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ABSTRACT.

A number of statistical implications of empirical scaling laws in form of power products obtained by linear regression are analysed. The sensitivity of the error against a change of exponents is described by a sensitivity factor and the uncertainty of predictions by a “range of predictions factor”. The sometimes small value of the former and corresponding large value of the latter are related to the existence of inner relations in the statistical material used. A procedure for identifying inner relations is outlined. The relations can be of physical nature or can be due to a deliberate or incidental selection of data. In the latter case finding inner relations can help the experimentalist in choosing the experimental variables with necessary and sufficient density in parameter space. Also discussed are the consequences of discarding variables, in particular the effect of eliminating the inner relations by a corresponding reduction of parameter space. Finally a recipe is given for the computations to be done. The whole is exemplified by considering scaling laws for the electron energy confinement time of ohmically heated tokamak plasmas.

CONTENTS

Introduction

It is often of interest to try a representation of a quantity f like the electron energy confinement time τ_{Ee} by a product of powers of certain other quantities f_σ, $\sigma = 1, \ldots, m$. In the case of the electron energy confinement time these could be plasma density n, temperature T and so on. The power product or "scaling law" for f is then

$$f = \prod_{\sigma=1}^{m} f_\sigma^{\mu_\sigma}, \quad f_1 = e = 2.718 \ldots \tag{1}$$

Taking f_1 as the natural number e allows one to express also a constant factor by an exponent, namely μ_1. The set of exponents μ_σ can be obtained for instance by linear regression or just by trial.

One should bear in mind that a power product representation lacks in many cases any theoretical foundation. It is then doubtful whether one is allowed to use relations like (1) for extrapolation. But granted that doing this is reasonable, there remains still the question of the accuracy of such an empirically found formula.

If we write f_{exp} for the experimental values of f, and f_{sc} for the "scaled" value of f obtained from (1) we can define an error S by

$$S = (\ln f_{exp} - \ln f_{sc})^2 \tag{2}$$

which is minimised in linear regression analysis in order to obtain the best set of exponents μ_σ. The brackets mean averaging over all experimental data sets E used in the linear regression analysis. For any quantity $\alpha = \alpha(E)$ they are defined by

$$\langle\alpha\rangle = \frac{\sum_E \gamma_E \alpha(E)}{\sum_E \gamma_E} \tag{3}$$

The sums run over all data sets E, the quantities γ_E are weight factors which allow, for example, to give different experimental devices a comparable weight

in spite of possibly vastly different numbers of available data sets.

From (2) one could infer an uncertainty of prediction when using (1) given by

$$e^{-\sqrt{S}} \; f_{sc} < f < e^{+\sqrt{S}} \; f_{sc} \qquad (4)$$

This, however, is often a too optimistic estimate, because the exponents μ_σ have some range of uncertainty. In this context it is of special importance, that the statistical material may have certain inner relations which can delude an error S being smaller than appropriate for an extrapolation to an experiment which does not necessarily fulfil some of these relations. An example for such an inner relation is one which exists for tokamak plasmas with ohmic heating only: the definition of the experimental confinement time τ_{exp} is nearly (except for different averages) identical with τ_{ohm} defined by

$$\frac{nT}{\tau_{ohm}} = \eta j^2 \qquad (5)$$

The r.h.s. is the ohmic power density. Using for η the Spitzer expression one has essentially $\eta \sim T^{-3/2} \; Z_{eff}$. Furthermore, $j \sim B_p/a$, where a is the minor plasma radius and B_p the poloidal field at the plasma boundary. B_p is related to the toroidal field B_t by $B_p = B_t/q_a \; (R/a)$ where R is the major torus radius and q_a the safety factor at the plasma boundary. With all this, (5) yields

$$\tau_{ohm} = \text{const} \; n \; T^{5/2} \; Z_{eff}^{-1} \; R^2 \; q_a^2 \; B_t^{-2} \qquad (6)$$

which is of the form (1). Since $\tau_{exp} \sim \tau_{ohm}$ by definition an empirically found τ_{sc} could be dominated by τ_{ohm} and so could be the error S. τ_{ohm}, however, cannot be used for extrapolation; it does not describe any (sought) loss mechanism but only reflects the ohmic heating relation (5).

Since $\tau_{exp} \sim \tau_{ohm}$ also $\tau_{sc} \sim \tau_{ohm}$ by number for ohmically heated plasmas but with different dependences on temperature, density and so on. Therefore, we have

$$\frac{\tau_{sc}}{\tau_{ohm}} \cong 1 \qquad (7)$$

which simply reflects an inner relation between n, T, etc.
From (7) it follows that

$$\hat{\tau}_{sc} = \tau_{sc} \left\{\frac{\tau_{sc}}{\tau_{ohm}}\right\}^{\alpha} \qquad (8)$$

is equivalent to τ_{sc} for <u>any</u> α and, therefore, the dependence on n, T, ... can differ appreciably from that appearing in τ_{sc}.

There could be more inner relations than this one, and in fact one can find three such relations as shown in a following report.

Finding inner relations is not only important for showing that "good" scaling laws might not possess an absolute meaning, but also in two other respects: by finding an inner relation one could have obtained a new physical relationship or one could have established that part of the parameter space was not yet sufficiently investigated. In the latter case finding inner relations could serve as a guide to the experimentalist for choosing interesting parameter sets.

1. Introduction of Related Quantities

Sometimes it is of interest to use, instead of the basic quantities f_σ, $\sigma = 1, \ldots, m$ and f, quantities g_i and g related to the former ones by

$$g_i = \prod_{\sigma=1}^{m+1} f_\sigma^{\alpha_{i\sigma}}, \quad f_{m+1} = f, \quad g_{n+1} = g \tag{9a}$$

$$i = 1, \ldots, n+1; \quad n \leq m$$

It is convenient to choose the $\alpha_{i\sigma}$ such that

$$g_1 = f_1 = e = 2.718 \ldots \tag{9b}$$

and that only $g_{n+1} = g$ contains $f_{m+1} = f$.

This requires

$$\alpha_{1\sigma} = \delta_{1\sigma}, \quad \alpha_{i,m+1} = \delta_{n+1,m+1} \tag{9c}$$

$$\delta_{ik} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

Relations of the form (9) can be used, for instance, to introduce dimensionless quantities like $\beta = nT\, 2\mu_o/B^2$. For this latter quantity we would have with $\beta = g_j$, $n = f_2$, $T = f_3$, $B = f_4$: $\alpha_{j1} = 1n\, 2\mu_o$, $\alpha_{j2} = 1$, $\alpha_{j3} = 1$, $\alpha_{j4} = -2$.

One could also discard certain basic quantities f_σ by means of (9) in choosing the corresponding $\alpha_{1\sigma} = 0$, $i = 1, \ldots, n+1$.

Instead of directly discussing power products of the form (1) for the quantity f we will first consider the quantity g

$$g = \prod_{i=1}^{n} g_i^{\nu_i} \tag{10}$$

In terms of the exponents ν_i of the related quantities g_i relation (1) becomes

$$f = \prod_{\sigma=1}^{m} f_\sigma^{\sum_{i=1}^{n} \nu_i \alpha_{i\sigma} - \alpha_{n+1,\sigma}} \tag{11a}$$

i.e.

$$\mu_\sigma = \sum_{i=1}^{n} \nu_i \alpha_{i\sigma} - \alpha_{n+1,\sigma} \tag{11b}$$

In the following the logarithm of relation (10) plays the dominant role. It can be written as

$$\ln g = \sum_{i=1}^{n} \nu_i \ln g_i = \underline{\nu} \cdot \underline{\ln g} \tag{12}$$

where a vector notation has been introduced:

$$\begin{aligned} \underline{\nu} &\equiv \nu_i, \; i = 1, \ldots, n \\ \underline{\ln g} &= \ln g_i, \; i = 1, \ldots, n \end{aligned} \tag{13}$$

2. Best Fit by Linear Regression

The error S defined in equation (2) can, because of (9c), also be written as

$$S = \langle (\ln g_{exp} - \ln g_{sc})^2 \rangle$$
$$= \underline{\nu} \cdot \underline{\underline{A}} \cdot \underline{\nu} - 2\, \underline{b} \cdot \underline{\nu} + c \qquad (14)$$

$\underline{\underline{A}} = [A_{ik}]$ is a real symmetric semidefinite n x n matrix, $\underline{b} = (b_i)$ an n-dimensional real vector and c a real positive scalar. They are defined by:

$$A_{ik} = \langle \ln g_i \ln g_k \rangle , \qquad i, k = 1, \ldots, n$$
$$b_i = \langle \ln g_i \ln g_{exp} \rangle , \qquad i = 1, \ldots, n \qquad (15)$$
$$c = \langle (\ln g_{exp})^2 \rangle$$

The quantities (15) together form the variance matrix

$$G_{ik} = \langle \ln g_i \ln g_k \rangle , \qquad i, k = 1, \ldots, n+1 \qquad (16)$$

Minimizing S with respect to $\underline{\nu}$ yields the optimum set of exponants $\underline{\nu}^{opt}$ and the corresponding optimum power product g_{sc}^{opt}:

$$\underline{\nu}^{opt} = \underline{\underline{A}}^{-1} \cdot \underline{b} , \qquad \ln g_{sc}^{opt} = \underline{\nu}^{opt} \cdot \underline{\ln g} \qquad (17)$$

and the minimum error S_{min}:

$$S_{min} = c - \underline{b} \cdot \underline{\underline{A}}^{-1} \cdot \underline{b} \qquad (18)$$

3. Range of Predictions for $S > S_{min}$

As outlined in the introduction one should, in general, assume $S > S_{min}$. In this case there exist many scaling laws with errors smaller or equal to S. If such scaling laws are applied to a special experiment characterized by $\underline{\ln g}$ they will generally yield different answers for the expected value of the quantity g. There will be one such scaling law leading to the largest of all these values, and there will be another one leading to the smallest of all these values. The total range of predictions will therefore be given by these extrema of g multiplied by the statistical error $\exp(\pm\sqrt{S})$.

The extrema of g are obtained from extremizing $\ln (g_{sc}/g^{opt})$ under the constraint that $\Delta S = S - S_{min}$ is prescribed:

$$\underline{\nu} = \underline{\nu}^{opt} + \underline{\Delta\nu} \tag{19}$$

we have

$$\ln (g_{sc}/g^{opt}) = \underline{\Delta\nu} \cdot \underline{\ln g} \tag{20}$$

and

$$\Delta S = S - S_{min} = \underline{\Delta\nu} \cdot \underline{\underline{A}} \cdot \underline{\Delta\nu} \tag{21}$$

The latter is the constraint. Taking it into account by a Lagrange multiplier we arrive eventually at

$$\underline{\Delta\nu} = \pm \sqrt{\frac{\Delta S}{\underline{\ln g} \cdot \underline{\underline{A}}^{-1} \cdot \underline{\ln g}}}\ \underline{\underline{A}}^{-1} \cdot \underline{\ln g} \tag{22}$$

The extrema g^{extr} follow from this and (20) to be

$$\ln(g^{extr}/g^{opt}) = \pm\sqrt{\Delta S}\ \sqrt{\underline{\ln g} \cdot \underline{\underline{A}}^{-1} \cdot \underline{\ln g}} \tag{23}$$

The scaling laws g_{sc} leading to these extrema of g depend on the experiment being considered. This can be the case, because it is not possible to discriminate between these scaling laws for a real error $S > S_{min}$.

If we combine (23) with the uncertainty $\pm \sqrt{S}$ connected with each $\ln g_{sc}$ we obtain for the total range of predictions instead of (4)

$$g^{opt} \exp\left\{ - S - \sqrt{\Delta S} \cdot \sqrt{\underline{\ln g} \cdot \underline{\underline{A}}^{-1} \cdot \underline{\ln g}} \right\} \leq g \leq$$

$$g^{opt} \exp\left\{ \sqrt{S} + \sqrt{\Delta S} \sqrt{\underline{\ln g} \cdot \underline{\underline{A}}^{-1} \cdot \underline{\ln g}} \right\} . \qquad (24)$$

4. Sensitivity of Error S Against Change of Exponents

Inner relations like $\tau_{sc}/\tau_{ohm} \cong 1$ allow modification of the scaling laws without changing S appreciably. In order to find out whether the exponent ν_i of a certain quantity g_i is well defined we determine the minimum error $\hat{S}_{min}$ and the optimum set of exponents $\underline{\hat{\nu}}^{opt}$ when a constraint

$$\underline{e} \cdot (\underline{\nu} - \underline{\nu}^{opt}) = \delta\nu \,, \qquad e^2 = 1 \tag{25}$$

is prescribed. The result is

$$\underline{\hat{\nu}}^{opt} = \underline{\nu}^{opt} + \frac{\underline{\underline{A}}^{-1} \cdot \underline{e}}{\underline{e} \cdot \underline{\underline{A}}^{-1} \cdot \underline{e}} \, \delta\nu \tag{26}$$

$$\hat{S}_{min} = S_{min} + \frac{(\delta\nu)^2}{\underline{e} \cdot \underline{\underline{A}}^{-1} \cdot \underline{e}} \tag{27}$$

It is of special interest to calculate the effect of changing the exponents of a particular basic quantity f_σ and optimizing again all the other exponents. From (11b) it follows:

$$\begin{aligned} \delta\mu_\sigma = \mu_\sigma - \mu_\sigma^{opt} &= \sum_{i=1}^{n} (\nu_i - \nu_i^{opt}) \, \alpha_{i\sigma} \\ &= \alpha_\sigma \, \underline{e}_\sigma \cdot (\underline{\nu} - \underline{\nu}^{opt}) = \alpha_\sigma \cdot \delta\nu \end{aligned} \tag{28}$$

In this expression we have introduced

$$\begin{aligned} \alpha_\sigma &= \sqrt{\sum_{i=1}^{n} \alpha_{i\sigma}} \\ \underline{e}_\sigma &= \left\{ \frac{\alpha_{i\sigma}}{\alpha\sigma} \,, \quad i = 1, \ldots, n \right\}; \qquad e_\sigma^2 = 1 \end{aligned} \tag{29}$$

Choosing

$$\underline{e} = \underline{e}_\sigma \,, \qquad \delta\nu = \frac{\delta\mu_\sigma}{\alpha_\sigma} \tag{30}$$

we find from (26)

$$\hat{\underline{\nu}}^{opt} = \underline{\nu}^{opt} + \frac{\underline{\underline{A}}^{-1} \cdot \underline{e}_\sigma}{\underline{e}_\sigma \cdot \underline{\underline{A}}^{-1} \cdot \underline{e}_\sigma} \frac{\delta_{\mu\sigma}}{\alpha_\sigma} \tag{31}$$

and from (27)

$$\hat{S}_{min} = S_{min} + \frac{1}{\alpha_\sigma^2 \, \underline{e}_\sigma \cdot \underline{\underline{A}}^{-1} \cdot \underline{e}_\sigma} (\delta\mu_\sigma)^2 \tag{32}$$

The factor in front of $(\delta\mu_\sigma)^2$ characterizes obviously the sensitivity of the error S against a change of the exponent μ_σ of a basic quantity f_σ with all other exponents re-optimised. Therefore, we call

$$R_\sigma = \frac{1}{\alpha_\sigma^2 \, \underline{e}_\sigma \cdot \underline{\underline{A}}^{-1} \cdot \underline{e}_\sigma} \tag{33}$$

the "sensitivity factor".

5. Inner Relations in the Statistical Material

As can be seen from (27) the maximum change of exponents is possible for an $\underline{e}$ maximizing $\underline{e} \cdot \underline{\underline{A}}^{-1} \cdot \underline{e}$. If one represents $\underline{\underline{A}}$ which is real, symmetric and positive semidefinite by the dyadic products of its orthonormal eigenfunctions $\underline{y}^{(\lambda)}$ and the corresponding eigenvalues a_λ as

$$\underline{\underline{A}} = \sum_{\lambda=1}^{n} a_\lambda \, \underline{y}^{(\lambda)} \, \underline{y}^{(\lambda)} \tag{34}$$

$$\underline{\underline{A}} \cdot \underline{y}^{(\lambda)} = a_\lambda \, \underline{y}^{(\lambda)} \quad , \quad a_\lambda \geq 0 \quad , \quad \underline{y}^{(\lambda)} \cdot \underline{y}^{(\nu)} = \delta_{\lambda\nu} \tag{35}$$

then we have

$$\underline{\underline{A}}^{-1} = \sum_{\lambda=1}^{n} \frac{1}{a_\lambda} \, \underline{y}^{(\lambda)} \cdot \underline{y}^{(\lambda)} \tag{36}$$

The maximum of $\underline{e} \cdot \underline{\underline{A}}^{-1} \cdot \underline{e}$ is, therefore, obtained for

$$\underline{e} = \underline{y}^{(\lambda_o)} \quad , \quad a_{\lambda_o} : \text{smallest eigenvalue} \tag{37}$$

With this choice of $\underline{e}$ (27) and (26) become

$$\hat{S}_{min} = S_{min} + a_{\lambda_o} \, (\delta\nu)^2 \tag{38}$$

$$\hat{\underline{\nu}}^{opt} = \underline{\nu}^{opt} + \delta\nu \, \underline{y}^{(\lambda_o)} \tag{39}$$

If $a_{\lambda_o} \ll 1$, appreciable values of $\delta\nu$ and therefore appreciable changes in the exponents are admissible (remember $|\underline{y}^{(\lambda_o)}| = 1$!). $a_{\lambda_o} = 0$ would mean no change of S at all for arbitrary $\delta\nu$. This implies that for all $\underline{\ln g}$ belonging to the statistical material used, the following would hold:

$$\underline{y}^{(\lambda_o)} \cdot \underline{\ln g} = 0$$

$$\text{or} \quad \prod_{i=1}^{n} g_i^{\,y_i^{(\lambda_o)}} = 1 \tag{40}$$

$$\text{or} \quad \prod_{\sigma=1}^{m} f_\sigma^{\sum_{i=1}^{n} y_i^{(\lambda_o)} \alpha_{i\sigma}} = 1$$

Relation (40) stands for an exact inner relation between the quantities g_i or equivalently the quantities f_σ. (Note: the component $y_1^{(\lambda_o)}$ ensures only that the r.h.s. value 1 is met; for non-dimensional quantities it depends on the units used).

If only

$$a_{\lambda_o} << 1 \tag{41}$$

is fulfilled, then (40) is accordingly only approximately true and represents a nearly inner relation being fulfilled the better the smaller a_{λ_o} is.

There might be more than one eigenvalue a_{λ_o} being small compared with one. In this case all the corresponding eigenvectors $\underline{y}^{(\lambda_o)}$ describe inner relations of the form (40).

It is clear from (36) that $\underline{e}_\sigma \cdot \underline{\underline{A}}^{-1} \cdot \underline{e}_\sigma$ in (33) is dominated by the small eigenvalues of $\underline{\underline{A}}$ unless $\underline{e}_\sigma$ is almost normal to the corresponding eigenvectors characterising inner relations. Similarly in (24) the expression $\underline{\ln g} \cdot \underline{\underline{A}}^{-1} \cdot \underline{\ln g}$ is dominated by the small eigenvalues of $\underline{\underline{A}}$ except for $\underline{\ln g}$ fulfilling sufficiently well the inner relations (40).

6. Discarding Variables

It is sometimes of interest, especially with respect to the real error S, to ask for the information in the statistical material "normal" to inner relations. This information is contained in a reduced space obtained by the projection operator:

$$\underline{\underline{P}} = \underline{\underline{1}} - \sum_{\lambda_o} \underline{y}^{(\lambda_o)} \, \underline{y}^{(\lambda_o)} \tag{42}$$

where the sum runs over all or some of the eigenvectors characterising inner relations. The quantities g_i occur then only in the reduced form:

$$\underline{\ln g}^R = \underline{\underline{P}} \cdot \underline{\ln g} \tag{43}$$

The corresponding reduced matrix $\underline{\underline{A}}^R$ and vector $\underline{b}^R$ are given by

$$\underline{\underline{A}}^R = \underline{\underline{P}} \cdot \underline{\underline{A}} \cdot \underline{\underline{P}} \, , \quad \underline{b}^R = \underline{\underline{P}} \cdot \underline{b} \, . \tag{44}$$

The inverse of $\underline{\underline{A}}^R$ is defined as the so-called Moore-Penrose or generalised inverse $\underline{\underline{A}}^{R-1}_{MP}$ fulfilling

$$\underline{\underline{A}}^R \cdot \underline{\underline{A}}^{R-1}_{MP} = \underline{\underline{A}}^{R-1}_{MP} \cdot \underline{\underline{A}}^R = \underline{\underline{P}} \tag{45}$$

With the eigenvector representation (34) we have simply

$$\underline{\underline{A}}^R = \sum_{\lambda \neq \lambda_o} a_\lambda \, \underline{y}^{(\lambda)} \, \underline{y}^{(\lambda)} \tag{46}$$

and

$$\underline{\underline{A}}^{R-1}_{MP} = \sum_{\lambda \neq \lambda_o} \frac{1}{a_\lambda} \, \underline{y}^{(\lambda)} \, y^{(\lambda)} \tag{47}$$

In the reduced space defined by (42) all the relations obtained in the previous chapters hold if one replaces the quantities by the reduced quantities. If in (42) the sum over λ_o includes all inner relations, all the big

uncertainties vanish since the small eigenvalues do not occur any more. But, of course, (43) is an incomplete description now.

If one calculates the error S using the reduced space description one finds an increase over S_{min} by

$$\Delta S_R = \underline{b} \cdot \sum_{\lambda_o} \frac{1}{a_{\lambda_o}} \underline{y}^{(\lambda_o)} \underline{y}^{(\lambda_o)} \cdot \underline{b} = \sum_{\lambda_o} \frac{b_{\lambda_o^2}}{a_{\lambda_o}} \tag{48}$$

with

$$b_{\lambda_o} = \underline{y}^{(\lambda_o)} \cdot \underline{b} \tag{49}$$

When the sum is extended over all inner relations, none are left, and $S_{min} + \Delta S_R$ does no more delude too small an error. Therefore, (48) represents an upper bound for ΔS.

In the literature [2] it is shown that discarding variables can be done in different ways, the results being not very sensitive to the chosen procedure. In the context of ohmically heated plasmas very often a space is chosen which does not contain the temperature. In that case

$$\underline{\underline{P}} = \underline{\underline{1}} - \underline{e}_T \underline{e}_T \tag{50}$$

with $\underline{e}_T = \underline{e}_\sigma$ following from (29) with σ corresponding to $f_\sigma = T$. The Moore-Penrose inverse for a $\underline{\underline{P}}$ of this form can be written as

$$\underline{\underline{A}}^{R-1}_{MP} = \underline{\underline{A}}^{-1} - \frac{\underline{\underline{A}}^{-1} \cdot \underline{e}_T \underline{e}_T \cdot \underline{\underline{A}}^{-1}}{\underline{e}_T \cdot \underline{\underline{A}}^{-1} \cdot \underline{e}_T} \tag{51}$$

The incompleteness of this kind of a description can be completed again by prescribing the temperature dependence. From (26) together with (29) one finds

$$\mu_T = \alpha_T \underline{e}_T \cdot \hat{\underline{\nu}}^{opt} = \alpha_T \underline{e}_T \cdot \underline{\nu}^{opt} + \delta\nu \cdot \alpha_T \tag{52}$$

and therefore

$$\hat{\underline{\nu}}^{opt} = \hat{\underline{\nu}}^{opt} - \frac{\underline{\underline{A}}^{-1} \cdot \underline{e}_T}{\underline{e}_T \cdot \underline{\underline{A}}^{-1} \cdot \underline{e}_T} \underline{e}_T \cdot \underline{\nu}^{opt} + \frac{\underline{\underline{A}}^{-1} \cdot \underline{e}_T}{\underline{e}_T \cdot \underline{\underline{A}}^{-1} \cdot \underline{e}_T} \frac{\mu_T}{\alpha_T}$$

$$= \hat{\underline{\nu}}^{opt} \ (\mu_T = 0) + \frac{\mu_T}{\alpha_T} \frac{\underline{\underline{A}}^{-1} \cdot \underline{e}_T}{\underline{e}_T \cdot \underline{\underline{A}}^{-1} \cdot \underline{e}_T} \qquad (53)$$

The vector $\hat{\underline{\nu}}^{opt}$ $(\mu_T = 0)$ is the optimum set of exponents for a T-independent scaling law. We can therefore write

$$\tau_{sc}^{opt} (\mu_T) = \tau_{sc}^{opt} (\mu_T = 0) \exp \left\{ \frac{\mu_T}{\alpha_T} \frac{\ln g \cdot \underline{\underline{A}}^{-1} \cdot \underline{e}_T}{\underline{e}_T \cdot \underline{\underline{A}}^{-1} \cdot \underline{e}_T} \right\} \qquad (54)$$

The range of predictions with this formula is for each value of μ_T given by relation (24) with $\underline{\underline{A}}^{-1}$ replaced by $\underline{\underline{A}}_{MP}^{R-1}$ of equation (51). This means generally a strong reduction compared with the full space evaluation at the expense that now the exponent μ_T of the temperature is chosen arbitrarily.

7. Summary in Form of a Recipe for Programming

Definitions

- Basic input quantities:

  f_σ, $\sigma = 1, \ldots, m+1$

  especially

  $f_1 = e = 2{,}718 \ldots$ = natural number

  f_{m+1} = empirical value of the quantity f which is to be represented by a scaling law:

- $$f_{sc} = \prod_{\sigma=1}^{m} f_\sigma^{\mu_\sigma}$$

  μ_σ, $\sigma=1, \ldots, m$ = set of exponents

- Basic covariance matrix:

  $$F_{\sigma\tau} = \langle \ln f_\sigma \ln f_\tau \rangle = \frac{\sum_E \gamma_E \ln f_\sigma \ln f_\tau}{\sum_E \gamma_E}$$

  $\sigma,\tau = 1, \ldots, m+1$

  E: single set of experimental basic input quantities ("one experiment")

  γ_E: weight factor for single set E, to be prescribed

- Related quantities:

  $$g_i = \prod_{\sigma=1}^{m+1} f_\sigma^{\alpha_{i\sigma}}, \quad i = 1, \ldots, n+1; \; n \leq m$$

  especially

  $$\alpha_{1\sigma} = \delta_{1\sigma}; \; \alpha_{i,\,m+1} = \delta_{i,\,n+1}; \; \delta_{ik} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

- Scaling law for quantity g (corresponding to g_{n+1})

  $$g_{sc} = \left(\prod_{i=1}^{n} g_i^{\nu_i} \right)$$

- Relation to f_{sc}

$$f_{sc} = \prod_{\sigma=1}^{m} f_\sigma \left(\sum_{i=1}^{n} \nu_i \alpha_{i\sigma} - \alpha_{n+1,\sigma} \right)$$

therefore

$$\mu_\sigma = \sum_{i=1}^{n} \nu_i \, \alpha_{i\sigma} - \alpha_{n+1,\sigma}, \quad \sigma = 1, \ldots, m$$

- Related covariance matrix

$$G_{ik} = \sum_{\sigma=1}^{m+1} \sum_{\tau=1}^{m+1} \alpha_{i\sigma} \, \alpha_{i\tau} \, F_{\sigma\tau} = \langle \ln g_i \ln g_k \rangle$$

$i,k = 1, \ldots, n+1$

- Splitting of related covariance matrix

$G_{ik} = A_{ik}$ for $i,k = 1, \ldots, n$

$G_{i,n+1} = b_i$ for $i = 1, \ldots, n$

$G_{n+1,n+1} = C$

Notes:

1) The transformation from f_σ variables to some g_i variables can be continued by using the already obtained g_i variables formally as f_σ variables.

2) The $\alpha_{i\sigma}$ can in particular be used to discard certain basic input quantities.

Computations

- $\alpha_\sigma = \sqrt{\sum_{i=1}^{n} \alpha_{i\sigma}^2}$, $\underline{e_\sigma} = \left\{ \frac{\alpha_{i\sigma}}{\alpha_\sigma} ; \; i = 1, \ldots, n \right\}$

- orthogonal eigenvectors $\underline{y}^{(\lambda)} = \left\{ y_i^{(\lambda)} ; \; i = 1, \ldots, n \right\}$,

and eigenvalues a_λ ; $\lambda = 1, \ldots, n$:

$$\underline{\underline{A}} \cdot \underline{y}^{(\lambda)} = a_\lambda \underline{y}^{(\lambda)}, \qquad \underline{y}^{(\lambda)} \cdot \underline{y}^{(\nu)} = \delta_{\lambda\nu}$$

$$\lambda, \nu = 1, \ldots, n$$

- inner relations:

$$\prod_{i=1}^{n} g_i^{y_i^{(\lambda_o)}} = 1$$

for all $\underline{y}^{(\lambda_o)}$ with $a_{\lambda_o} \leqq \varepsilon << 1$

exponents μ_σ^o of f_σ, $\sigma = 1, \ldots, m$ for an inner relation $\underline{y}^{(\lambda_o)}$:

$$\prod_{\sigma=1}^{m} f_\sigma^{\mu_\sigma^o} = 1, \quad \mu_\nu^o = \sum_{i=1}^{n} y_i^{(\lambda_o)} \alpha_{i\sigma}$$

Inverse of $\underline{\underline{A}}$ e.g. by

$$\underline{\underline{A}}^{-1} = \sum_{\lambda=1}^{n} \frac{1}{a_\lambda} \underline{y}^{(\lambda)} \underline{y}^{(\lambda)} = \left\{ \sum_{\lambda=1}^{n} \frac{1}{a_\lambda} y_i^{(\lambda)} y_k^{(\lambda)}; \; i, k = 1, \ldots, n \right\}$$

- Vector of exponents ν_i for scaling law g_{sc}:

$$\underline{\nu} = \underline{\underline{A}}^{-1} \cdot \underline{b}$$

- Exponents for scaling law f_{sc}:

$$\mu_\sigma = \sum_{i=1}^{n} \nu_i \alpha_{i\sigma} - \alpha_{n+1,\sigma}; \qquad \sigma = 1, \ldots, m$$

- Minimum error:

$$S_{min} = c - \underline{b} \cdot \underline{\underline{A}}^{-1}, \underline{b}$$

- Sensitivity factor:

$$R_\sigma = \frac{1}{\alpha_\sigma^2 \, \underline{e}_\sigma \cdot \underline{\underline{A}}^{-1} \cdot \underline{e}_\sigma}$$

- Admissable change of exponents:

$$\delta\mu_\sigma = \sqrt{\frac{S - S_{min}}{R_\sigma}}$$

$S - S_{min}$: to be prescribed e.g. by

$$S - S_{min} = \sum_{\lambda_o} \frac{b_{\lambda_o}^2}{a_{\lambda_o}}, \quad b_{\lambda_o} = \underline{y}^{(\lambda_o)}\ \underline{b}$$

sum runs over all inner relations λ_o

- Application to a special experiment with basic input quantities f_σ^o:

- $f_{sc}^o = \prod_{\sigma=1}^{m} f_\sigma^o\, \mu_\sigma$

- range of prediction factor:

$$\exp\left\{\sqrt{S} + \sqrt{S - S_{min}}\ \underline{\ln g}^o \cdot \underline{\underline{A}}^{-1} \cdot \underline{\ln g}^o\right\}$$

$$\underline{\ln g}^o = \left\{ \ln g_i^o ; \qquad i = 1, \ldots, n \right\}$$

$$= \left\{ \sum_{\sigma=1}^{m} \alpha_{i\sigma} \ln f_\sigma^o , \ i = 1, \ldots, n \right\}$$

Discarding variables

- Discarding inner relations $\underline{y}^{(\lambda_o)}$:

replace

$$\underline{\underline{A}}^{-1} \text{ by } \sum_{\lambda \neq \lambda_o} \frac{1}{a_\lambda}\ \underline{y}^{(\lambda)}\ \underline{y}^{(\lambda)}$$

repeat all previous calculations beginning with the vector for exponents.

- Discarding quantity f_{σ_o} :

  replace

$$\underline{\underline{A}}^{-1} \text{ by } \underline{\underline{A}}^{-1} - \frac{\underline{\underline{A}}^{-1} \cdot \underline{e}_{\sigma_o} \, \underline{e}_{\sigma_o} \cdot \underline{\underline{A}}^{-1}}{\underline{e}_{\sigma_o} \, \underline{\underline{A}}^{-1} \, \underline{e}_{\sigma_o}}$$

  repeat all previous calculations beginning with the vector of exponents.

- Reintroduction of quantity f_{σ_o} with prescribed exponents μ_{σ_o} and application to experiment characterized by a set $\{f_\sigma\}$:

$$f_{sc}\,(\mu_{\sigma_o}) = f_{sc}\,(\mu_{\sigma_o} = 0)\, \exp\left\{ \mu_{\sigma_o} \frac{\sum_\sigma \ln f_\sigma \, \underline{e}_\sigma \cdot \underline{\underline{A}}^{-1} \cdot \underline{e}_{\sigma_o}}{\underline{e}_{\sigma_o} \cdot \underline{\underline{A}}^{-1} \cdot \underline{e}_{\sigma_o}} \right\}$$

  $\underline{\underline{A}}^{-1}$: original one

$$S_{min}\,(\text{new}) = S_{min} + R_{\sigma_o} \cdot (\mu_{\sigma_o} - \mu_{\sigma_o}^{opt})^2$$

- Range of prediction as for f_{σ_o} discarded (but μ_{σ_o} not determined).

References

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This report represents a considerable extension and a summary of work which has been previously reported in:

1. Skalierungsgeseke für Elektronen-Energietransport in toroidalen plasmen. D. Düchs, D. Pfirsch, H. Ringler and H. Wobig. Verh. der DPG, 5, p.746, (1980).

2. Scaling of Energy Confinement Time: Error Estimates and Dependence on Plasma Temperature.
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3. Confinement and Ignition.
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