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Growth estimates, control and structures in a two-field model of the scrape-off layer

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\title{Growth estimates, control and structures in a two-field model of the
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\begin{abstract}
{
Some properties of a reduced two-dimensional two-fluid (density plus vorticity)
model of the scrape-off layer (SOL) are studied analytically. The model
is built around a conservative system describing transport perpendicular to the
magnetic field in a slab geometry, to which
terms are added to account for diffusion and parallel losses and to mimic plasma
flow from the core.
Nonlinear estimates for growth rates are derived, showing the growth in the
density gradient to be controlled by
the vorticity gradient, and vice-versa, thus suggesting
a nonlinear instability in the model. Control of fluctuations by means of a
biasing potential is assessed, being shown negative polarisations are
more effective in decreasing plasma turbulence in the SOL,
hence explaining what is seen in experiments. Exact
solutions for the conservative part of the equations are obtained in the form of
travelling waves, which might be the conservative ancestors of the collective
structures (so-called blobs) observed both in experiments and in numerical
simulations.}
\end{abstract}


\section{Introduction}
Transport in the plasma scrape-off layer (SOL) is one of the main issues
pertaining to the operation of future fusion machines, the
physics of particle and energy flows in this boundary controlling both the
machine performance and the life expectancy of plasma-facing materials
[1,2]. %\cite{Stangeby,Naulin,Roth,Boedo}.
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Modelling of the SOL being extremely complex
%\cite{Zeiler,Garcia,Scott,Ricci1,Ricci2,Kendl}
[3], numerical codes have been developed
with recourse to reduced two-field models and conservation laws, which greatly
simplifies the analysis of SOL transport while retaining the fundamental
properties of the underlying physics
%\cite{Garbet,Philippe1,Philippe2,Figarella,Colin}
[4-6]. The broad picture emerging is that, although small-scale turbulence
dominates,
there may be large-scale coherent structures (associated with
intermittent events known as blobs) that are observed both in experiments and in
computations, and greatly contribute to the nature of transport in the SOL
%\cite{Naulin,Boedo,Garcia,Ricci2,Kendl,Philippe1,Philippe2,Figarella,Krasheninn
ikov,Furno,Bodi,Theiler,Maggs}%
[2,3,5-7]. Such blobs generate machine-scale flows that lead to thermal
charge asymmetries, impurity deposition and, through nonlinear interaction
with small-scale turbulence, change in the local transport properties.

Here one will approach these
questions through the study of a reduced two-dimensional (2D) fluid model
for plasma dynamics (electrons plus current) perpendicular to the
magnetic field, built from a conservative model [8] %\cite{Izacard}
(itself a cold-ion, reduced version of more complete four-field models
[9]) %\cite{Morrison,Hazeltine}
to which phenomenological terms  are added  accounting
for particle and current diffusion, parallel losses and a core-plasma source
[4-6].
%\cite{Philippe1,Philippe2,Figarella,Colin}
Emphasis will be on
obtaining nonlinear growth estimates for the physical quantities
and on discussing the eventual control of fluctuations by polarization [5,10].
%\cite{Philippe2,Figarella,Colin,Silva1,Silva2}
Focusing on the conservative
part of the equations, one will also aim at deriving exact solutions that
might be the conservative ancestors of the collective structures related
with blobs.

\section{The model}

\label{SecII}

The starting point is the two-field (density plus vorticity) model
\begin{align}
\partial Ln/\partial t&=-\left[ \phi ,Ln\right] -\left[ \phi ,1/B\right] +\left[
Ln,1/B\right]  \nonumber \\
\partial \nabla ^{2}\phi /\partial t&=-\left[ \phi ,\nabla ^{2}\phi %
\right] +\left[ Ln,1/B\right]    \label{Eq.1}
\end{align}
describing dynamics perpendicular to the magnetic field, which can be
extracted from a four-field model including finite-Larmor-radius effects,
drift-velocity ordering and gyroviscous terms [9]%\cite{Morrison}
, provided the
dynamics along the magnetic field (the so-called parallel dynamics) is
suppressed, the poloidal magnetic fluctuations are neglected and the
cold-ion limit is taken [8]. %\cite{Izacard}
Hereabove, $Ln\left(x_{1},x_{2},t\right)$ stands
for the logarithm of the normalised density $n\left( x_{1},x_{2},t\right) $,
$\phi \left(x_{1},x_{2},t\right)$ for the normalised electrostatic potential,
and $B(x_{1})$
for the normalised magnetic field, $x_{1}=\left( r-a\right) $ and $%
x_{2}=a\theta $ are the radial and poloidal coordinates, respectively, with $%
a$ the minor radius of the core plasma (space being rescaled by the ion
sonic Larmor radius and time by the ion cyclotron frequency), the canonical
Poisson
```

bracket $\left[f,h\right] $ is given by
%\begin{equation}
$\left[f,h\right] =\partial _{1}f\partial _{2}h-\partial _{2}f\partial _{1}h$,
%\label{Eq.3}
%\end{equation}%
and the del operator reads $\nabla=\left(\partial_{1},\partial_{2}\right)$, with
$\partial _{i}=\partial /\partial x_{i}$.

To this conservative model one adds, as is usually found in the literature
[5,6],
%\cite{Philippe1,Philippe2,Figarella,Colin}
diffusive terms, governed by the diffusion coefficients D for particles and $%
\nu $ for vorticity, as well as the parallel losses to the walls or
limiters, determined (from sheath physics [1,4]) %\cite{Stangeby,Garbet})
by the plasma
floating potential $\Lambda $ and some $\sigma _{||}$ measuring the
characteristic time for parallel transport, and also a source term $S(x_{1})$,
accounting for the plasma
flow from the core. Hence, and putting
$\tilde{g}=g=\partial _{1}\left(1/B\right) $,
(\ref{Eq.1}) becomes
\begin{align}
\partial Ln/\partial t&=-\left[\phi ,Ln\right] -\tilde{g}\partial
_{2}\left(Ln-\phi\right) +D\left(\nabla^{2} Ln+\left| \nabla
Ln\right|^{2}\right) -\sigma _{\Vert }e^{\left(\Lambda -\phi
\right)} + S   \nonumber\\
\partial \nabla^{2} \phi /\partial t&=-\left[\phi ,\nabla^{2} \phi \right]
-g\partial _{2}Ln +\nu \nabla ^{4}\phi +\sigma _{\Vert }\left[1-e^{\left(
\Lambda -\phi
\right)}
\right], \label{Eq.5}
\end{align}%
which accounts for particle balance (for electrons) and charge conservation,
respectively. The model in (\ref{Eq.5}) is very much the same as a 2D model
used for flux-driven uniform-temperature (so-called FDUT) simulations [6], and
also very similar to the
extensively studied model TOKAM-2D [4,5],
%\cite{Garbet,Philippe1,Philippe2,Figarella,Colin}
the difference with the latter lying in the
presence hereabove of the term associated with the magnetic field
inhomogeneity $\tilde{g}$ in the continuity equation, so all
results in this paper apply to TOKAM-2D by making $\tilde{g}=0$. The
compressibility term in (\ref{Eq.5}), namely, $\tilde{g}\partial
_{2}\left(Ln-\phi\right) $, being of the same order as the coupling term $%
g\partial _{2}Ln$ in the vorticity equation, one has decided to
keep it in the model.

\section{Nonlinear growth estimates}

\label{SecIII}

To estimate the dependence of the growth rate of
physical quantities, (\ref{Eq.5}) will be simplified by assuming the difference
$\phi \left(x_{1},x_{2},t\right)
-\Lambda $ between the electric and plasma floating potentials
is negligible, and that the parallel current losses are compensated
by the source $S\left(x_{1}\right)$ (a quasi-equilibrium hypothesis). Therefore,
one expects
the following equations to display the same qualitative behavior as those in
(\ref{Eq.5}):
\begin{align}
\partial Ln/\partial t&=-\left[\phi ,Ln\right] -\tilde{g}
\partial _{2}\left(Ln-\phi\right) +D\left(\nabla^{2} Ln+\left| \nabla
Ln\right|^{2}\right)   \nonumber \\

```
\partial \nabla^{2} \phi /\partial t&  =-\left[ \phi ,\nabla^{2} \phi \right]
-g\partial _{2}Ln+\nu \nabla ^{4}\phi   .\label{Eq.8}
\end{align}
Then, defining
$\nabla ^{\bot }=\left( \partial _{2},-\partial _{1}\right)$,
which is such that $\left[ f,h\right] = \nabla f \cdot \nabla^{\bot} g$,
(\ref{Eq.8}) becomes
\begin{align}
\partial Ln/\partial t&=-\nabla \phi \cdot \nabla ^{\bot }Ln-\tilde{g}\partial
_{2}\left(Ln-\phi\right)+D\left( \nabla^{2}
Ln+\left|\nabla Ln\right|^{2}\right)  \nonumber \\
\partial  \nabla^{2} \phi/\partial t
&=-\nabla \phi \cdot \nabla ^{\bot }\nabla^{2} \phi -g\partial _{2}Ln+\nu \nabla
^{4}\phi . \label{Eq.10}
\end{align}%

One now assumes the solution of (\ref{Eq.10}) to be defined in a domain $D$ with
either periodic boundary conditions or vanishing values for all functions at its
boundary, meaning, in practice, that
boundary terms can be dropped in all partial integrations.
%This simplifying assumption may not seem fully appropriate to the boundary
between the SOL and the core plasma (at $x_{1}=0$),
%yet any eventual non-vanishing boundary terms would not be dynamical, but fixed
instead, and hence one does not expect them to qualitatively change the
%inequalities obtained below.
The growth rates thus estimated are global as well as nonlinear, in the sense
that they proceed from the model equations without any previous linearisation of
the latter
(unlike in the approach followed below when addressing the control of
fluctuations).
So, multiplying the second of equations (\ref{Eq.10}) by $\phi $ on the left and
integrating over $D$ yields
\begin{equation}
\frac{1}{2}\frac{d  }{d t}\int_{D}\left| \nabla \phi \right|
^{2}=\int_{D}g\phi  \partial_{2}Ln -\nu \int_{D}\left( \nabla^{2} \phi \right)
^{2},\label{Eq.11}
\end{equation}%
whence the inequality
\begin{equation}
\frac{d }{d t}\int_{D}\left| \nabla \phi \right|
^{2}\leq \int_{D}\left| \nabla Ln\right| ^{2}+ \int_{D}\left(g\phi \right)^{2},
\label{Eq.12}
\end{equation}%
which means the growth of $|\nabla \phi |$ is partially
controlled by the gradient $|\nabla Ln|$ of the logarithm of the electron
density.
Proceeding to the first of equations (\ref{Eq.10}) to establish a bound on the
growth of $|\nabla Ln| $, one takes the gradient of this equation, makes the
inner product with $\nabla Ln$ and integrates by
parts on the domain $D$ to obtain
\begin{equation}
\frac{1}{2}\frac{d }{d t}\int_{D}\left|\nabla Ln\right|^{2}= \int_{D}\left(
\nabla \phi \cdot\nabla ^{\bot }Ln -D\left| \nabla Ln\right| ^{2}\right)
\nabla^{2}Ln +\int_{D}\tilde{g}\partial _{2}\left( Ln-\phi \right) \nabla^{2} Ln
- \int_{D}D\left( \nabla^{2} Ln\right) ^{2},\label{Eq.14}
\end{equation}%
so the following inequality follows
\begin{equation}
 \frac{d }{d t}\int_{D}\left| \nabla Ln\right| ^{2}\leq
3\int_{D}\left| \nabla \phi \right| ^{2}+\int_{D}\left(2+D\left|\nabla
Ln\right| ^{2}\right)\left|\nabla
Ln\right| ^{2}+\int_{D}\left( \tilde{g}^{2}
+D+\left\vert \nabla Ln\right\vert ^{2}\right) \left( \nabla^{2} Ln\right)
^{2} . \label{Eq.15}
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\end{equation}
The growth of $\left\vert \nabla Ln\right\vert $ is thus bounded by
$\left\vert \nabla \phi \right\vert $, whose growth is itself bounded
by $\left\vert \nabla Ln\right\vert $, suggesting a nonlinear instability (at
least for some domains of
the configuration space).

\section{Control of fluctuations} \label{SecIV}

A route for plasma control in the SOL
%(highly desirable for tokamak operation)
is via polarisation,
either of the wall (biased limiter or divertor elements) or using biased probes,
an approach that has been
tested both numerically [5] and experimentally [10].
%\cite{Silva1,Silva2}
%and by means of numerical simulations [5].
%\cite{Philippe2,Figarella,Colin}.
%Here, the model in (\ref{Eq.5}) will be used to understand how
%control of fluctuations might be achieved with the aid of biasing.
So, let $S(x_{1})$ and $V_{\rm{bias}}\left(x_{1},t\right) $ be such that
\begin{align}
Ln^{(0)}\left(x_{1}\right) &=Ln_{0}-\mu^{-1}x_{1} \nonumber \\
\phi ^{(0)}\left(x_{1},t\right) &=\Lambda +V_{\rm bias}\left(x_{1},t\right)
\label{Eq.17}
\end{align}
is a solution of (\ref{Eq.5}) with Λ replaced by $\Lambda+V_{\rm
bias}\left(x_{1},t\right)$, μ giving the scale length for the exponential
decay of density in the SOL.
One expects $V_{\rm bias}\left(x_{1},t\right) $ to be of the
form%
\begin{equation}
V_{\rm bias}\left(x_{1},t\right) =U\left(x_{1},t\right) e^{-\left(
Z-x_{1}\right) /\lambda _{D}} , \label{Eq.19}
\end{equation}%
Z being the radial width of the SOL, $\lambda _{D}$ the
Debye length, and $U\left(x_{1},t\right) $ a slowly varying function of $%
x_{1} $ and t. The intensity $U\left(x_{1},t\right) $ of the potential taken
at
the wall or at a probe would be a control parameter, whilst the decaying term in
$V_{\rm{bias}}\left(x_{1},t\right) $
takes into account the screening properties of the plasma away from $x_{1}=Z$.
If one now looks for fluctuations around the unperturbed solution (\ref{Eq.17}),
one may write up to first order,
\begin{align}
Ln\left(x_{1},x_{2},t \right) &= Ln ^{(0)}\left(x_{1}\right) +\delta
Ln\left(x_{1},x_{2},t \right) \nonumber \\
\phi\left(x_{1},x_{2},t \right) &= \phi ^{(0)}\left(x_{1},t\right) +\delta
\phi\left(x_{1},x_{2},t \right) , \label{Eq.21}
\end{align}
and subsequently plug (\ref{Eq.17}) and (\ref{Eq.21}) into (\ref{Eq.5}) to
obtain,
\begin{align}
\partial \delta Ln /\partial t& \simeq -\,\partial _{1}\phi ^{(0)}\partial
_{2}\delta Ln-\mu^{-1}\partial_{2}\delta \phi -\tilde{g}\partial _{2}\left(
\delta Ln-\delta \phi\right)
+\sigma
_{\Vert }\delta \phi +D\left(\nabla^{2} \delta Ln-2\mu^{-1}\partial_{1}\delta
Ln \right) \nonumber \\
\partial \nabla^{2} \delta \phi /\partial t &\simeq -\,\partial _{1}\phi
^{(0)}\partial _{2}\nabla^{2} \delta \phi +\partial _{1}^{3}\phi ^{(0)}\partial
_{2}\delta \phi-g\partial _{2}\delta Ln+ \sigma _{\Vert

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}\delta \phi
+\nu \nabla ^{4}\delta \phi.\label{Eq.23}
\end{align}
Fourirer transforming (\ref{Eq.23}), keeping only the lowest-order terms in
convolution products, setting $D\simeq \nu \simeq \sigma _{\Vert } \simeq0$,
and putting $k=\sqrt{k_{1}^{2}+k_{2}^{2}}$ yields for the
damping/growth rates
\begin{equation}
\gamma_{\pm} \simeq \pm
\left( \left| k_{2}\right| /2 k \right)\sqrt{4g\left( \mu^{-1}-\tilde{g}\right)
-\left(\tilde{g}
 k - \partial _{1}^{3}\phi^{(0)}_{0}/k\right)^{2}}, \label{Eq.35}
\end{equation}
whenever the quantity under the square root is non-negative.
The solutions of (\ref{Eq.23}) will not grow indefinitely (and become unstable)
provided
\begin{equation}
4g\left( \mu^{-1}-\tilde{g}\right) k^{2} \le\left(\tilde{g}k^{2} - \partial
_{1}^{3}\phi ^{(0)}_{0}\right)
^{2} ,\label{Eq.36}
\end{equation}
which recovers the driving effect of a negative density gradient
(a positive $\mu$) and the stabilising role of the third derivative of the
potential that have been known for $\tilde{g}=0$ \cite{Philippe2}.

A physical interpretation of (\ref{Eq.36}) is possible if, from (\ref{Eq.17})
and  (\ref{Eq.19}), one notices
 $\phi^{(0)}_{0} =\Lambda +U_{0}e^{-Z/\lambda _{D}}$ is the potential at a
distance $Z$ from the polarisation probe. Introducing the distance to the
probe as a parameter, assuming a negative density gradient in the SOL,
and neglecting the derivatives of the slowly varying $U\left(x_{1},t \right)$ in
(\ref{Eq.19}), (\ref{Eq.36}) reads %can be expressed in terms of physical
variables as
\begin{equation}
4 k ^{2}%
\partial_{1}\left( 1/B\right)
\left[ |\partial_{1}n|/n-\partial_{1}\left( 1/B\right) \right]
  \le \left[ k^{2}\partial_{1}\left( 1/B \right)
-U_{0}e^{-Z/\lambda_{D}}/\lambda _{D}^{3}
\right] ^{2}.
\label{Eq.37}
\end{equation}
It follows from (\ref{Eq.37}) that
control of unstable modes is a local effect and becomes increasingly
difficult further away from the probe (because of the $e^{-Z/\lambda_{D}}$
term), and also that
a negative polarisation $U_{0}$ is more favourable than a positive
one (accounting for the fact that, in general,
$\partial_{1}\left(1/B\right)\ge0$ in a tokamak), whereas
for positive $U_{0}$ the less controllable modes occur around $k^{2}\simeq
U_{0}e^{-Z/\lambda _{D}}/\lambda _{D}^{3}
\partial_{1}\left( 1/B\right) $.
In agreement with the advantage predicted here of using a negative $U_{0}$,
experiments have indeed shown that negative biasing leads not only
to a larger improvement in particle confinement, but it also reduces the
propagation of large-scale events (or blobs) and lowers the amplitude of
fluctuations [10]. %\cite{Silva1,Silva2}.
That it is preferable to apply a negative (as opposed to a positive) bias could
not be predicted from the TOKAM-2D model because
$\tilde{g}=0$ there [5], %\cite{Garbet,Philippe1,Philippe2,Figarella,Colin}
and with $\tilde{g}=0$ (\ref{Eq.37}) would read $4 k ^{2}
\partial_{1}\left( 1/B\right) \left|\partial_{1}n\right|/n
  \le \left(U_{0}e^{-Z/\lambda_{D}}/\lambda _{D}^{3}
\right) ^{2}$ instead, a condition where the sign of $U_{0}$ has no influence
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whatsoever.

\section{Structures and conservative dynamics} \label{SecV}

Many dynamical systems of physical interest have both conservative and dissipative components, having been proven finite-dimensional vector fields always correspond to a
superposition of Hamiltonian and gradient components \cite{Vilela1}. Identifying the Hamiltonian component is important because it often happens that in
some regions of phase space the effect of the non-conservative components cancels out along a neighbourhood of some of the Hamiltonian orbits, implying the full system ends up displaying deformed versions of the latter,
which led to the notion of ``constants of motion in dissipative systems"
\cite{Vilela1}.
It is conceivable that a similar situation might apply in infinite dimensions as one of the tools used for the finite-dimensional vector field decomposition, namely the Hodge--De Rahm theorem, can be extended to infinite dimensions
\cite{Arai}.

For the physical problem dealt with in this paper, the implication is that looking for solutions of the conservative part of the model in (\ref{Eq.10}), which reads
\begin{align}
\partial Ln/\partial t&=-\nabla \phi \cdot \nabla ^{\bot }Ln-\tilde{g}\partial _{2}\left(Ln-\phi\right) \nonumber \\
\partial \nabla^{2} \phi/\partial t &
=-\nabla \phi \cdot \nabla ^{\bot }\nabla^{2} \phi -g\partial _{2}Ln,
\label{Eq.39}
\end{align}
one might at
least obtain the ancestors of coherent structures that might also exist in the full model. So, after transforming according to $Ln\left(x_{1},x_{2},t
\right) = \widetilde{Ln}\left(x_{1},x_{2},t \right) -\tilde{g}x_{1}$ and
$\phi \left(x_{1},x_{2},t \right) = \widetilde{\phi } \left(x_{1},x_{2},t
\right) -\tilde{g}x_{1} $ whilst assuming $g=\tilde{g}$ constant,
looking for travelling-wave solutions of the form
$\widetilde{Ln} \left(x_{1},x_{2},t \right) =\widetilde{Ln}\left(x_{1}-v_{1}t, x_{2}-v_{2}t\right)$ and
$\widetilde{\phi }\left(x_{1},x_{2},t \right)
=\widetilde{\phi }\left(x_{1}-v_{1}t, x_{2}-v_{2}t\right) $,
defining $y_{i}=x_{i}-v_{i}t$ and making $\partial_{i}=\partial/\partial y_{i}$ henceforth, and putting
$\widetilde{\phi }\left(y_{1},y_{2} \right)=F\left(y_{1},y_{2} \right)
+v_{2}y_{1}-v_{1}y_{2}$,
(\ref{Eq.39}) reduces to%
\begin{align}
\partial _{1}F\partial _{2}\widetilde{Ln} &= \partial_{1}\widetilde{Ln}
\partial _{2}F \nonumber \\
\partial _{1}F\partial _{2}\nabla^{2} F - \partial _{1}\nabla^{2}F \partial _{2}F&
=-g\partial _{2}\widetilde{Ln}+\tilde{g}\partial _{2}\nabla^{2} F. \label{Eq.45}
\end{align}
The first equation in (\ref{Eq.45}) can be seen to be satisfied for
$\widetilde{Ln}=f\left(F\right) $, with f an arbitrary differentiable function, so substitution in the second equation yields
\begin{equation}
\partial _{1}F\partial _{2}\nabla^{2} F-\partial _{1}\nabla^{2}
F\partial _{2}F+g\partial _{2}f\left(F\right) -\tilde{g}\partial _{2}\nabla^{2}
F=0,
\label{Eq.47}
\end{equation}%
which admits a variety of solutions, depending on the choice of
$f\left(F\right)$. The symmetry exhibited by (\ref{Eq.45}) allows one to
identify

still another type of solutions, those of the form
$F\left(y_{1},y_{2}\right) =F_{ \rm s}\left(y_{1},y_{2}\right) $ and
$\widetilde{Ln}=\Theta_{ \rm s}F_{\rm s}$, with $\Theta_{\rm s}$ and $F_{\rm s}$
an operator and a function symmetric in y_{1} and y_{2}, meaning
$F_{\rm s}\left(y_{2},y_{1}\right) =F_{\rm s}\left(y_{1},y_{2}\right) $
and
$\Theta_{\rm s}F_{\rm s}\left(y_{2},y_{1}\right) =\Theta_{\rm s}F_{\rm
s}\left(y_{1},y_{2}\right)$, in which case (\ref{Eq.45}) reduces to
\begin{equation}
g\partial _{2}\Theta_{\rm s}F_{\rm s}-\tilde{g}\partial _{2}\nabla^{2} F_{\rm s}
=0.\label{Eq.46d}
\end{equation}
Particular solutions can be obtained by making
$f\left(F \right)=\alpha F$,
with α constant, and writing
$F\left(y_{1},y_{2}\right) = \widetilde{F}\left(y_{1},y_{2}\right)
+\tilde{g}y_{1} $,
in which case (\ref{Eq.47}) takes the form
\begin{equation}
\partial _{1}\widetilde{F}\partial _{2}\nabla^{2} \widetilde{F}-\partial
_{1}\nabla^{2}
\widetilde{F}\partial _{2}\widetilde{F}=-\alpha g\partial _{2}\widetilde{F}.
\label{Eq.50}
\end{equation}
Two obvious solutions to the homogeneous counterpart of (\ref{Eq.50}) are
$\widetilde{F}_{0}\left(y_{1},y_{2}\right)=Ae^{\left(k_{1}y_{1} + k_{2}y_{2}
\right) }$ and
$\widetilde{F}_{0}\left(y_{1},y_{2}\right)=A\cos\left(k_{1}y_{1} +
k_{2}y_{2} \right)
+B\sin\left(k_{1}y_{1} + k_{2}y_{2} \right) $
so that, plugging into (\ref{Eq.50}) the \emph{ansatz}
$\widetilde{F}\left(y_{1},y_{2}\right)
=\widetilde{F}_{0}\left(y_{1},y_{2}\right) +
\widetilde{H}\left(y_{1}\right)\mp(\alpha g/k^{2}) y_{1}$, one gets
\begin{equation}
\partial _{1}^{3}\widetilde{H} \mp k^{2}\partial_{1}\widetilde{H}=0,
\label{Eq.55}
\end{equation}
whose solutions can be trivially found.
Eventually more localised solutions to (\ref{Eq.39}) can be derived from
(\ref{Eq.46d}) by noting the Laplacian ∇^{2} is symmetric in y_{1} and
y_{2},
so putting $\Theta_{\rm s} F_{\rm s} = (\tilde{g}/g)\nabla^{2}F_{s} $ and
choosing (amongst various possibilities)
$F_{\rm s}\left(y_{1},y_{2}\right) =Ae^{-\gamma(y_{1}^{2} + y_{2}^{2}) /2}$
leads to
\begin{align}
Ln\left(x_{1},x_{2},t \right) &=-\frac{\tilde{g}}{g}\gamma
Ae^{-\gamma[(x_{1}-v_{1}t)^{2} + (x_{2}-v_{2}t)^{2}] /2}
\nonumber \left\{2-\gamma\left[\left(x_{1}-v_{1}t\right)^{2}
+\left(x_{2}-v_{2}t\right)^{2}\right]\right\} -\tilde{g}x_{1} \nonumber \\
\phi\left(x_{1},x_{2},t \right) &=Ae^{-\gamma[(x_{1}-v_{1}t)^{2} +
(x_{2}-v_{2}t)^{2}] /2} +v_{2}\left(x_{1}-v_{1}t\right)
-v_{1}\left(x_{2}-v_{2}t\right)- \tilde{g}x_{1}. \label{Eq.65}
\end{align}

The time evolution for a possible $Ln\left(x_{1},x_{2},t \right)$
given by (\ref{Eq.65}) is shown in Fig.~\ref{Fig.1}, which depicts a large-scale
structure that is initially located at the inner SOL region and
starts moving essentially outwards.
To retain here is the existence of conservative solutions that move
concentrations of particles and energy along both radial- and poloidal-like
directions in a 2D cross-section, eventually
ejecting them from the core to the wall, basically mimicking the behaviour of

blobs observed in experiments and
in simulations [2,3,5-7].
%\cite{Naulin,Boedo,Garcia,Ricci2,Kendl,Riva,Philippe2,Bisai1,Bisai2,Krasheninnikov,Furno,Bodi,Theiler}.
\begin{figure}[t]
\centering
\includegraphics[width=0.8\textwidth]{Gaussian_blob_gama50.eps}
\caption{Snapshots with contour plots of the logarithmic density function Ln for the solution (\ref{Eq.65}) of the conservative system (\ref{Eq.39}), for $A=-1.0$, $v_{1}=0.03$,
$v_{2}=0.05$, $\gamma=50.0$ and $g=\tilde{g}
=0.1$, the magnitude of contour levels decreasing linearly from red to violet across the rainbow pattern.} \label{Fig.1}
\end{figure}
Of course, these are solutions only to the conservative part of the equations, with
diffusion, parallel losses and source not being accounted for.
However, inspection of (\ref{Eq.5}) suggests that, in the first equation, the role of the
unspecified source term is to compensate for the parallel losses and that, in the second
equation, the longitudinal conductance term will not play a determinant
role as long as fluctuations away from the plasma potential are not
very large. Therefore, it is not unlikely that the overall structure of the complete
solutions will be mostly determined by the conservative dynamics induced by (\ref{Eq.39}).

\section{Summary and conclusions}

\label{SecVI}

A theoretical analysis has been provided on various aspects of a two-fluid model describing SOL turbulence
in slab geometry that retains the magnetic field inhomogeneity terms in the continuity equation.
The model equations have a conservative kernel governing transport across magnetic field lines, plus extra terms
that account for diffusion, longitudinal losses (along the magnetic field) and a plasma core source.
It has been shown that an upper bound for the growth rate of the vorticity gradient depends on (hence is controlled by) the density gradient and that, inversely,
an upper bound for the growth rate of the latter depends on the former, which seems to indicate the presence in the model of a nonlinear instability,
with both quantities working together to pull their gradients further and further up.
The possibility of controlling the turbulent fluctuations in model quantities by means of a biasing potential has also been assessed,
having been demonstrated that negative is more favourable than positive bias,
thus providing a theoretical explanation for experimental observations pointing in the same direction.
Finally, model equations have been analytically solved for their conservative part and
exact solutions of the travelling-wave type have been derived, some of which propagate from the inner to the outer plasma layers and might, therefore,
be interpreted as the conservative ancestors of the collective, large-scale structures identified as blobs in experiments and in computations.

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\begin{thebibliography}{99}

\bibitem{Stangeby} STANGEBY, P.C., The Plasma Boundary of Magnetic
Fusion Devices, IOP Publishing, Bristol (2000).

\bibitem{Naulin} NAULIN, V., J. Nucl. Mater. 363--5 (2007) 24; ROTH, J., et al.,
J. Nucl. Mater. 390--1 (2009) 1; BOEDO, J.A., J. Nucl. Mater. 390--1 (2009) 29.

\bibitem{Zeiler} ZEILER, A., et al., Phys. Plasmas 4 (1997) 2134; GARCIA, O.E.,
et al., Phys. Rev. Lett. 92 (2004) 165003; SCOTT, B.D., Contrib. Plasma Phys. 46
(2006) 714; RICCI, P., et al., Phys. Plasmas 18 (2011) 032109; RICCI, P., et
al., Plasma Phys. Control. Fusion 54 (2012) 124047; KENDL, A., Plasma Phys.
Control. Fusion 57 (2015) 045012; RIVA, F., et al., Plasma Phys. Control. Fusion
58 (2016) 044005.

\bibitem{Garbet} GARBET, X., et al., Nucl. Fusion 31 (1991) 967; SARAZIN, Y.,
GHENDRIH, Ph., Phys. Plasmas 5 (1998) 4214.

\bibitem{Philippe2} GHENDRIH, Ph., et al., Nucl. Fusion 43 (2003) 1013;
FIGARELLA, C.F., et al., J. Nucl. Mater. 337--9 (2005) 342; COLIN, C., et al.,
Contrib. Plasma Phys. 54 (2014) 543.

\bibitem{Bisai1} BISAI, N., et al., Phys. Plasmas 11 (2004) 4018; BISAI, N., et
al., Phys. Plasmas 12 (2005) 102515.

\bibitem{Krasheninnikov} KRASHENNINNIKOV, S.I., et al., J. Plasma Phys. 74
(2008) 679; FURNO, I., et al., Phys. Rev. Lett. 100 (2008) 055004; BODL, K., et
al., Contrib. Plasma Phys. 48 (2014) 63; THEILER, C., et al., Phys. Rev. Lett.
103 (2009) 065001; MAGGS, J.E., MORALES, G.J., Plasma Phys. Control. Fusion 54
(2012) 124041.

\bibitem{Izacard} IZACARD, O., et al., Phys. Plasmas 18 (2011) 062105.

\bibitem{Morrison} HAZELTINE, R.D., et al., Phys. Fluids; HAZELTINE, R.D., et
al., Phys. Fluids 30 (1987) 3204.

\bibitem{Silva1} SILVA, C., et al., J. Nucl. Mater. 337--9 (2005) 415; SILVA,
C., et al., Plasma Phys. Control. Fusion 48 (2006) 727.

\bibitem{Vilela1} VILELA MENDES, R., DUARTE, J.T., J. Math. Phys. 22 (1981)
1420; DUARTE, J.T., VILELA MENDES, R., J. Math. Phys. 24 (1983) 1772.

\bibitem{Arai} ARAI. A., MITOMA, I., Math. Ann. 291 (1991) 51.
\end{thebibliography}

%\end{thebibliography}

\end{document}

\begin{figure}[tc]
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\includegraphics[width=1.0\textwidth]{Wave_cos2.eps}
\caption{Six snapshots with contour plots of the logarithmic density function
$Ln$ for the solution (\ref{Eq.57}) of the conservative system (\ref{Eq.39}),
with
$B=C_{0}=C_{1}=C_{2}=0$, $\alpha=A=1$,  $v_{1}=v_{2}=0.1$, $k_{1}=0.2$,
$k_{2}=1$ and $g=\tilde{g}
=0.1$, the magnitude of contour levels decreasing linearly from red to violet
across the rainbow pattern.} \label{Fig.1}
\end{figure}


\begin{figure}[tc]
\centering
\includegraphics[width=1.0\textwidth]{Wave_exp2.eps}
\caption{Six snapshots with contour plots of the logarithmic density function
$Ln$ for the solution (\ref{Eq.62}) of the conservative system (\ref{Eq.39}),
for
$C_{0}=C_{1}=C_{2}=0$, $\alpha=A=1$,  $v_{1}=-0.2$, $v_{2}=-0.1$, $k_{1}=-0.5$,
$k_{2}=-1$ and $g=\tilde{g}
=0.1$, the magnitude of contour levels decreasing linearly from red to violet
across the rainbow pattern.} \label{Fig.2}
\end{figure}

Another solution to the homogeneous counterpart of (\ref{Eq.50}) is
\begin{equation}
\widetilde{F}_{0}\left( y_{1},y_{2}\right)=Ae^{\left(k_{1}y_{1} + k_{2}y_{2}
\right) } ,
  \label{Eq.58}
\end{equation}
hence, putting
\begin{equation}
\widetilde{F}\left( y_{1},y_{2}\right)
=\widetilde{F}_{0}\left( y_{1},y_{2}\right) +
\widetilde{H}\left( y_{1}\right)- \frac{\alpha g}{k^{2}} y_{1}, \label{Eq.59}
\end{equation}
(\ref{Eq.50}) becomes
\begin{equation}
\partial _{1}^{3}\widetilde{H} -k^{2}\partial_{1}\widetilde{H}=0, \label{Eq.60}
\end{equation}
whose general solution reads [see (n) in the appendix]
\begin{equation}
\widetilde{H} \left( y_{1}\right) = C_{0} + C_{1} e^{ k y_{1}} + C_{2} e^{ -k
y_{1}}. \label{Eq.61}
\end{equation}
On account of this, the functions
\begin{equation}\eqalign{
\fl Ln\left(x_{1},x_{2},t \right) =\alpha\left[ \phi\left(x_{1},x_{2},t
\right)-v_{2}\left(  x_{1}-v_{1}t\right)+v_{1}\left(  x_{2}-v_{2}t\right)\right]
+\left(\alpha -1\right) \tilde{g}x_{1}
\cr
\fl\phi\left(x_{1},x_{2},t \right) = \left(v_{2}+\tilde{g}-\frac{\alpha g}
{k^{2}}\right) \left(  x_{1}-v_{1}t\right)
-v_{1}\left(  x_{2}-v_{2}t\right)- \tilde{g}x_{1} \fl
\\ \!\!\!\,\,
{}+Ae^{\left[k_{1}\left(  x_{1}-v_{1}t\right)
+k_{2}\left(  x_{2}-v_{2}t\right) \right]}  +C_{0}+
C_{1}e^{k\left( x_{1}-v_{1}t\right)  }+C_{2}e^{-k\left( x_{1}-v_{1}t\right)  }
} \label{Eq.62}
\end{equation}
are also solutions to (\ref{Eq.39}), snapshots of $Ln\left(x_{1},x_{2},t \right)
$ hereabove being given in figure~\ref{Fig.2}
for $C_{0}=C_{1}=C_{2}=0$, $\alpha=A=1$,  $v_{1}=-0.2$, $v_{2}=-0.1$,
$k_{1}=-0.5$, $k_{2}=-1$ and $g=\tilde{g}
=0.1$. One sees, also in this case, a large-scale structure that is initially
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located at the inner SOL region and subsequently starts moving essentially outwards.
The important point to retain here is the existence of solutions that move concentrations of particles and energy along both (radial- and poloidal-like) directions in a 2D cross-section, eventually
ejecting them from the core to the wall, hence mimicking the basic, gross behaviour depicted by blobs that is observed in experiments and
n
\cite{Naulin,Boedo,Garcia,Ricci2,Kendl,Riva,Philippe2,Bisai1,Bisai2,Krasheninnikov,Furno,Bodi,Theiler}.

\begin{figure}[tc]
\centering
\includegraphics[width=1.0\textwidth]{Gaussian_blob_gama15.eps}
\caption{Six snapshots with contour plots of the logarithmic density function Ln for the solution (\ref{Eq.65}) of the conservative system (\ref{Eq.39}), for $A=-1.0$, $v_{1}=0.03$,
$v_{2}=0.05$, $\gamma=15.0$ and $g=\tilde{g}
=0.1$, the magnitude of contour levels decreasing linearly from red to violet across the rainbow pattern.} \label{Fig.3}
\end{figure}