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MHD Flow Instability in Presence of Resistive Wall

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A conducting fluid flowing along a magnetic field parallel to a conducting wall is unstable if the flow velocity exceeds a critical value.

The uniform flow of an ideally conducting mhd fluid along a uniform magnetic field is stable at all velocities. This can be seen by transforming to the frame of the fluid. In this frame the fluid is stationary and only the familiar stable waves can occur. If the flow is sheared a variant of the Kelvin-Kelmholtz instability is possible, the relative velocity of the fluid layers then providing a source of free energy. If, for the uniform flow case, an ideally conducting wall parallel to the flow is introduced the flow remains stable, the boundary condition for perturbations at the wall being independent of the flow velocity. However, if the finite conductivity of the wall is allowed for, the physics is changed. Now the flow velocity affects the magnetic field perturbation at the wall and, conversely, the wall provides a frame with respect to which the fluid flow has a free energy.

Configurations of this type can be unstable, and this is illustrated here by considering the simplest case, that of the uniform flow of an incompressible fluid in slab geometry. For low flow velocities there are three stable waves, two forward waves propagating in the direction of the flow and a backward wave propagating against the flow. If the flow velocity is increased, instability appears for a given wave number when the phase velocity of the "backward" wave changes sign and the wave propagates in the direction of the flow. The critical velocity is a function of the wave number, the Alfvén velocity, the fluid-wall separation and the electrical resistance of the wall.

THE MODEL

The geometry is illustrated in Figure 1 together with the choice of symbols.

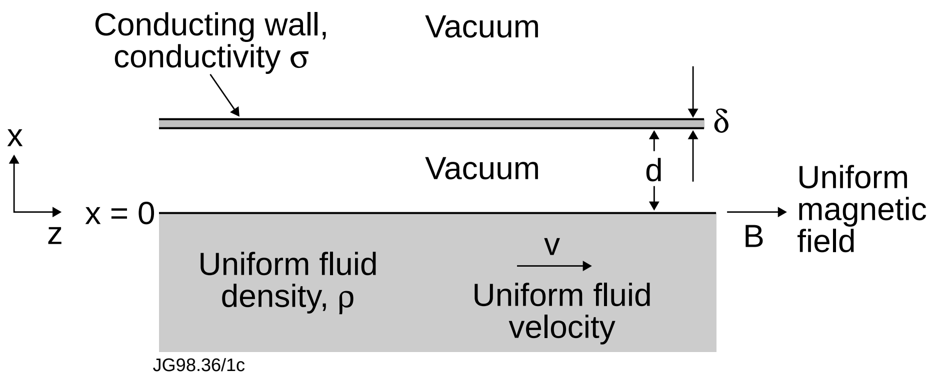

Figure 1. The geometry

The wall is taken to have the form of a "thin" shell to simplify the algebra. The fluid is in compressible and the perturbed quantities will be taken to have the form $f(x)\exp i(-\omega t + kz)$. The fluid, vacuum and wall regions will now be dealt with in turn.

FLUID

The basic equations are the equations of motion

$$\rho\left(\frac{\partial}{\partial t}+\mathbf{v}.\nabla\right)\mathbf{v} \;=\; -\nabla p+\mathbf{j}\times\mathbf{B}$$

and of incompressibility

$$\nabla.\mathbf{v} \;=\; 0$$

with Ampères law

$$\mathbf{j} \;=\; \frac{1}{\mu_o}\;\nabla\times\mathbf{B}$$

and Faraday's equation with the assumption of perfect conductivity, $\mathbf{E}+\mathbf{v}\times\mathbf{B}=0$,

$$\frac{\partial\mathbf{B}}{\partial t} \;=\; \nabla\times\mathbf{v}\times\mathbf{B}\;,$$

together with the definition of the flux function

$$\mathbf{B} \;=\; -\nabla\times\psi\mathbf{i}_y\;.$$

Linearising, these equations become

$$\rho i(\omega-kv)v_x \;=\; \frac{dp}{dx}-j_yB$$

$$\rho(\omega-kv)v_z \;=\; kp$$

$$\frac{\partial v_x}{\partial x}+ikv_z \;=\; 0$$

$$B_x \;=\; ik\psi$$

$$B_z \;=\; -\frac{\partial\psi}{\partial x}$$

$$j_y \;=\; \frac{1}{\mu_o}\left(\frac{\partial^2\psi}{\partial x^2}-k^2\psi\right) \qquad (1)$$

$$(\omega-kv)\,\psi \;=\; iBv_x\;,$$

the symbols ρ, v and B representing the equilibrium quantities.

These linearised equations give the equation for ψ,

$$\left((\omega-kv)^2-k^2V_A^2\right)\left(\frac{d^2\psi}{dx^2}-k^2\psi\right) \;=\; 0\;. \qquad (2)$$

The eigenmode solution with $\psi \to 0$ as $x \to -\infty$ is

$$\psi = \psi_s e^{kx} \qquad x < 0 \tag{3}$$

where ψ_S is the value of ψ at the surface, x = 0.

VACUUM

In the vacuum regime the equation for ψ is Laplace's equation

$$\frac{d^2\psi}{dx^2} - k^2 \psi = 0 \tag{4}$$

and the solution for the region between the fluid and the wall takes the form

$$\psi = \alpha e^{kx} + \beta e^{-kx} \qquad 0 < x < b \ . \tag{5}$$

WALL

In the conducting wall the governing equation is the diffusion equation

$$\frac{\partial\psi}{\partial t} = \frac{1}{\mu_o\sigma}\frac{\partial^2\psi}{\partial x^2} \ .$$

Using the thin wall approximation, integration across the shell gives

$$\left.\frac{d\psi}{dx}\right|_{d}^{d+\delta} = -\mu_o\sigma\delta i\omega\psi$$

and defining the velocity $c_w = (\mu_o\sigma\delta)^{-1}$,

$$\left.\frac{d\psi}{dx}\right|_{d}^{d+\delta} = -\frac{i\omega}{c_w}\psi \qquad x = d \ . \tag{6}$$

EXTERNAL VACUUM

In the external vacuum region $x > d + \delta$ the solution of the vacuum equation (4) is

$$\psi = \psi_w e^{-k(x-d)} \qquad x > d + \delta \ . \tag{7}$$

CONNECTING THE SOLUTIONS

Solutions (5) and (7) are connected by equation (6), so that

$$\frac{\beta}{\alpha} = -\frac{2ikc_w}{\omega} e^{2kd} \tag{8}$$

and, using continuity of ψ at x = 0, equations (3), (5) and (8) give the solution for the first vacuum region

$$\psi = \psi_s \frac{e^{-2kd}\, e^{kx} - \left(1 + \frac{2ikc_w}{\omega}\right) e^{-kx}}{e^{-2kd} - \left(1 + \frac{2ikc_w}{\omega}\right)} \; . \tag{9}$$

DISPERSION RELATION

The dispersion relation is now obtained by joining the solutions through the interface at the surface of the fluid. The interface condition is that of force balance across the surface and, using the linearised magnetic pressure BB_z/μ_o, this is given by

$$\left(p + \frac{BB_z}{\mu_o}\right)_{x=o^-} = \left(\frac{BB_z}{\mu_o}\right)_{x=o^+} \; .$$

With the linearised forms of the basic equations this becomes

$$\left(k^2V_A^2 - (\omega - kv)^2\right) \psi'_{x=o^-} = k^2V_A^2 \; \psi'_{x=o^+} \; , \tag{10}$$

where primes indicate differetials with respect to x.

Thus, using equations (3) and (9) to calculate ψ', equation (10) gives the dispersion relation

$$(\omega - kv)^2\left(\omega\left(1 - e^{-2kd}\right) + 2ikc_w\right) = 2k^2V_A^2\left(\omega + 2ikc_w\right) . \tag{11}$$

STABILITY

The stability boundary in parameter space is given by the condition that the imaginary part of ω be zero. From equation (11) this condition occurs for $\omega_r = 0$, so that $\omega = 0$, and

$$v = \pm\sqrt{2}V_A \; . \tag{12}$$

Stability is now explored by expanding about the marginal point, $\omega = 0$. Thus, keeping only the linear terms in ω in equation (11) and putting $v^2 \simeq 2V_A^2$

$$\omega = \frac{v^2 - 2V_A^2}{8c_w^2 + e^{-4kd}\, V_A^2}\left(\pm 2\sqrt{2}\,\frac{c_w}{V_A} + i\, e^{-2kd}\right) kc_w \; . \tag{13}$$

The condition for instability is that $\omega_i > 0$ and it follows from equation (13) that instability occurs for $|v| > \sqrt{2}\, V_A$. Equation (13) also shows the required stability in the two limiting cases $d \to \infty$ and $c_W \to 0$ ($\sigma \to \infty$).

For a highly conducting wall c_W is small and the growth rate, γ, becomes

$$\gamma = \frac{v^2 - 2V_A^2}{V_A^2} e^{2kd} kc_w \tag{14}$$

MECHANICS OF THE INSTABILITY

Instability arises when the flow velocity exceeds the critical velocity, $\sqrt{2}\, V_A$. However, this is only a necessary condition, instability also requiring the presence of the conducting wall. In the absence of a wall, changing the flow velocity is just equivalent to changing the frame of observation, and this leaves the velocity of the fluid with respect to the wave, v - ω/k, unchanged.

In the presence of a wall with finite conductivity a change of flow velocity brings about a change of flow velocity relative to the wave velocity, and this allows an unstable transfer of energy from the equilibrium flow to the perturbations of the magnetic field and the fluid. From equation (13) the rate of change of phase velocity with respect to fluid velocity at marginal stability is

$$\frac{d(\omega_r / k)}{dv} = -\frac{c_w^2}{c_w^2 + \frac{1}{8} e^{-4kd} V_A^2}$$

showing that with no wall ($d \to \infty$) we have $d(\omega_r/k)/dv = 1$, but that with a highly conducting wall ($c_W \to 0$), $d(\omega_r/k)/dv \to 0$.

The driving force for the instability is illustrated in Figure (2). In Figure 2(a) it is seen how, from the Bernoulli effect, the pressure is increased in the expanded region where the flow velocity is decreased. The increase in pressure is largest toward the surface, giving a pressure gradient force away from the surface. This force balances the centrifugal force of the fluid as it flows round the convex deformation. The opposite effects operate in the “necked” region.

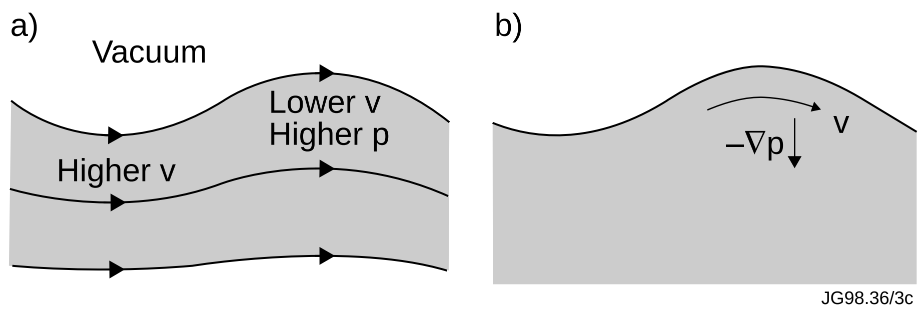

Figure 2(a) shows the higher pressure in the expanded region and 2(b) shows how the centrifugal force is balanced by the pressure gradient.

In the bulk of the fluid no electric currents are generated. At the surface, however, there is a sheet current. The magnetic force, j×B, of this current balances the drop in the perturbed fluid pressure across the surface as illustrated in Figure (3). This surface current represents the destabilising effect of the flow through the intermediary of the pressure gradient.

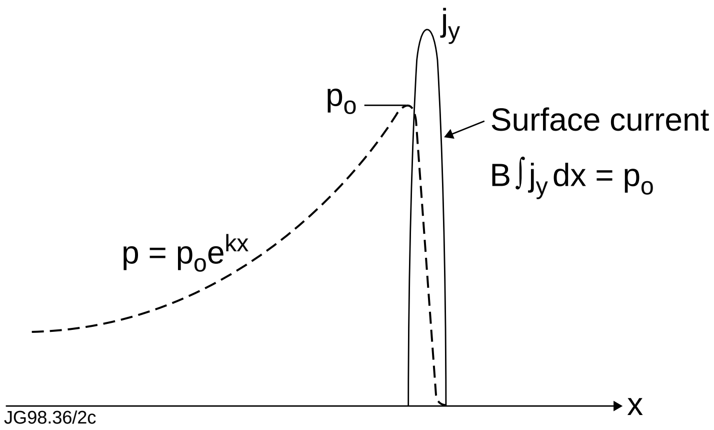

Figure 3. The force of the surface current balances the fluid pressure.

Integrating the x component of the equation of motion from $x \to -\infty$ to $x = 0^+$ gives the relation between the fluid motion and the surface current J_s

$$\rho i(\omega - kv)\int_{-\infty}^{o} v_x dx \;=\; -J_s B \tag{15}$$

where

$$J_s \;=\; \int_{o^-}^{o^+} j_y \; dx \;.$$

Close to marginal stability equation (15) together with $v_x = ikv\psi/B$ gives

$$J_s \;=\; -\frac{k}{\mu_o}\frac{v^2}{V_A^2}\,\psi_s \;. \tag{16}$$

The current induced in the conducting wall is

$$J_w \;=\; \int_{d^-}^{d^+} j_y \; dx \;. \tag{17}$$

The y-component of the electric field is ψ and Ohm's law gives $j_y = \sigma\dot{\psi}$. Close to marginal stability ($\omega \to 0$) equation (9) gives $\psi_w = e^{-kd}\,\psi_s$ so that equation (17) becomes

$$J_w = -\frac{\gamma\, e^{-kd}}{\mu_o c_w}\psi_s \; . \tag{18}$$

At marginal stability $\gamma = 0$ and so no current is induced in the conducting wall. If the flow velocity is reduced below the critical value a current flows in the conducting wall, and the perturbation decays due to the resulting dissipation.

If the flow velocity is increased above the critical value the increased driving force overcomes the dissipation and produce an unstable growth.

ENERGY BALANCE

The magnetic energy density of the perturbation is

$$W_M \;=\; \frac{1}{2\mu_o}\left(B_x^2 + B_z^2\right)$$

and, using the definition of ψ, the total magnetic energy per unit length is

$$W_M \;=\; \tfrac{1}{2}\int_{-\infty}^{+\infty} \frac{1}{2\mu_o}\left(\psi'^2 + k^2\psi^2\right) dx$$

where the first factor $^1/_2$ arises from the average value of $\cos^2 kz$. Integrating by parts and using equation (1)

$$\frac{1}{\mu_o}\int_{-\infty}^{+\infty}\left(\psi'^2 + k^2\psi^2\right)dx \;=\; -\int_{-\infty}^{+\infty} \psi\, j_y\, dx \; .$$

Since $j_y = 0$, except at the plasma surface and in the wall, this becomes

$$\frac{1}{\mu_o}\int_{-\infty}^{+\infty}\left(\psi'^2 + k^2\psi^2\right)dx \;=\; -\left(\psi_s\, J_s + \psi_w\, J_w\right) \tag{19}$$

and the magnetic energy in seen to be balanced by the energy drive of the surface current and the energy associated with the wall current.

The eigenfunction for $x < 0$ is $\psi = \psi_s\, e^{kx}$, and close to marginal stability $\psi = \psi_s\, e^{-kx}$ for $x > 0$, so that

$$\int_{-\infty}^{+\infty}\left(\psi'^2 + k^2\psi^2\right)dx \;=\; 2k\,\psi_s^2 \; . \tag{20}$$

Thus, using equation (20) together with equations (16) and (18) for J_S and J_W, equation (19) becomes

$$2 = \frac{v^2}{V_A^2} - \frac{\gamma}{kc_w} e^{-2kd},$$

and from this we recover the growth rate given by equation (14)

$$\gamma = \frac{v^2 - 2V_A^2}{V_A^2} e^{2kd} kc_w$$

SUMMARY

A conducting fluid flowing uniformly along a uniform magnetic field is normally stable. This is easily seen by moving to the frame of the flow. In the then stationary plasma all that remains are stable waves. The presence of a perfectly conducting wall parallel to the flow leaves the argument, and the conclusions, unchanged.

However, a finitely conducting wall provides a frame of reference for the flow which can no longer be transformed away. If the fluid is stationary there are two Alfvén waves and a further wave resulting from the finite conductivity of the wall. When the velocity of flow is increased above the natural mhd velocity of the system, $\sqrt{2}\, V_A$, instability occurs, the fluid flow providing the energy source.

Around marginal stability the propagation velocity of the instability is small. In this almost stationary mode the instability simply drives the magnetic field through the wall at a rate determined by the resistivity of the wall. The mechanism of the drive for the instability can be seen as the centrifugal force of the flow around the distorted magnetic field. The destabilising force is transmitted via the fluid pressure to a sheet current at the surface of the fluid. This current then provides the magnetic perturbation which presses on the wall, the growth rate depending on the rate of resistive diffusion through the wall.

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REFERENCE

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