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Efficient Time Dependent Modelling of the Physics of NBI Heating in JET

W G F Core, K-D Zastrow.

JET Joint Undertaking, Abingdon, Oxfordshire, OX14 3EA, UK.

ABSTRACT

The time dependent modelling used in the interpretation and validation of data obtained from charge exchange spectroscopy measurements of the JET plasma during neutral beam injection heating is presented. The results obtained are primarily intended to form the basis for the efficient calculation of the effect of neutral beam injection heating on the plasma in transport simulations and in the interpretation of JET data.

1. INTRODUCTION

In neutral injection heating of a tokamak plasma fast ions are initially injected as neutrals, which after ionisation slow down to thermal energy through Coulomb scattering on the background plasma ions and electrons. The total energy of the plasma, including that carried by fast particles, is enhanced at the moment of ionisation. In this context the injected power is instantaneous. During the slowing down process the fast ions transfer their energy to the thermal plasma. For calculations of the thermal energy confinement time in transient phases of the discharge one has to use this delayed heating power. The injected power would be used in combination with the total plasma energy to obtain the total energy confinement time. During the slowing down process, the fast ions in addition to heating the plasma impart momentum to the system, and furthermore the fast ions can undergo thermonuclear reactions with the plasma ions or in self collisions. The efficient calculation of these quantities is not only important in the interpretation of data, but also in transport studies of the evolving JET plasma [1].

In this report these quantities are calculated using models obtained from a Fokker-Planck treatment of the slowing down process in which the plasma thermal species density and temperature are assumed to be slowly varying functions of time $t > \tau_S$, where τ_S is the Spitzer slowing down time. From the time dependent solution models of the fast ion energy, transfer rates of energy and momentum to the plasma bulk ion and electron species are first obtained. Then approximations for the beam-plasma and beam-beam thermonuclear reaction rates are calculated. Finally a simplified model of the effect of plasma rotation on these quantities is included.

In the next chapter we introduce the ion Fokker Planck equation and its solution. The energy balance and transfer rates are presented in chapter 3, and the momentum transfer rates are given in chapter 4. Thermonuclear reactions are treated in chapter 5. Inchapter 6 we introduce a model to include the effect of rotation on the calculated quantities. Typical results for the analysis of JET data are shown in chapter 7.

2. THE FAST ION FOKKER PLANCK EQUATION AND SOLUTION

The Fokker Planck equation describing the slowing down process and evolution of the fast ion distribution in the velocity space variables v, $\zeta = v_{\parallel} / v$, where $v_{\parallel}$ is the projection of the fast ion velocity along the magnetic field is [2]

$$\tau_s \frac{\partial f}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}\left\{\left(v^3 + v_c^3\right)f\right\} + \beta\left(\frac{v_c}{v}\right)^3 \frac{\partial}{\partial \zeta}\left(1-\zeta^2\right)\frac{\partial f}{\partial \zeta}$$

$$+\frac{S\tau_s}{2\pi}\frac{\delta(v - v_o)}{v^2}\delta(\zeta - \zeta_o) \qquad (1)$$

where the Spitzer time

$$\tau_s(t) = \frac{3v_e^3 m_e m}{16\sqrt{\pi}e^4 Z^2 n_e \ln\Lambda}$$

the critical velocity

$$v_c(t) = \left(\frac{3}{4}\sqrt{\pi}\frac{m_e}{m_i}\right)^{1/3} v_e$$

and the pitch angle scattering parameter is

$$\beta = \frac{m_i}{2m}\bar{Z}.$$

where

$$\bar{Z} = \sum_j \frac{n_j Z_j^2}{n_e}$$

is the effective charge of the plasma, m_i, m_e, m are the plasma ion, electron and fast ion mass respectively, S(t) is the rate of injection of fast beam ions onto a magnetic surface, $\ln\Lambda$ is the Coulomb logarithm and finally, v_o, $\zeta_o\left(= v_{\parallel o} / v_o\right)$ are the initial fast ion velocity and pitch respectively. The electron density n_e and temperature T_e, where $\frac{1}{2} m_e v_e^2 = T_e$ are assumed to be slowly varing functions of time t. And a solution of Eq. (1) based upon a Laplace transform in the time variable is indicated.

Following a Laplace transform in time and a Legendre polynomial decomposition in the cosine of the pitch-angle

$$f(v,\ \zeta,\ t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\lambda \exp(\lambda t)\sum_{n=o}^{\infty} f_n(v,\ \lambda)P_n(\zeta) \tag{2}$$

the solution of Eq. (1) is obtained in a straight forward manner and the fast ion distribution function takes the form

$$f(v,\ \zeta,\ t) = \frac{\tau_s}{4\pi}\frac{1}{v^3+v_c^3}\sum_{n=o}^{\infty}(2n+1)\left(\frac{v}{v_o}\right)^{\beta n(n+1)} u^{\beta n(n+1)\beta}$$

$$P_n(\zeta_o)P_n(\zeta)S\left\{t - \frac{\tau_s}{3}\ln(u)\right\} \tag{3}$$

where $u = \left(v_o^3 + v_c^3\right)/\left(v^3 + v_c^3\right)$.

3. CALCULATION OF FAST ION ENERGY BALANCE AND ENERGY TRANSFER TO PLASMA BULK IONS AND ELECTRONS

Using the results of the previous section the fast ion energy balance and energy transfer rates to the bulk plasma ion and electron species can be calculated. Multiplying each term of Eq. (1) by $\frac{1}{2}mv^2$ and integrating over spherical coordinates in velocity space $d^3\underline{v} = v^2 dv d\zeta d\phi$ gives

(i). The Fast ion Energy Content of the Plasma

$$E_f(t) = \frac{1}{2}m\int v^2 f(v,\ \zeta,\ t)d^3\underline{v} \tag{4}$$

Substitution of the distribution function Eq. (3) and integrating over the gyro-phase ϕ and the pitch ζ gives

$$E_f(t) = \frac{1}{2}m\tau_s\int_{v(t)}^{v_o} v^4 dv \frac{S\left\{t - \frac{\tau_s}{3}\ln\frac{v_o^3+v_c^3}{v^3+v_c^3}\right\}}{v^3+v_c^3} \tag{5}$$

where the lower limit of integration is defined by

$$v(t)^3 = \begin{cases} \left(v_o^3+v_c^3\right)e^{-3t/\tau s} - v_c^3 : & t < t_s = \frac{\tau_s}{3}\ln\frac{v_o^3+v_c^3}{v_c^3} \\ 0 & : \ t \geq t_s \end{cases}$$

Under the change of variable

$$\tau = t - \frac{\tau_s}{3}\ln\frac{v_o^3 + v_c^3}{v^3 + v_c^3}$$

$$E_f(t) = \frac{1}{2}m\int_{t_o(t)}^{t} d\tau\ S(\tau)\left\{\left(v_o^3 + v_c^3\right)e^{-3(t-\tau)/\tau s} - v_c^3\right\}^{2/3}$$

and

$$\frac{E_f(t)}{E_o} = \int_{t_o(t)}^{t} d\tau\ S(\tau)\left\{\left(1 + \frac{v_c^3}{v_o^3}\right)e^{-3(t-\tau)/\tau s} - \frac{v_c^3}{v_o^3}\right\}^{2/3} \tag{6}$$

where in the above the lower limit of integration is defined as

$$t_o(t) = \begin{cases} o & ;\ t \leq t_s = \frac{\tau_s}{3}\ln\frac{v_o^3 + v_c^3}{v_c^3} \\ t - t_s & ;\ t > t_s \end{cases}$$

and the initial fast ion energy $E_o = \frac{1}{2}mv_o^2$.

(ii) The Transfer of Fast Ion Energy to the Plasma Electrons

$$P_e(t) = \frac{1}{2}m\frac{1}{\tau_s}\int \frac{\partial}{\partial v}\left\{v^3 f(v,\ \zeta,\ t)\right\}d^3\underline{v}$$

and

$$E_e(t) = \frac{1}{2}m\int \frac{\partial}{\partial v}\left\{v^3 f(v,\ \zeta,\ t)\right\}d^3\underline{v}$$

Following the above procedure of reduction and a further integration by parts over velocity yields

$$E_e(t) = m\tau_s\int_{v(t)}^{v_o} v^4 dv \frac{S\left\{t - \frac{\tau_s}{3}\ln\frac{v_o^3 + v_c^3}{v^3 + v_c^3}\right\}}{v^3 + v_c^3} \tag{7}$$

and

$$\frac{E_e(t)}{E_o} = 2\int_{t_o(t)}^{t} d\tau S(\tau)\left\{\left(1 + \frac{v_c^3}{v_o^3}\right)e^{-3(t-\tau)/\tau s} - \frac{v_c^3}{v_o^3}\right\}^{2/3} \tag{8}$$

(iii) The Transfer of Fast Ion Energy to the Plasma Ions

$$P_i(t) = \frac{1}{2} m \frac{1}{\tau_s} \int \frac{\partial}{\partial v} \left\{ v_c^3 f(v, \zeta, t) \right\} d^3\underline{v}$$

and

$$E_i(t) = \frac{1}{2} m \int \frac{\partial}{\partial v} \left\{ v_c^3 f(v, \zeta, t) \right\} d^3\underline{v}$$

Following the reduction procedure of (ii) we obtain

$$E_i(t) = m v_c^3 \tau_s \int_{v(t)}^{v_o} v dv \frac{S\left\{ t - \frac{\tau_s}{3} \ln \frac{v_o^3 + v_c^3}{v^3 + v_c^3} \right\}}{v^3 + v_c^3} \tag{9}$$

and finally

$$\frac{E_i(t)}{E_o} = 2m \left(\frac{v_c}{v_o} \right)^3 \int_{t_o(t)}^{t} d\tau \; S(\tau) \left\{ \left(1 + \frac{v_c^3}{v_o^3} \right) e^{-3(t-\tau)/\tau_s} - \frac{v_c^3}{v_o^3} \right\}^{-1/3} \tag{10}$$

4. PLASMA ROTATION AND THE TRANSFER OF FAST BEAM ION MOMENTUM TO THE PLASMA IONS AND ELECTRONS

The equations describing the evolution of the plasma toroidal rotation during neutral beam injection are in the large aspect ratio tokamak approximation [2]

$$m_i n_i \frac{dv_\phi}{dt} = F_i^b - \frac{1}{r} \frac{\partial}{\partial r} r \; D_\pi^i \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{\tau_\pi^i} - R_{ie} \tag{11}$$

and

$$m_e n_e \frac{dv_\phi}{dt} = F_e^b - \frac{1}{r} \frac{\partial}{\partial r} r \; D_\pi^e \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{\tau_\pi^e} - R_{ei} \tag{12}$$

where v_ϕ is the toroidal velocity, m_i, m_e n_i, n_e are the plasma ion and electron mass and density respectively, D_π^i, D_π^e, τ_π^i, τ_π^e, are the diffusion coefficients and viscous damping times which are obtained from the viscosity tensor $\underline{\pi}$, R_{ie} is the momentum gained by the plasma ions through collisions with the electrons, $R_{ie} + R_{ei} = o$, and finally F_i^b, F_e^b are the rates of momentum transfer from the fast ions to the plasma ions and electrons.

In the high field, $T_i = T_e$ approximation $D^i_\pi >> D^e_\pi$, $\tau^e_\pi >> \tau^i_\pi$ and taking $m_e << m_i$, $n_i \sim n_e$ Eq. (11, 12) can be combined to give

$$n_i m_i \frac{dv_\phi}{dt} = \left(F_i^b + F_e^b\right) - \frac{1}{r}\frac{\partial}{\partial r} r \; D^i_\pi \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{\tau^i_\pi} \tag{13}$$

To calculate the rate of momentum input from the fast ions Eq. (1) is first multiplied by $mv_\parallel$ and then following integration over velocity space the following momentum transfer rates are obtained.

(i) The Transfer of Fast Ion Momentum to the Plasma Electrons

$$F_e^b = \frac{m}{\tau_s}\int \frac{\zeta}{v}\frac{\partial}{\partial v}\left\{v^3 f(v,\zeta,t)\right\} d^3\underline{v} \tag{14}$$

Substitution of the fast ion distribution function, Eq (3) into Eq (14) followed by integration over ϕ, and ζ gives

$$F_e^b = mP_1(\zeta_0)\int_{v(t)}^{v_0} v^3 dv \frac{S\left\{t - \frac{\tau_s}{3}\ln\frac{v_0^3 + v_c^3}{v^3 + v_c^3}\right\}}{v^3 + v_c^3}\left\{\frac{v_o^3 + v_c^3}{v^3 + v_c^3}\frac{v^3}{v_o^3}\right\}^{2\beta/3} \tag{15}$$

and

$$F_e^b = mv_o P_1(\zeta_o)\left\{\frac{v_o^3 + v^3}{v_o^3}\right\}^{2/3(\beta+1/2)}$$

$$\int_{t_o(t)}^{t} d\tau \; S(\tau) e^{-(t-\tau)/\tau_s}\left\{1 - \frac{v_c^3}{v_o^3 + v_c^3} e^{3(t-\tau)/\tau s}\right\}^{2/3(\beta+1/2)} \tag{16}$$

(ii) The Transfer of Fast Ion Momentum to the Plasma Ions

$$F_i^b = \frac{mv_c^3}{\tau_s}\int \frac{\zeta}{v}\frac{\partial}{\partial v}\left\{f(v,\ \zeta,\ t)\right\} d^3\underline{v} \tag{17}$$

$$= mv_c^3 P_i(\zeta_o)\int_{v(t)}^{v_0} dv \frac{S\left\{t - \frac{\tau_s}{3}\ln\frac{v_o^3 + v_c^3}{v^3 + v_c^3}\right\}}{v^3 + v_c^3}\left\{\frac{v_o^3 + v_c^3}{v^3 + v_c^3}\frac{v^3}{v_o^3}\right\}^{2\beta/3} \tag{18}$$

and

$$F_i^b = \frac{mv_c^3}{v_o^2} P_1(\zeta_o) \left\{ \frac{v_o^3 + v_c^3}{v_o^3} \right\}^{2/3(\beta-1)}$$

$$\int_{t_o(t)}^{t} d\tau \;\; S(\tau) e^{2(t-\tau)/\tau_s} \left\{ 1 - \frac{v_c^3}{v_o^3 + v_c^3} e^{2(t-\tau)/\tau_s} \right\}^{2/3(\beta-1)} \qquad (19)$$

and finally

(iii) The Transfer of Fast Ion Momentum to the Plasma Impurities

$$F_{imp}^b = \frac{mv_c^3}{\tau_s} \beta \int \frac{\zeta}{v^2} \frac{\partial}{\partial \zeta} \left(1 - \zeta^2\right) \frac{\partial f}{\partial \zeta} d^3\underline{v} \qquad (20)$$

$$= 2mv_c^3 \;\; \beta P_i(\zeta_o) \int_{v(t)}^{v_0} dv \;\; \frac{S\left\{ t - \frac{\tau_s}{3} \ln \frac{v_o^3 + v_c^3}{v^3 + v_c^3} \right\}}{v^3 + v_c^3} \left\{ \frac{v_o^3 + v_c^3}{v^3 + v_c^3} \frac{v^3}{v_o^3} \right\}^{2\beta/3} \qquad (21)$$

and

$$F_{imp}^b = \frac{2mv_c^3}{v_o^2} \beta P_1(\zeta_o) \left\{ \frac{v_o^3 + v_c^3}{v_o^3} \right\}^{2/3(\beta-1)}$$

$$\int_{t_o(t)}^{t} d\tau \;\; S(\tau) e^{2(t-\tau)/\tau_s} \left\{ 1 - \frac{v_c^3}{v_o^3 + v_c^3} e^{3(t-\tau)/\tau_s} \right\}^{2/3(\beta-1)} \qquad (22)$$

5. CALCULATION OFTHERMONUCLEAR RATE COEFFICIENTS $<\sigma v>$ IN NBI HEATED PLASMA

The fast injected beam ions in addition to heating the plasma ions and electrons through the Coulomb scattering processes can in particular heating systems such as D(T, n)^4H_e undergo fusion reactions with the plasma ions and in self collisions during slowing down. These reactions are a significant component in the total thermonuclear yield of the plasma.

The thermonuclear reaction rate (R_{ij}) in a plasma containing interacting ion species of type (i) and (j) is given by

$$R_{ij} = \frac{n_i n_j}{1 + \delta i_j} \langle \sigma v \rangle \qquad (23)$$

where n_i and n_j are the number densities of the interacting particles, δ_{ij} is the Kronecker delta, and

$$\langle \sigma v \rangle = \int d^3\underline{v} \int d^3\underline{v}' \, f_i(\underline{v},t) f_j(\underline{v}',t) \sigma(|\underline{v}-\underline{v}'|) |\underline{v}-\underline{v}'| \tag{24}$$

where σ is the cross-section and $|v - v'|$ is the impact velocity.

For the particle distribution functions in spherical co-ordinates in velocity space

$$f_i(v,\ t) = \sum_{n=o}^{\infty} a_m(v,t) P_m(\zeta) \tag{25}$$

$$f_j(v,\ t) = \sum_{n=o}^{\infty} b_n(v,t) P_n(\zeta) \tag{26}$$

Substitution of Eq. (25, 26) in Eq. (24) gives after some reduction

$$n_i n_j \langle \sigma v \rangle = 8\pi^2 \sum_{n=o}^{\infty} \frac{1}{2n+1} \int_o^{\infty} v dv \int_o^{\infty} v' dv' \, a_n(v,t) b_n(v',t) \int_{|v-v'|}^{v-v'} u^2 du \ \sigma(u) P_n(z) \tag{27}$$

where $v^2 + v'^2 - u^2 = 2vv'z$

(i) Calculation of Beam Plasma Thermonuclear Rate Coefficient $\langle \sigma v \rangle$

For fast beam ions slowing down in a warm plasma in thermal equilibrium we have for the plasma ions

$$a_o(v,t) = \frac{1}{\pi^{3/4} v_{th}^3} \exp\left(-v^2 / v_{th}^2\right); \tag{28}$$

and

$$a_n(v,t) = 0, \ \ n \neq 0. \tag{29}$$

and Eq. (27) simplifies considerably, we have

$$n_f \langle \sigma v \rangle = \tau_s \int_{v(t)}^{v_0} v^2 dv \frac{S\left\{ t - \frac{\tau_s}{3} \ln \frac{v_0^3 + v_c^3}{v^3 + v_c^3} \right\}}{v^3 + v_c^3} \langle \sigma v \rangle \tag{30}$$

where

$$(\sigma v) = \frac{2}{\sqrt{\pi}} \frac{e^{-(v/v_{th})^2}}{v v_{th}} \int_0^{\infty} u^2 du \ \sigma(u) \ \sinh\left(2uv / v_{th}^2\right) e^{-(v/v_{th})^2} \tag{31}$$

n_f is the fast ion density, and $\sigma(u)$ is the cross-section for the particular fusion reaction under consideration.

When the reaction cross-section $\sigma(u)$ is only available in tabulated or functional forms which are too complex for an analytical treatment, the integral has to be evaluated numerically. However, because of the difficulty of dealing with such problems, and when a high degree of accuracy is not too important, analytical approximations can be derived and used to provide results which are well within acceptable error. For this purpose we take the Gamov type of cross section representation deduced from Quantum mechanical consideration

$$\sigma(E) = \frac{Q(E)}{E} \exp\left(-B / E^{1/2}\right) \tag{32}$$

where $E = \frac{1}{2}\mu u^2$, is the energy in the c.m. frame, B is a constant depending upon the reaction under consideration, and the Astrophysical function

$$Q(E) \ = \ \frac{A_1 \ + \ E \ (A_2 + E(A_3 + E(A_4 + EA_5)))}{1 \ + \ (B_1 + E(B_2 + E(B_3 + EB_4)))} \tag{33}$$

is the corresponding Bosch and Hale Padé approximation fit to the experimental data [3].

Eq. (31) now takes the form

$$(\sigma v) = \frac{2}{\sqrt{\pi}} \frac{e^{-v^2}}{v v_{th}} \frac{1}{\mu} \int_0^{\infty} du \ \ Q\left(\frac{1}{2}\mu v_{th}^2 u^2\right) \exp\left\{-\frac{\alpha}{u} \pm 2uv - u^2\right\} \tag{34}$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass and $\alpha = \left(2B^2 / \mu v_{th}\right)^{1/2}$.

To evaluate Eq. (34) we first note that the Q(E) function in Eq. (32) is a slowly varying function of energy and from the form of Eq. (34) the method of integration by steepest descents is applicable. Integration through the dominant saddle point gives

$$\langle \sigma v \rangle \simeq \frac{2}{\mu v} Q\left(\frac{1}{2}\mu v_{th}^2 u_1^2 \ \right) \sqrt{\frac{u_1^3}{u_1^3 + \alpha}} \ \exp\left\{ -\left(u_1 - \frac{v}{v_{th}}\right)^2 - \frac{\alpha}{u_1} \right\} \tag{35}$$

where u_1 is the appropriate root of the saddle point equation $2u^2(u - v/v_{th}) = \alpha$.

Substitution of Eq. (35) in Eq. (30) gives the beam-plasma reaction rate

$$\langle \sigma v \rangle = \frac{1}{n_f} \int_{t_o(t)}^{t} d\tau S(\tau)(\sigma v) \tag{36}$$

where the fast ion density

$$n_f = \tau_s \int_{v(t)}^{v_o} \frac{v^2 dv}{v^3 + v_c{}^3} \left\{ t - \frac{\tau_s}{3} \ln \frac{v_o^3 + v_c^3}{v_3 + v_c^3} \right\} \tag{37}$$

or

$$n_f(t) = \int_{t_o(t)}^{t} d\tau S(\tau) \tag{38}$$

and (σv) is to be calculate with the velocity

$$v = \left\{ \left(v_o^3 + v_c^3 \right) e^{-3(t-\tau)/\tau s} - v_c^3 \right\}^{1/3}.$$

(ii) Calculation of Beam - Beam Thermonuclear Rate Coefficient <σv>

Substituting the cross section Eq. (32) into Eq. (27) gives

$$n_i n_j \langle \sigma v \rangle = \frac{16\pi^2}{\mu} \sum_{n=o}^{\infty} \frac{1}{2n+1} \int_0^{\infty} v^2 dv \int_0^{\infty} v'^2 dv' \, a_n(v,t) b_n(v',t)$$

$$\int_{-1}^{1} dz \; Q\left(\frac{1}{2} \mu u \right) \frac{e^{-\frac{\alpha}{\sqrt{u}}}}{\sqrt{u}} P_n(z) \tag{39}$$

where in the inner integral $u = (v^2 + v'^2 - 2vv'z)$.

Further reduction of Eq. (39) is difficult however some progress can be made. Consider the inner integral over the variable z as a function of the velocity variable v, v'. The contribution of the integral to the reaction rate is largest when u takes on its maximum value that is when v = v' is greatest. For two interacting beams this velocity is when the fast beam ion velocities are at or near the injection velocity v_o. Close to the injection velocity the fast ion distribution functions Eq. (25, 26) are sharply peaked about the initial pitch ζ_{oi} and to a good approximation take the form

$$f_i(v,\ \zeta,\ t) = \frac{S_i(t)}{2\pi}\frac{1}{v^3+v_c^3}\delta(\zeta-\zeta_{oi})$$

$$= \frac{S_i(t)}{4\pi}\frac{1}{v^3+v_c^3}\sum_{n=o}^{\infty}(2n+1)P_n(\zeta_{oi})P_n(\zeta) \tag{40}$$

Substitution of equations of the form of Eq. (40) for the two interacting beams into Eq. (39) inverting the order of integration and summation, using the result derived from the addition theorem for the Legendre polynomials, appendix 2

$$\sum_{n=o}^{\infty}\left(n+\frac{1}{2}\right)P_n(z_1)P_n(z_2)P_n(z) = \frac{1}{\pi}\int_0^{\pi}d\phi\ \delta(z-Z) \tag{41}$$

where $Z = z_1 z_2 - \left(1-z_1^2\right)^{1/2}\left(1-z_2^2\right)^{1/2}\cos\phi$, then completing the integration over the velocity variables v, v' gives

$$n_1 n_2\langle\sigma v\rangle = \frac{2\tau_{s1}\tau_{s2}}{\pi\mu}\int_{v_1}^{v_{01}} v^2 dv\frac{S_1(v)\tau_{s1}}{v^3+v_c^3}\int_{v_2}^{v_{02}} v'^2 dv'\frac{S_2(v')}{v'^3+v_c^3}(\sigma v) \tag{42}$$

where now

$$(\sigma v) = \int_o^{\pi}d\phi\ Q\left(\frac{1}{2}\mu u\right)\ \frac{e^{-\frac{\alpha}{\sqrt{u}}}}{\sqrt{u}} \tag{43}$$

and

$$u = v^2 + v'^2 - 2vv'\left\{\zeta_{01}\zeta_{02} - \left(1-\zeta_{01}^2\right)^{1/2}\left(1-\zeta_{02}^2\right)^{1/2}\cos\ \phi\right\}$$

n_1, n_2 are the two fast ion beam densities, $S_1(v)$, $S_2(v')$ are the fast ion injection rates, and v_{01}, v_{02} ζ_{01}, ζ_{02} are the initial velocities and pitch of the injected beam ions respectively. The lower limits of integration are given by

$$v_1(t)^3 = \begin{cases}\left(v_o^3+v_c^3\right)e^{-3t/\tau s}-v_c^3 : & t<t_s \\ 0 & : t\geq t_s=\frac{\tau_s}{3}\ln\frac{v_{01}^3+v_c^3}{v_c^3}\end{cases}$$

and similarly for $v_2(t)$ with v_{01} replaced by v_{02}.

By assuming $Q\left(\frac{1}{2}mu\right)$ to be a slowly varying function of u the following approximations for the function (σv) Eq (43) are readily obtained.

$$(\sigma v) \simeq -Q\left(\frac{1}{2}mu_o\right)\frac{\partial}{\partial\alpha}\frac{e^{-\frac{\alpha}{\sqrt{u_o}}}}{\alpha}I_o\left(\frac{\bar{\alpha}}{2u_o^{3/2}}\right) \tag{44}$$

where $u_o = v^2 + v'^2 - 2vv'\zeta_{01}\zeta_{02}$, $\bar{\alpha} = 2\alpha vv'\left(1-\zeta_{01}^2\right)^{1/2}\left(1-\zeta_{02}^2\right)^{1/2}$, and $\alpha u_o >> \bar{\alpha}$.

and when $\alpha u_o \simeq \bar{\alpha}$

$$(\sigma v) \simeq \frac{1}{\pi}\sqrt{\frac{2}{u_o}}Q\left(\frac{1}{2}mu_o\right)K_o\left(\frac{\alpha}{\sqrt{2u_o}}\right) \tag{45}$$

where $I_0(x)$, $K_0(x)$ are the modified Bessel functions of order zero.

alternatively

$$n_1n_2\langle\sigma v\rangle \simeq \frac{2\tau_{s1}\tau_{s2}}{\mu}\int_{t_1(t)}^{t} S_1(\tau_1)d\tau_1 \int_{t_2(t)}^{t} S_2(\tau_2)d\tau_2(\sigma v) \tag{46}$$

6. MODELLING THE EFFECT OF ROTATION

The input of momentum from the beam ions to the plasma gives rise to plasma rotation preferentially in the toroidal direction. Fast ions deposited in passing orbits moving in a co-rotational direction in the laboratory frame will in the rotating plasma frame suffer an apparent reduction in energy, and conversely, passing particles moving in counter rotation will have their energy increased. The fast ion slowing down time and consequently the density is then either reduced or increased. The bulk plasma rotational velocities impact on the beam heating efficiency, the thermonuclear reactivity, and consequently on the interpretation of the measured fusion yields.

For a fast ion initially produced on a magnetic surface with initial velocity and pitch (v_o, ζ_o) the transformation of particle velocity and pitch from the laboratory to the rotating frame $\left(v_o^*, \zeta_o^*\right)$ is

$$v_o^{*2} = v_o^2 - 2\sigma v_o v_\phi \zeta + v_\phi^2 \tag{47}$$

and

$$\zeta_o^* = \frac{v_\phi \zeta - \sigma v_\phi}{v_o^*} \tag{48}$$

where v_ϕ is the plasma toroidal rotation velocity, $\sigma = 1$ for injection into the passing region and $\sigma = \pm 1$ for injection into the trapped particle band.

By considering the flux of fast ions across the trapped-passing boundary in velocity space $\zeta = \pm\sqrt{2\varepsilon}$, where $\varepsilon = r/R_0$ to be small, and if under this transformation, orbit distortions are small, and the fast ion distribution function $f\left(v^*,\zeta^*\right) \simeq f(v,\zeta)$ then the model calculations of the quantities considered in the previous sections are applicable but with the source function centred on the initial velocity and pitch $\left(v_o^*, \zeta_o^*\right)$ given by Eq. (47, 48).

When transferring the calculated quantities back to laboratory coordinates, additional terms arise in the ion torque density and the power density. The additional torque corresponds to the addition of angular momentum to the bulk after the fast particles have slowed down. The additional term in the power balance represents the power necessary to maintain the toroidal rotation and is calculated as the product of the torque density and the frequency of toroidal rotation.

7. APPLICATION OF THE MODELS TO JET

The calculations presented in the previous chapters have been implemented for the analysis of JET data as part of the **CH**arge **E**xchange **A**nalysis **P**ackage (**CHEAP**)[4]. We have implemented the integrations in velocity space rather than the integrations in time. The latter integrands have singularities for $v \rightarrow 0$ in the power and momentum transferred to ions, and are very steep for velocities close to the injection velocity for the beam-thermal neutron rate. Therefore it has proven to be more efficient to perform the integrations in velocity space.

The code uses a hybrid electron density profile, taking the radial shape from LIDAR and the absolute calibration and time evolution from interferometer data, mapped onto flux surfaces based on the equilibrium code EFIT. The electron temperature is taken from LIDAR.

The code provides the source of fast particles, $S(\rho,t)$ where $S = r/a$ is the normalised minor radius, from which all quantities are calculated as outlined in the previous chapters. The number of lost beam neutrals along the beam path elements, $\xi(\rho,t)$, is mapped onto the corresponding flux volume decrements by cutting the beam in intervals of finite length. This avoids singularities at the flux surfaces that are tangented by the beam path.

$$S(\rho) = \sum_{k=1}^{3} \frac{f_k}{e} \frac{kP}{E} \frac{\Delta\xi_k(\rho)}{\Delta V(\rho)} \tag{49}$$

where f_k, k=1,3 is the fraction of beam power P at full, half and one third of the beam energy, E. The neutral beam attenuation factor along the beam path *ds* through the plasma depends exponentially on the electron density profile, atomic stopping cross-sections for each ion species *j* (electron and ion ionisation and charge exchange collisions) and their local concentrations, c_j.

$$\xi_k(\rho) = \exp\left(-\int ds(\rho) n_e(\rho) \sum_{j \geq 1} \sigma_j c_j(\rho) \right) \tag{50}$$

The beam attenuation part of the CHEAP code is self consistent in that it provides the donor density for the quantitative analysis of the charge exchange spectra. The resulting concentrations are in turn used in the attenuation code.

As an application of the models to JET we show results obtained with the CHEAP code for shot (#32919). In Figure 1 we show the time evolution of central electron temperature and density to illustrate the parameter range of the calculation. With these data τ_s varies between ≈ 1 sec at the beginning of the beam injection to ≈ 0.5 s towards the end. The critical velocity is of the order 3-4 10^6m/s. In Figures 2 - 6 we show the results of the calculations presented in the previous chapters. The calculations refer to the injection of $\approx$140keV deuterium ions. The effects of rotation have been included in the calculations. Note in particular the reduction of direct heating (Figure 4) due to the decreased injection energy of the fast ions. The total power corresponds to the injected power, after shine through corrections, when the power that is invested into rotation is included.

The torque integrated over the calculated profile is compared to the total torque, including shine through, in Figure 5.

$$M = \sum_{k=1}^{3} f_k \frac{P}{c} \sqrt{\frac{2km_b c^2}{E}} R_{imp} \tag{51}$$

where $m_b c^2$ is the mass of the beam ions in eV and R_{imp} is the impact parameter of the beam path with the torus axis. JET is typically operated with M/P in the range of 1.0-1.6 Nm/MW. The torque is calculated from the rates of momentum transfer (see chapter 4 (I)) by multiplication with the radius of beam absorption.

To validate and determine the range of applicability of the approximations for the function (σv), Eq. (35, 44), used in the calculation of fusion reactivities a comparison with an alternative method of calculation is necessary. For this purpose consider the interaction $D(T,n)^4He$. For the range of centre of mass energies $0<E_{cm}<250$keV the Padé-Coefficients for this reaction are presented in Table 1.

Consider first the approximation used in the calculation of the beam-thermal neutron rate, Eq. (35). In Figure (7) the results for fast ions slowing down in a warm plasma are compared with those obtained from a Monte-Carlo calculation. It is seen that below a fast ion energy of 50keV and for the particular finite bulk ion temperatures considered (T_i = 2.5, 10keV) the effect of the plasma ion temperature on the reaction rate increases and significant departure from the cold target reactivity results.

Consider now the approximation used in calculations of beam-beam reaction rates. The comparison between a numerical evaluation and Eq. (43) is shown in Figure (8). Below a centre of mass energy E_{cm}<52keV the analytical approximation deviates considerably from the numerical evaluation and additional corrections in the form of an expansion of the astrophysical function Q(E) in the analytical evaluation is required. However, because the contribution of this reaction to the total thermonuclear yield is small this hardly seems worth while.

8. SUMMARY

Using a time dependent solution of the neutral beam injection Fokker-Planck equation expressions for calculating the effect of fast ions slowing down on the bulk plasma have been given. The results for power and torque are to be used in calculations of the thermal energy and toroidal angular momentum confinement time. This is in contrast to calculations of the total energy confinement time, for which the absorbed power has to be used. There is no equivalent measure of the total toroidal angular momentum to be used with the absorbed momentum. For the calculation of thermonuclear reactivity in beam-thermal and beam-beam interactions the analytic approximations are in good agreement with Monte-Carlo and numerical methods of calculation.

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TABLE 1

Padé - Coefficients for the Q(E) Function

Coefficient	$D(T,n)^4He$
B ($\sqrt{KeV}$)	34.3827
A_1	6.927×10^4
A_2	7.454×10^8
A_3	2.050×10^6
A_4	5.2002×10^4
A_5	0.0
B_1	6.38×10^1
B_2	-9.95×10^{-1}
B_3	6.981×1^{-5}
B_4	1.728×10^{-4}

APPENDIX 1

Calculation of Steady State Fast Ion Energy Content of the Plasma

The fast ion energy content of the plasma is

$$E_f(t) = \frac{1}{2} m \int v^2 f(v,\zeta,t) d^3\underline{v} \tag{A1}$$

Following substitution of the fast ion distribution function. Eq (2) in the limit t $\rightarrow \alpha$ of the main text into Eq. (A1) and integration over gyro-phase φ and pitch ζ we obtain

$$E_f(t) = \frac{1}{2} m\tau_s S \int_o^{v_o} \frac{v^4 dv}{v^3 + v_c^3} \tag{A2}$$

Evaluation of (A2) is straightforward and we obtain

$$E_f(t) = \frac{1}{2} m\tau_s S \left[\frac{1}{2} v_o{}^2 - \frac{1}{3} v_c{}^2 \left\{ \frac{1}{2} \ln \frac{v_o^2 - v_o v_c + v_c^2}{v_o^2 + 2 v_o v_c + v_c^2} + \sqrt{3} \tan^{-1} \frac{\sqrt{3} v_o}{2 v_c - v_o} \right\} \right] \tag{A3}$$

where here S = S(t) is the constant level of particle injection rate onto a magnetic surface.

APPENDIX 2

$$\sum_{n=o}^{\infty}\left(n+\frac{1}{2}\right)P_n(z_1)P_n(z_2)P_n(z)=\frac{1}{\pi}\int_o^{\pi}d\varphi\ \ \delta(z-Z)$$

To prove this result we take as starting point the addition theorem for the Legendre polymials

$$P_n\left\{z_1z_2+\left(1-z_1^2\right)^{1/2}\left(1-z_2^2\right)^{1/2}\cos\ \varphi\right\}=P_n(z_1)P_n(z_2)$$
$$+\sum_{m=1}^{m=n}(-1)^m\frac{(n-m)!}{(n+m)!}P_n^m(z_1)P_n^m(z_2)\cos\ m\varphi$$

From which we get

$$P_n(z_1)P_n(z_2)=\frac{1}{2\pi}\int_o^{2\pi}P_n\left\{z_1z_2+\left(1-z_1^2\right)^{1/2}\left(1-z_2^2\right)^{1/2}\cos\ \varphi\right\}d\varphi$$

and the sum can then be written

$$\sum_{n=0}^{\infty}=\frac{1}{2\pi}\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)P_n(z)\int_o^{2\pi}P_n\left\{z_1z_2+\left(1-z_1^2\right)^{1/2}\left(1-z_2^2\right)^{1/2}\cos\ \varphi\right\}d\varphi$$

Inverting the order of integration and summation and noting the delta function representation

$$\delta(\zeta-\zeta_1)=\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)P_n(\zeta_1)P_n(\zeta)$$

gives

$$\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)P_n(z_1)P_n(z_2)P_n(z)=\frac{1}{\pi}\int_o^{\pi}d\varphi\ \ \delta(z-Z)$$

where $Z=z_1z_2-\left(1-z_1^2\right)^{1/2}\left(1-z_2^2\right)^{1/2}\cos\ \varphi$

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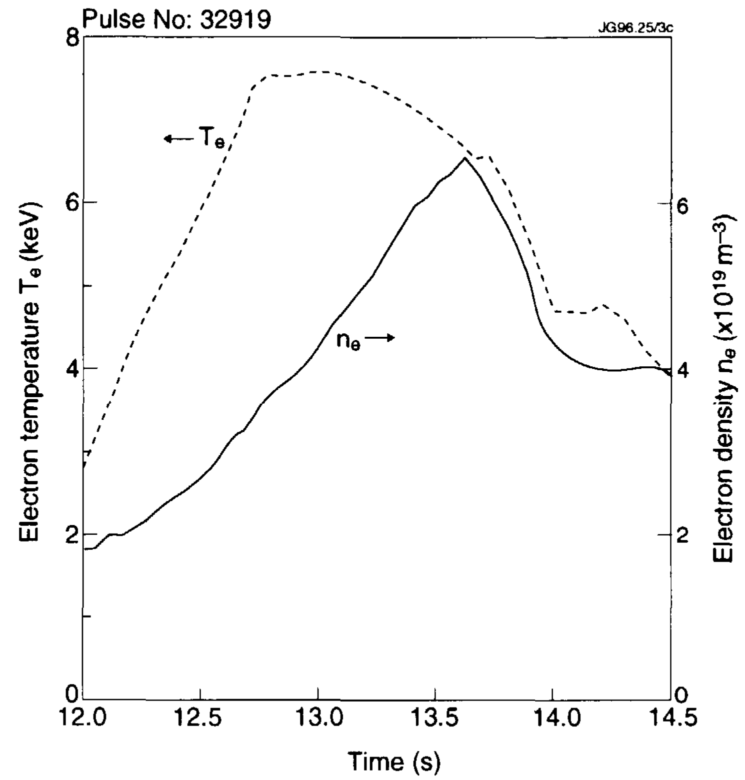

Fig.1: The observed time dependence of the central plasma electron density $n_e(t)$ and electron temperature $T_e(t)$ during deuterium injection heating of the JET plasma used in the calculations.

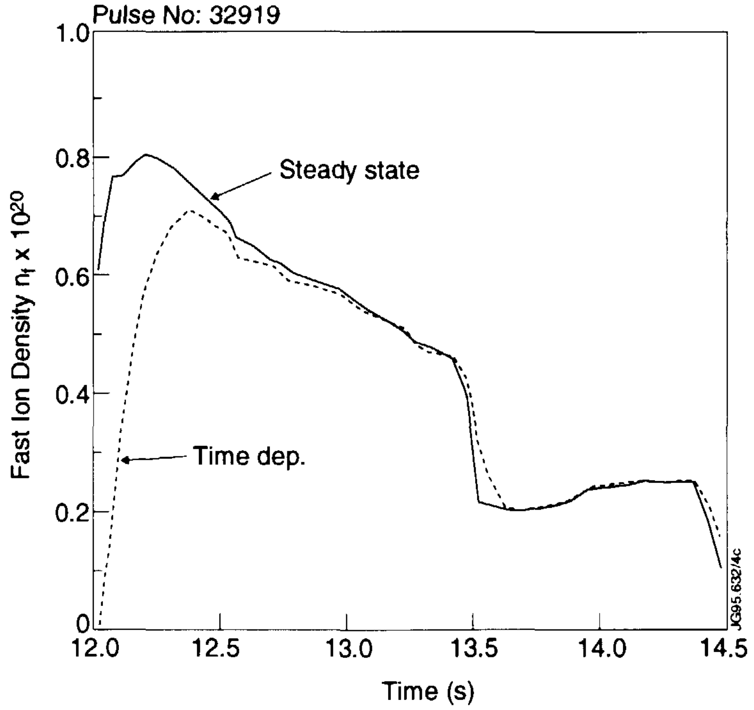

Fig.2: The calculated time dependent growth of the fast ion density within the plasma is compared with the steady state calculation.

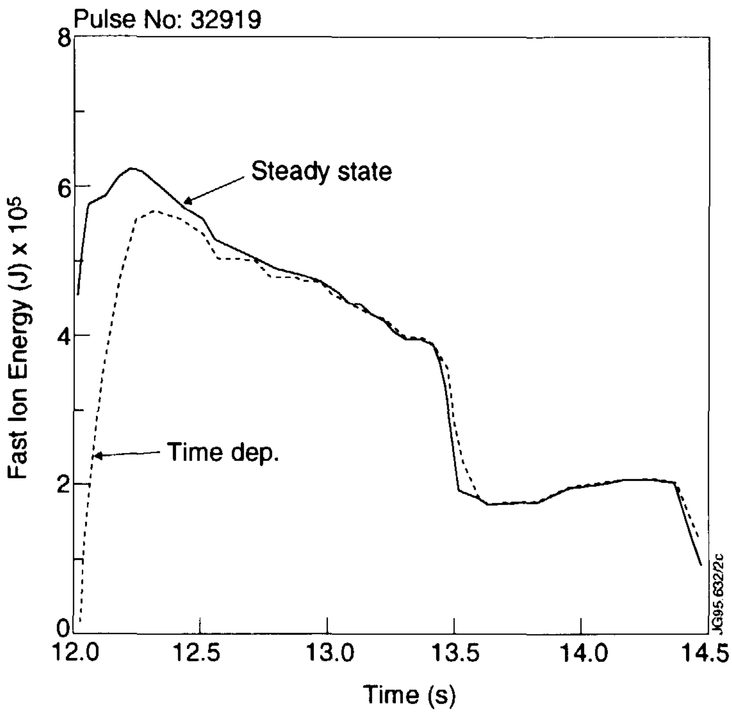

Fig.3: The calculated time dependent development and steady state fast ion energy content of the plasma.

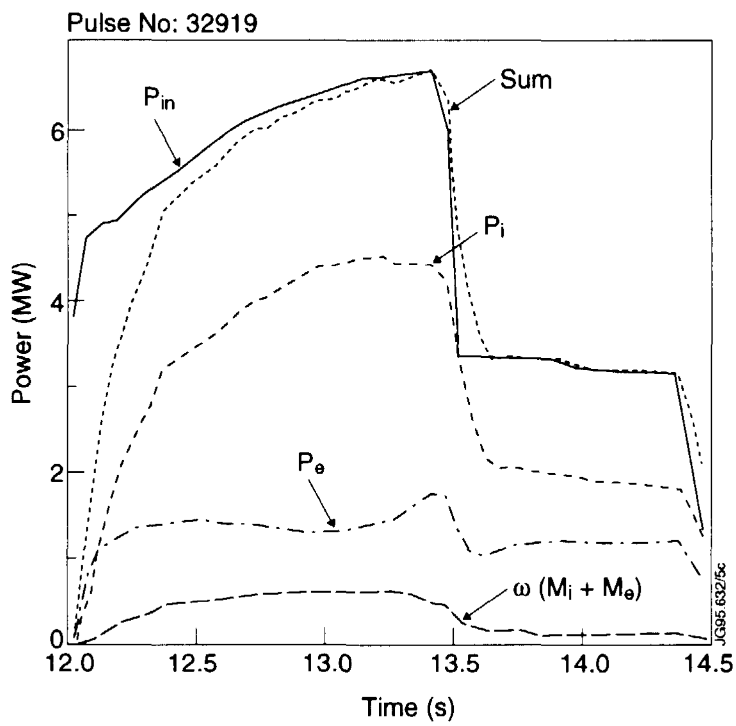

Fig.4: The partition of the injected beam power between the bulk plasma ion and electron species. ω is the angular rotation of the plasma and M_i, M_e are the torque from the fast ions to the bulk plasma ions and electrons respectively.

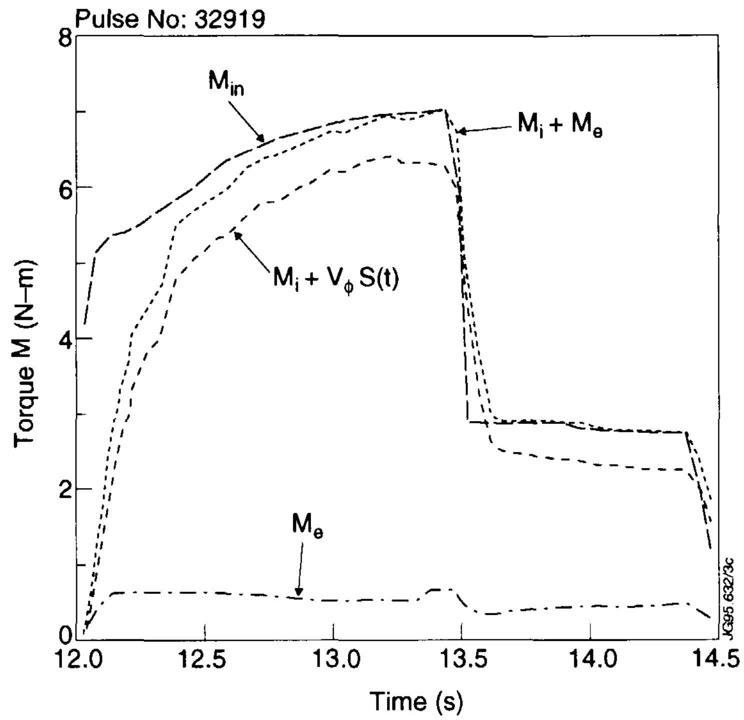

Fig.5: The torque of the injected fast ions to the bulk plasma ion and impurities and electrons. V_ϕ is the plasma toroidal velocity.

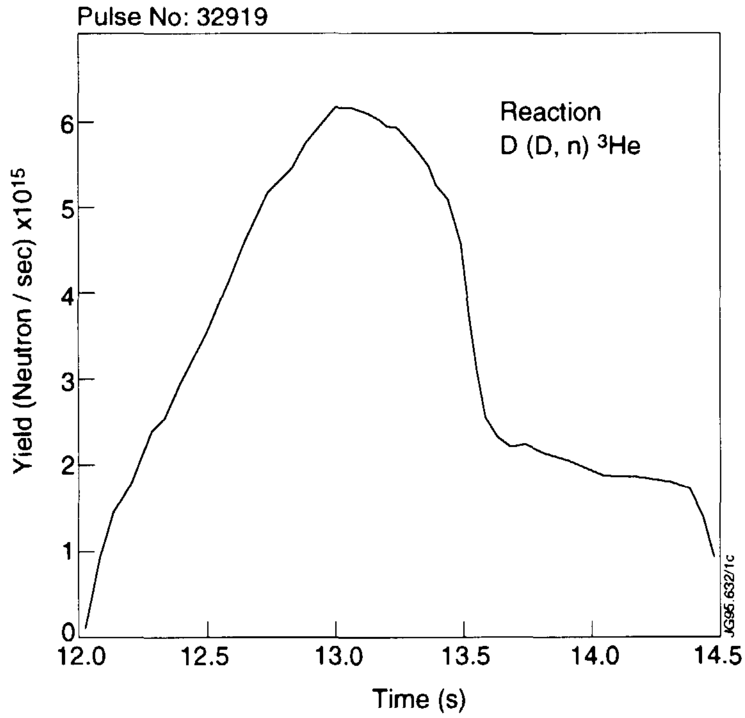

Fig.6: The time evolution of the beam-thermal neutron yield.

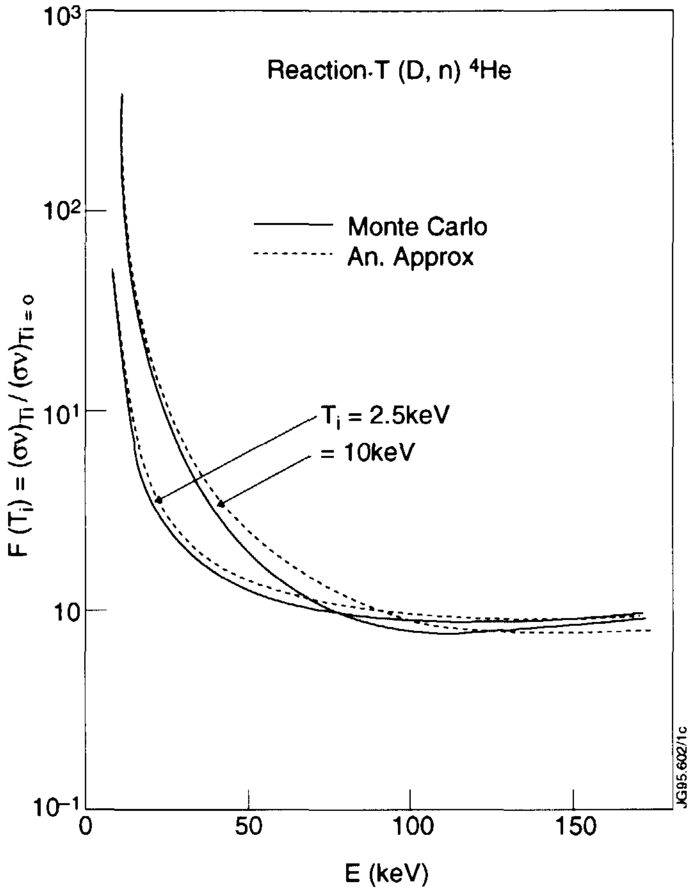

Fig.7: The effect of finite bulk plasma ion temperature on the function (σv) is compared with results obtained from a Monte-Carlo simulation.

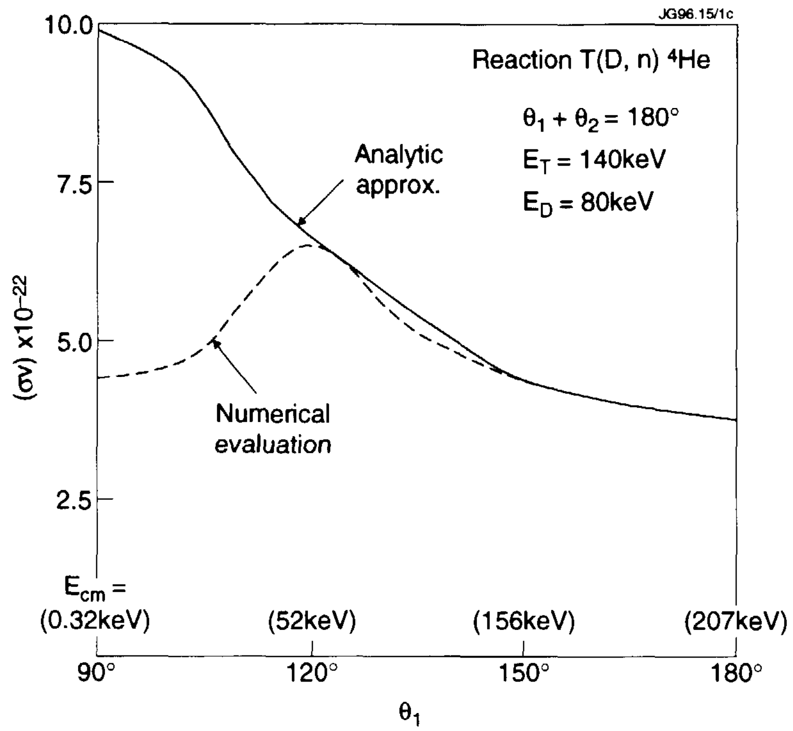

Fig.8: The comparison of the analytic approximation and a numerical evaluation of the function $\langle \sigma v \rangle$ for beam-beam fusion interactions. For parallel beams moving perpendicular to the magnetic field the calculations differ by a factor 2.