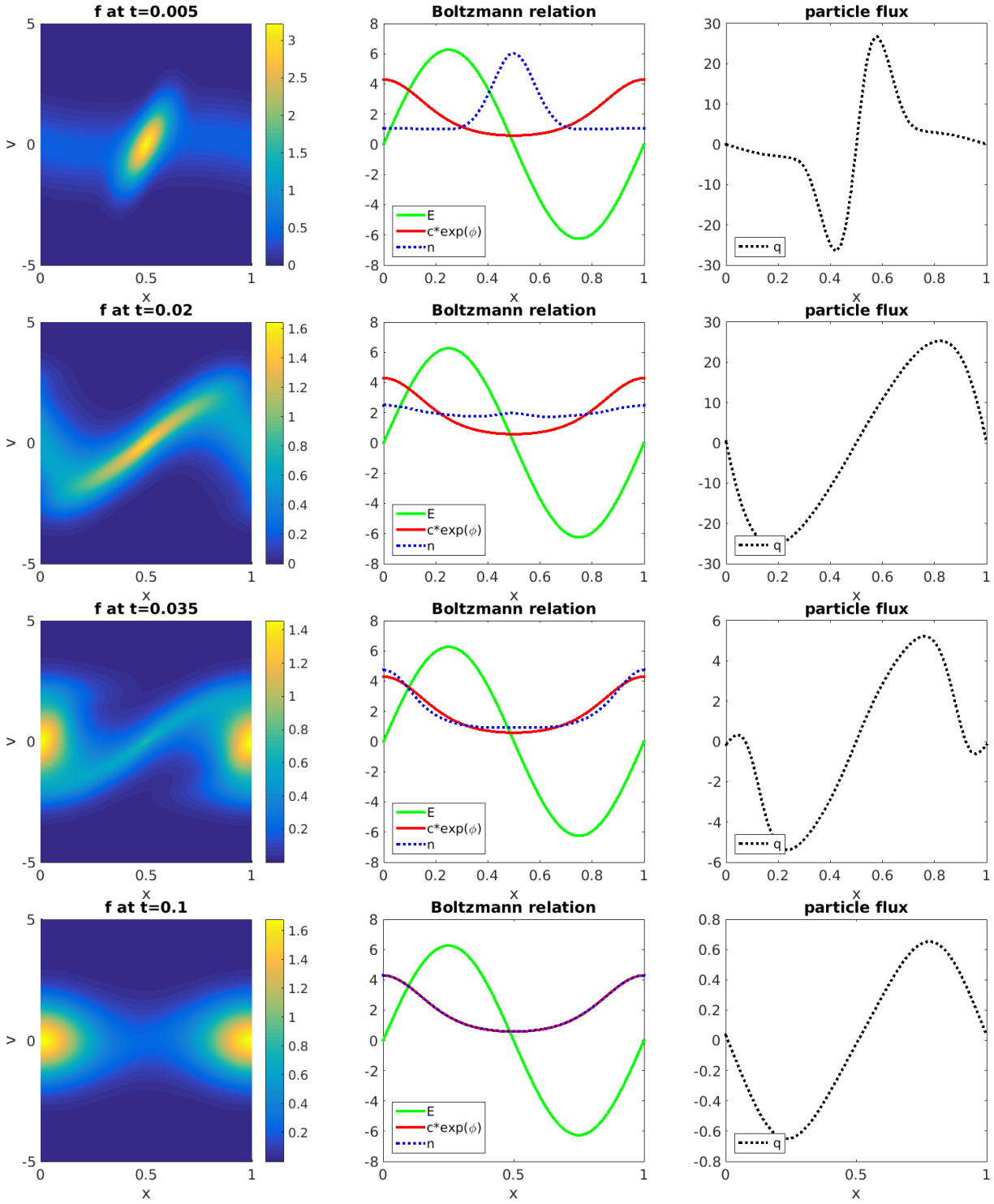

FIGURE 1. Snapshots of the distribution function f at four instances in time, obtained with the first AP-scheme (75) for $\varepsilon = 10^{-2}$. The middle column depicts the correponding particle density n, compared to the Boltzmann relation (99), and the right column contains the particle flux $q = nu$.

three different schemes: the AP1-scheme (75), the AP2-scheme (80) and the explicit upwind Vlasov-BGK solver (solution 'REF'). The value of ε was 10^{-2}, such that the explicit solver could be run with $N_x = 201$ and $N_v = 1001$. This result shows that the two AP-schemes yield meaningful solutions in the (relatively) large-ε regime.

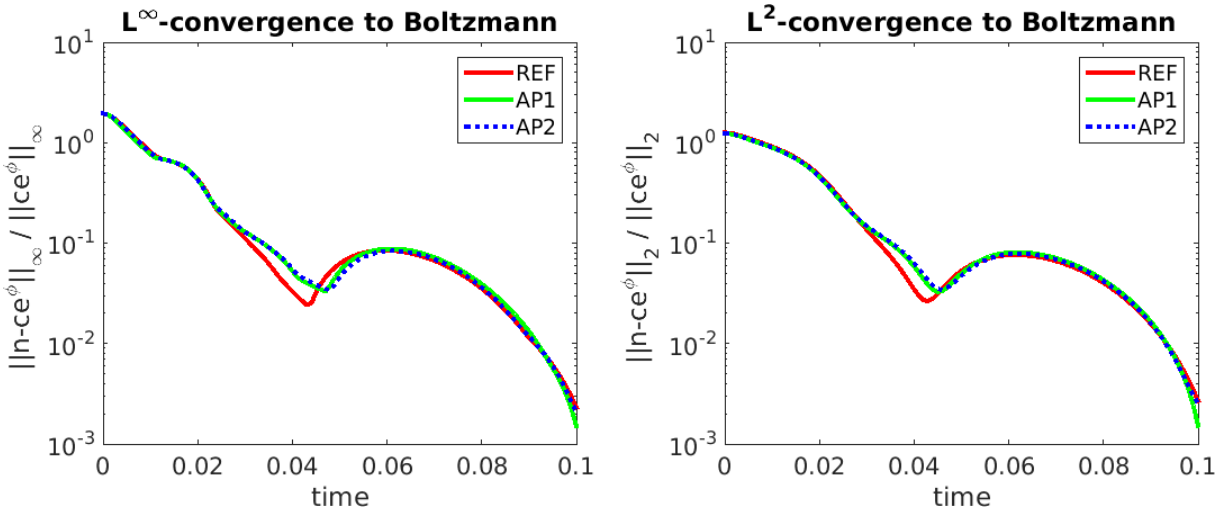

FIGURE 2. Convergence over time towards the Boltzmann relation (99) for solutions from the two AP-schemes and a reference solution 'REF', for $\varepsilon = 10^{-2}$. The reference solution has been obtained from an explicit upwind Vlasov-BGK solver on a finer mesh with $N_x = 201$ and $N_v = 1001$.

What happens when the value of ε is decreased? Figure 3 shows the convergence towards the Boltzmann relation for different values $10^{-6} \leq \varepsilon \leq 10^{-1}$, obtained with the two schemes AP1 and AP2. We observe an oscillatory convergence, with a frequency that increases as ε becomes smaller. The convergence is also faster for smaller ε, fact that shows that ε is related to the chosen time scale via the Mach number. For $\varepsilon = 10^{-6}$, the initial state almost instantaneously jumps into the Boltzmann state (boundary layer at $t = 0$). Note also that the scheme AP2 seems to be more accurate than the scheme AP1, which is seen by comparing the minimal errors reached with each scheme.

Our next objective is to show the good asymptotic properties of the AP-schemes. For this we plot in Figure 4 the steady state solutions of the distribution function f at time $t = 0.1$ along with its macroscopic moments, obtained with the AP-schemes and with the explicit upwind solver. Good agreement between the three schemes is obtained for $f(t = 0.1)$ and for the electron density $n(t = 0.1)$, which is in the Boltzmann state. However, the electron flux $q(t = 0.1)$ differs from the explicit solution to the AP solutions. In fact, a computation of the electron flux is not needed in the explicit Vlasov solver. Rather, once f^k is known,

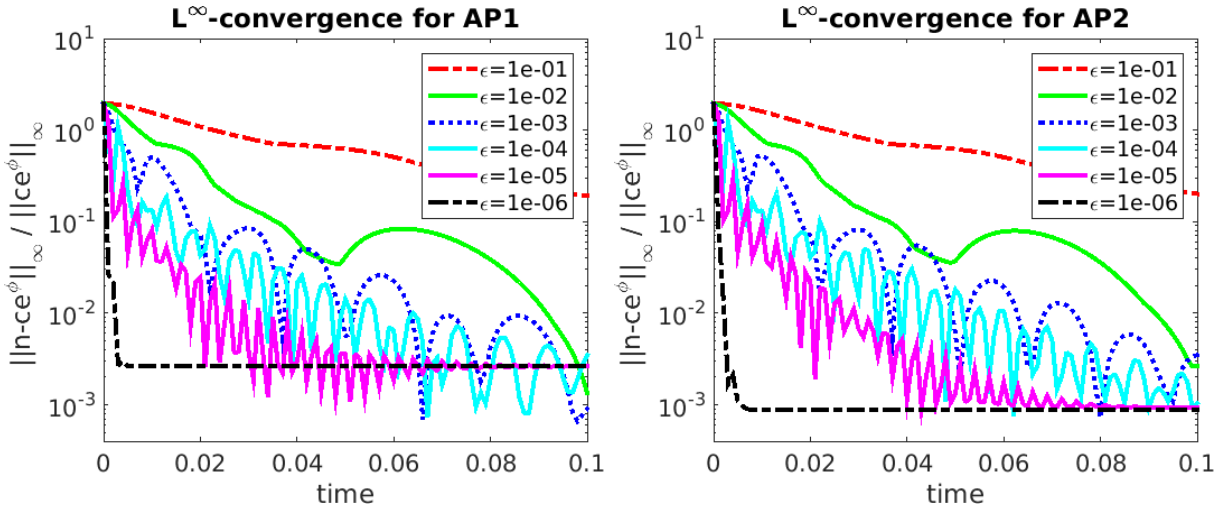

FIGURE 3. Convergence over time towards the Boltzmann relation (99) for solutions from the two AP-schemes for $10^{-6} \leq \varepsilon \leq 10^{-1}$.

one computes

$$q^k = \frac{1}{\sqrt{\varepsilon}} \int v f^k dv \,. \tag{100}$$

Here, small numerical errors in f^k are divided by $\sqrt{\varepsilon}$ and thus (for $\varepsilon \ll 1$) become large errors in q^k (note also that one introduces additional discretization errors due to approximation of the velocity integral, which are amplified as well). We conclude that for small ε, the calculation of q^k from the Vlasov-solution f^k is not viable. The AP-schemes, in which q is an unknown, yield the correct solution in the asymptotic limit $\varepsilon \to 0$.

Let us give further evidence that the standard explicit solver "breaks down" for small values of ε. In Table 1 we compare, in a quantitative manner, the explicit upwind scheme with the two AP-schemes as $\varepsilon \to 0$. Several remarks are to be made:

- Considering the L^∞- and L^2-distances with respect to the Boltzmann relation, we already know from Figures 2 and 3 that the system is very close to the adiabatic electron state at $t = 0.1$ for $\varepsilon \leq 10^{-2}$, which is confirmed here for $\varepsilon \to 0$. The scheme AP2 shows the best accuracy in this regime.

- Regarding CPU-time, the AP-schemes become more efficient than the explicit solver for $\varepsilon \leq 10^{-5}$. Remark in particular the sudden decrease in CPU-time of the AP-schemes for $\varepsilon = 10^{-6}$; at this threshold the kinetic CFL condition is relaxed, c.f. (91), because $\sqrt{\varepsilon}\,\gamma^k - 1$ becomes negative, and the fluid CFL condition is applied. The performance of the AP-schemes can be further improved by via a more implicit discretization of the g-equation, in order to avoid the strict kinetic CFL condition

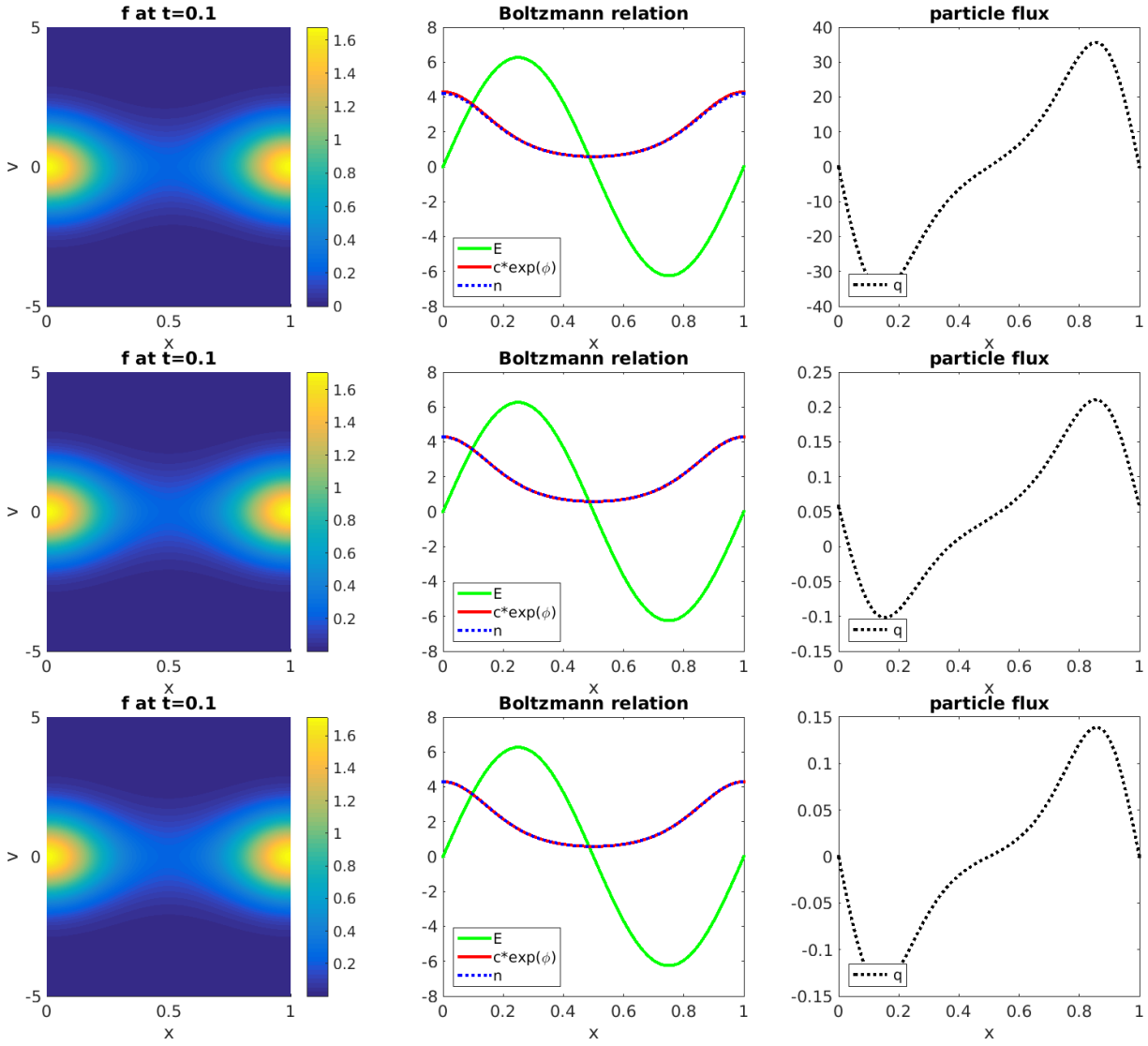

FIGURE 4. Steady-states reached at $t = 0.1$ for $\varepsilon = 10^{-5}$ with the explicit upwind scheme (first line), the AP1-scheme (second line) and the AP2-scheme (third line). Note the good agreement of the distribution functions $f(t = 0.1)$ and the erroneous result for the electron flux q obtained with the explicit upwind scheme.

when $\sqrt{\varepsilon}\,\gamma^k - 1 \sim \mathcal{O}(1)$.

- Another important point is the behaviour of the condition numbers of the system matrices in the two AP-schemes. The fact that they do not explode as $\varepsilon \to 0$ is the typical characteristic of the asymptotic-preserving property. The numerical

singularity has been avoided via the a priori problem reformulation in the two AP-schemes.

- Finally, remark the increase of the L^∞- as well as L^2-errors for $\varepsilon = 10^{-5}$ in the explicit upwind scheme, which shows the break-down of the explicit solver for small ε. This is a hint of the fact that the formal limit $\varepsilon \to 0$ in the kinetic equation (44) leads to an ill-posed problem.

To summarize, the numerical tests we performed here demonstrated the asymptotic preserving properties of our schemes, in particular an ε-independent condition number, an ε-independent CFL-condition and the "ability" to recover the Boltzmann relation in the limit $\varepsilon \to 0$, for fixed discretization parameters Δt, Δx, Δv.

TABLE 1. Comparing the AP-schemes with the explicit upwind solver: L^∞- and L^2-error with respect to the Boltzmann relation (99) at $t = 0.1$, largest condition number over time of the system matrix (only for the AP-schemes), L^∞-norm of the particle flow q^ε at the time $t = 0.1$ and normalized CPU-time.

ε	L^∞-error	L^2-error	$\|\|q^\varepsilon(t=0.1)\|\|_\infty$	cond.nr.$(*10^7)$	CPU-time
			expl.-upwind		
10^{-0}	0.8865	0.9072	2.30	-	1.00
10^{-1}	0.2129	0.2269	2.13	-	3.22
10^{-2}	0.0217	0.0244	1.36	-	7.92
10^{-3}	0.0096	0.0105	3.62	-	25.41
10^{-4}	0.0087	0.0079	11.38	-	83.77
10^{-5}	0.0208	0.0191	35.64	-	255.17
			AP1		
10^{-0}	0.7553	0.8819	2.25	0.0002	2.02
10^{-1}	0.1916	0.1745	2.06	0.0007	5.35
10^{-2}	0.0013	0.0013	0.65	0.0148	15.80
10^{-3}	0.0009	0.0010	0.21	0.1000	46.04
10^{-4}	0.0035	0.0028	0.17	0.7918	118.15
10^{-5}	0.0027	0.0021	0.21	2.2951	110.18
10^{-6}	0.0026	0.0020	0.16	5.6453	2.14
10^{-7}	0.0026	0.0020	0.15	4.5821	1.87
10^{-8}	0.0026	0.0020	0.15	2.2940	1.83
$10^{-\infty}$	0.0026	0.0020	0.15	2.2940	1.82
			AP2		
10^{-0}	0.7750	0.8841	2.22	0.0001	1.75
10^{-1}	0.2010	0.1807	2.10	0.0001	5.04
10^{-2}	0.0026	0.0027	0.66	0.0001	15.07
10^{-3}	0.0035	0.0027	0.21	0.0008	41.95
10^{-4}	0.0010	0.0012	0.19	0.0084	107.76
10^{-5}	0.0009	0.0010	0.14	0.0843	110.54
10^{-6}	0.0009	0.0009	0.15	0.8235	2.10
10^{-7}	0.0009	0.0009	0.15	7.5709	1.85
10^{-8}	0.0009	0.0009	0.15	1.1497	1.84
$10^{-\infty}$	0.0009	0.0009	0.15	0.0842	1.77

7. Conclusion

The aim of this work was first to identify, via an appropriate scaling, some macroscopic models for electrons and ions arising in plasma physics, obtaining hence a hybrid model describing tokamak plasmas with disparate masses. In a second part we constructed two asymptotic-preserving schemes for the uniform numerical treatment of the transition from the kinetic to the adiabatic electron regime (Boltzmann electrons) along magnetic field lines. The small parameter ε responsible for this transition represents the mass ratio m_e/m_i and the Knudsen number. A suitable reformulation of the original kinetic problem, based on micro-macro techniques, permitted to overcome the difficulties induced by stiff terms, which make standard schemes break down. Numerical examples that demonstrate the efficiency and applicability of the here proposed AP-schemes were presented.
A natural extension of this work is use the here introduced numerical ideas married with those of our previous work [36] in order to treat a physically more involved problem, three-dimensional and considering also the temperature and the magnetic field B. This work is currently under investigation.

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Appendix A. Analysis of the CFL-condition on the micro-equation

To understand by some means why the time-discretizations chosen in the kinetic equation of the AP-reformulations (75) respectively (80) preserve the AP-property, in particular why the time step Δt can be chosen independently on the ε-parameter, let us study in this subsection the simplified kinetic equation

$$\partial_t g + \frac{1}{\sqrt{\varepsilon}} v\, \partial_x g - \frac{1}{\sqrt{\varepsilon}} E\, \partial_v g = -\frac{1}{\varepsilon} g\,, \tag{101}$$

and its semi-discretization in time

$$\frac{g^{k+1}-g^k}{\Delta t} + \frac{1}{\sqrt{\varepsilon}}\, v\, \partial_x g^k - \frac{1}{\sqrt{\varepsilon}}\, E^k\, \partial_v\, g^k = -\frac{1}{\varepsilon} g^{k+1}. \tag{102}$$

Concerning the space discretization, a simple Lax-Friedrich's scheme is sufficient for our purpose, *i.e.*

$$\begin{aligned} g^{k+1}_{\alpha,\beta} &= \frac{1}{4}\left[g^k_{\alpha+1,\beta} + g^k_{\alpha-1,\beta} + g^k_{\alpha,\beta+1} + g^k_{\alpha,\beta-1}\right] - \frac{\Delta t\, v_\beta}{2\sqrt{\varepsilon}\,\Delta x}\left(g^k_{\alpha+1,\beta} - g^k_{\alpha-1,\beta}\right) \\ &\quad + \frac{\Delta t\, E^k_\alpha}{2\sqrt{\varepsilon}\,\Delta v}\left(g^k_{\alpha,\beta+1} - g^k_{\alpha,\beta-1}\right) - \frac{\Delta t}{\varepsilon}\, g^{k+1}_{\alpha,\beta}\,, \quad \alpha,\beta \in \mathbb{Z}^2\,, \end{aligned}$$

which can be also put into the form

$$\begin{aligned} \left(1+\frac{\Delta t}{\varepsilon}\right) g^{k+1}_{\alpha,\beta} &= \frac{1}{4}\left[g^k_{\alpha+1,\beta} + g^k_{\alpha-1,\beta} + g^k_{\alpha,\beta+1} + g^k_{\alpha,\beta-1}\right] - \frac{\Delta t\, v_\beta}{2\sqrt{\varepsilon}\,\Delta x}\left(g^k_{\alpha+1,\beta} - g^k_{\alpha-1,\beta}\right) \\ &\quad + \frac{\Delta t\, E^k_\alpha}{2\sqrt{\varepsilon}\,\Delta v}\left(g^k_{\alpha,\beta+1} - g^k_{\alpha,\beta-1}\right)\,. \end{aligned} \tag{103}$$

The stability criterion of this scheme is obtained by using "Von Neumann method", which is based on Fourier analysis and amounts to showing that there exists, under certain conditions, a constant $0 < \nu < 1$ such that

$$||g^{k+1}||_2 \le \nu ||g^k||_2\,, \quad \text{with} \quad ||g^k||_2 := \left(\Delta x\, \Delta v \sum_\alpha \sum_\beta |g^k_{\alpha,\beta}|^2\right)^{1/2}\,,$$

estimate which provides the stability of the scheme.
To obtain this stability condition, we insert the single grid wave-function

$$g^k_{\alpha,\beta} := e^{\mathbf{i} x_\alpha\, \xi}\, e^{\mathbf{i} v_\beta\, \eta} = e^{\mathbf{i}\,\alpha\,\Delta x\,\xi}\, e^{\mathbf{i}\,\beta\,\Delta v\,\eta}\,, \qquad (\xi,\eta) \in \mathbb{R}^2\,, \tag{104}$$

into the finite difference scheme (103) to obtain the recurrence relation

$$g^{k+1}_{\alpha,\beta} = a(\xi,\eta)\, g^k_{\alpha,\beta}\,, \qquad \forall k \in \mathbb{N}\,,\ \forall (\alpha,\beta) \in \mathbb{Z}^2\,,$$

where $a(\xi,\eta)$ is the so-called amplification factor. A scheme is called stable if $|a(\xi,\eta)| \le \nu < 1$ for all $(\xi,\eta) \in \mathbb{R}^2$. If this criterion is violated for some (ξ,η), then the Fourier

components with wave-number (ξ, η) will be amplified during the time-iterations, leading to an explosion of the scheme.

Inserting (104) into (103) yields the following expression for the amplification factor

$$a(\xi,\eta) = \frac{\varepsilon}{\varepsilon+\Delta t}\left\{\frac{1}{2}\left[\cos(\Delta x\,\xi)+\cos(\Delta v\,\eta)\right] - \mathbf{i}\frac{\Delta t\, v_\beta}{\Delta x\,\sqrt{\varepsilon}}\sin(\Delta x\,\xi) + \mathbf{i}\frac{\Delta t\, E^k_\alpha}{\Delta v\,\sqrt{\varepsilon}}\sin(\Delta v\,\eta)\right\}.$$

Thus, one obtains the estimate

$$|a(\xi,\eta)|^2 \;\leq\; \frac{\varepsilon}{2(\varepsilon+\Delta t)^2}\left[\varepsilon\cos^2(\Delta x\,\xi) + 4\nu^2_{x,\beta}\sin^2(\Delta x\,\xi) + \varepsilon\cos^2(\Delta v\,\eta) + 4\nu^2_{v,\alpha}\sin^2(\Delta v\,\eta)\right],$$

with

$$\nu_{x,\beta} := \frac{v_\beta\,\Delta t}{\Delta x}, \quad \nu_{v,\alpha} := \frac{E^k_\alpha\,\Delta t}{\Delta v}.$$

Denoting simply

$$\begin{aligned}
\nu_x &:= \tfrac{v_M\,\Delta t}{\Delta x}, \qquad v_M := \max\{|v_{min}|, |v_{max}|\},\\
\nu_v &:= \tfrac{E_M\,\Delta t}{\Delta v}, \qquad E_M := \max_{k\in\mathbb{N},\,\alpha\in\mathbb{Z}}\{|E^k_\alpha|\},
\end{aligned}$$

then one obtains

$$|a(\xi,\eta)|^2 \leq \frac{\varepsilon}{2(\varepsilon+\Delta t)^2}\left(\max\{\varepsilon, 4\,\nu_x^2\} + \max\{\varepsilon, 4\,\nu_v^2\}\right).$$

What can be observed from the estimate of $|a(\xi,\eta)|^2$ is that :

- in the adiabatic asymptotics $\varepsilon \to 0$, one has

  $$|a(\xi,\eta)|^2 \to_{\varepsilon\to 0} 0 \quad \text{for fixed } \Delta t > 0,$$

  signifying that no CFL-condition is required in this case;
- in the kinetic asymptotics $\varepsilon \to 1$, one gets

  $$|a(\xi,\eta)|^2 \leq \frac{1}{2(1+\Delta t)^2}\left(\max\{1, 4\,\nu_x^2\} + \max\{1, 4\,\nu_v^2\}\right),$$

  which signifies that one needs for the regime $\varepsilon \sim 1$ the usual CFL-condition

  $$\max\{|\nu_x|,\, |\nu_v|\} < \frac{1}{2} \quad \Rightarrow \Delta t < \frac{1}{2}\min\left\{\frac{\Delta x}{v_M}, \frac{\Delta v}{E_M}\right\},$$

  in order to have $|a(\xi,\eta)| < 1$;
- in the intermediate regime one obtains finally

  $$\begin{aligned}
  \theta(\varepsilon) &:= \frac{\varepsilon}{2(\varepsilon+\Delta t)^2}\left(\max\{\varepsilon, 4\,\nu_x^2\} + \max\{\varepsilon, 4\,\nu_v^2\}\right)\\
  &= \frac{\varepsilon}{2(\varepsilon/\Delta t+1)^2}\left(\max\{\frac{\varepsilon}{(\Delta t)^2}, 4\,\frac{v_M^2}{(\Delta x)^2}\} + \max\{\frac{\varepsilon}{(\Delta t)^2}, 4\,\frac{E_M^2}{(\Delta v)^2}\}\right),
  \end{aligned}$$

such that one has $|\theta(\varepsilon)| < 1$ if

$$\frac{4\varepsilon v_M^2/(\Delta x)^2}{(\varepsilon/\Delta t+1)^2}+\frac{4\varepsilon E_M^2/(\Delta v)^2}{(\varepsilon/\Delta t+1)^2}<2\,,$$

leading to the condition

$$\frac{\varepsilon}{\Delta t}>\sqrt{2\varepsilon\left(\frac{v_M^2}{(\Delta x)^2}+\frac{E_M^2}{(\Delta v)^2}\right)}-1=:\sqrt{\varepsilon}\,\gamma-1\,, \tag{105}$$

which induces exactly the choice of the time step given in (91).
It is worth remarking at this point that for $\varepsilon\sim\Delta t$, the condition (105) inquires

$$1>\sqrt{\Delta t}\gamma-1>0\quad\Rightarrow\quad 2\left[\frac{v_M^2}{(\Delta x)^2}+\frac{E_M^2}{(\Delta v)^2}\right]^{-1}>\Delta t>\frac{1}{2}\left[\frac{v_M^2}{(\Delta x)^2}+\frac{E_M^2}{(\Delta v)^2}\right]^{-1},$$

which is the most pessimistic CFL-condition one can get and which seems of "parabolic" type, coming essentially from the interplay between the transport operator and the collision operator.

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