JET-P(95)75

Electron Landau Damping of Toroidal Alfvén Eigenmodes

J Candy

JET Joint Undertaking, Abingdon, Oxfordshire, OX14 3EA, UK.

Preprint of a paper to be submitted for publication in
Plasma Physics and Controlled Fusion (letter)

December 1995

Abstract

A method to calculate the electron Landau damping of toroidal Alfvén eigenmodes (TAE) is presented. Due to the presence of a small current produced by thermal electrons near the trapped–passing boundary, a jump in the electrostatic potential near the $k_{\|} = 0$ surface occurs. An analytic expression for this jump is derived in the low-β, large-aspect-ratio limit, and is suitable for incorporation into either numerical eigenvalue codes or analytic models. In the limit of small magnetic shear, the calculation is carried through analytically, giving a simple local expression for the damping.

1. Introduction

In previous analyses of the toroidal Alfvén eigenmode (TAE) [1-4], a large-aspect-ratio, boundary layer approximation was employed to simplify the rather difficult task of determining eigenvalues for multiple-gap eigenmodes. The advantage of such an approach was that all physics beyond cylindrical, reduced magnetohydrodynamics (MHD) had to be retained *only* within "inner" (*i.e.*, TAE gap) equations. Away from these gap regions the simple "outer" equation $\mathcal{L}_m y_m = 0$, with

$$\mathcal{L}_m y_m \equiv \frac{\partial}{\partial r}\left[r^3\left(\frac{\omega^2}{v_A^2} - k_{\|m}^2\right)\frac{\partial y_m}{\partial r}\right] - r y_m\left(m^2-1\right)\left(\frac{\omega^2}{v_A^2} - k_{\|m}^2\right) + r^2 y_m \frac{\partial}{\partial r}\left(\frac{\omega^2}{v_A^2}\right) \quad , \tag{1}$$

was used. Here, $y_m \equiv \phi_m/r$ with ϕ_m the electrostatic potential and $k_{\|m} = (n - m/q)/R_0$ the parallel wavevector. In solving the inner/outer matching problem, the physics of the outer region is reduced to a calculation of a set of parameters (Δ_m) which characterize the variation of $\phi_m(r)$ from boundary to gap, or from gap to gap. In terms of the radial variable $x_m \equiv nq - m$, these gaps are centered at $x_m = \pm\omega/\omega_A$, with $\omega_A = v_A/qR_0$. Local theory gives $\omega \sim \omega_A/2$, so that $x_m = \pm 1/2$, although profile variation will typically shift this value somewhat.

In addition to the intrinsic (*i.e.*, continuum) damping, there are a variety of additional mechanisms which can stabilize the TAE. Indeed, the first calculation of the ion Landau damping (ILD) was given by Betti and Freidberg [5], who considered the response of passing ions ($v_i \ll v_A$) far from the trapped–passing boundary. This approach is justified as only a small fraction of the thermal ions, far out on the tail of the Maxwellian, are energetic enough to be resonant.

It was pointed out by Rosenbluth [6] that this approach *cannot* be applied directly to the calculation of electron Landau damping (ELD) in ignited plasmas, for which we expect $T_e \sim 20$ keV. In this regime, one finds $v_e \gg v_A$ — the limit which is considered in the this note. For electrons in this energy range, a consideration of the distribution near the trapped-passing boundary is required. Moreover, to satisfy the resonance condition $\omega - k_\| v_e \sim 0$, the electron dissipation will be restricted to radii where the parallel wavelength is nearly zero. Unlike ILD [5], which becomes exponentially

small as $v_i/v_A \to 0$, ELD is shown to decrease only linearly with the small parameter v_A/v_e. However, since the resonant layer where $k_\parallel \sim (v_A/v_e)/2qR_0 \sim 0$ typically lies *between* TAE gaps, the mode amplitude will generally be reduced somewhat from its peak value in the gap region. This point is illustrated in Sec. 3, where it is shown that for small values of the magnetic shear ($s \ll 1$) both the mode amplitude and ELD vanish exponentially at $k_\parallel = 0$.

2. The Electron Response

For the mode y_m, we have indicated that the electron resonance points occur very close to $x_m = \pm 1$. If the TAE gap and electron resonance points are not close to overlapping, only the outer TAE mode structure is affected. A qualtitative sketch clarifying this picture is shown in Fig. 1. To calculate this effect, we rewrite Eq. (1) including the perturbed electron pressure $\mathcal{P}$:

$$\mathcal{L}_m y_m = 4\pi i \omega r^2 \left\langle \nabla \cdot \left(\frac{\mathbf{B} \times \nabla \cdot \mathcal{P}}{B^2} \right) \right\rangle_m \quad , \tag{2}$$

with the m subscript denoting a projection onto the m-th poloidal harmonic, and $\mathcal{P} = P_\perp \mathbf{1} + (P_\parallel - P_\perp)\hat{\mathbf{b}}\hat{\mathbf{b}}$. The projection may be obtained using the integral operator

$$\langle \cdot \rangle_m \equiv \frac{1}{2\pi} \int_0^{2\pi} e^{im\theta} \cdot d\theta \quad . \tag{3}$$

An expansion of the RHS of Eq. (2) to lowest order in ϵ shows

$$\nabla \cdot \left(\frac{\mathbf{B} \times \nabla \cdot \mathcal{P}}{B^2} \right) \sim -\frac{1}{B_0 R_0} \left[\sin\theta \, \frac{\partial}{\partial r}(P_\perp + P_\parallel) + \frac{\cos\theta}{r} \, \frac{\partial}{\partial \theta}(P_\perp + P_\parallel) \right] \quad , \tag{4}$$

where the pressure components are defined in the usual way through the moment

$$P_\perp + P_\parallel = \int_{\mathcal{R}} d^3v \, \frac{m_e v^2}{2} \left(1 + \frac{v_\parallel^2}{v^2} \right) \mathrm{g} \quad . \tag{5}$$

$\mathcal{R}$ denotes the region of velocity space accessible to passing particles, and g is the nonadiabatic part of the electron distribution function. To facilitate the velocity space integration, we introduce a pitch-angle-like velocity variable $\lambda \equiv \mu B_0/\mathcal{E} = (v_\perp^2/v^2)(B_0/B)$ with $\mathcal{E} \equiv v^2/2$. This effects the transformation

$$\frac{d^3v}{2\pi} \longrightarrow \frac{\mathcal{E}\, d\mathcal{E}\, d\lambda}{|v_\parallel|} + O(\epsilon) \quad , \tag{6}$$

with the RHS to be summed over both signs of $v_\parallel$. The projection of the quantity inside square brackets in Eq. (4) becomes

$$\left\langle \sin\theta \frac{\partial}{\partial r}(P_\perp + P_\parallel) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}(P_\perp + P_\parallel) \right\rangle_m = \\ -i\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\left(\langle h\rangle_{m+1} - \langle h\rangle_{m-1}\right) - i\,\frac{m}{r}\left(\langle h\rangle_{m+1} + \langle h\rangle_{m-1}\right) \quad , \tag{7}$$

with h the θ-dependent velocity integral

$$h \equiv \int_{\mathcal{P}} \frac{\mathcal{E}\, d\mathcal{E}\, d\lambda}{v_\parallel} m_e \mathcal{E}(2-\lambda)\mathrm{g} + O(\epsilon) \quad . \tag{8}$$

In Eq. (7), the sum over parallel velocities has been included, so that subsequent calculations are restricted to $v_\parallel > 0$. Also, the velocity integration is clearly over all energies subject to the constraint $0 \le \lambda \le 1/(1+\epsilon)$. Passing the projection operator through the $\mathcal{E}$ and λ variables leaves a result in terms of transit averages of g:

$$\left\langle \frac{\mathrm{g}}{v_\parallel} \right\rangle_m = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{v_\parallel} e^{im\theta}\mathrm{g} \quad . \tag{9}$$

However, since the line element along a flux tube is $d\ell = qR_0 d\theta$, we can also write

$$\left\langle \frac{\mathrm{g}}{v_\parallel} \right\rangle_m = \frac{1}{2\pi q R_0}\oint dt\, e^{im\theta}\mathrm{g} \qquad \text{with} \qquad \tau \equiv \oint \frac{d\ell}{v_\parallel} = \oint dt \quad . \tag{10}$$

A straightforward integration yields an explicit expression for the orbit period,

$$\tau = \pi q R_0 \left(\frac{z}{\lambda\epsilon\mathcal{E}}\right)^{1/2} K(z) \qquad \text{with} \qquad z \equiv \frac{2\lambda\epsilon}{1-\lambda(1-\epsilon)} \quad , \tag{11}$$

and $0 \le z \le 1$. Also, K is a normalized elliptic integral:

$$K(z) \equiv \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{(1 - z\sin^2\theta)^{1/2}} \quad . \tag{12}$$

The relevant kinetic equation for g contains only the magnetic curvature drift, $\mathbf{v}_D \cdot \nabla\phi$, as a source term. For large-aspect-ratio, circular equilibria, this equation takes the form

$$\frac{\partial \mathrm{g}}{\partial t} + v_{\|}\frac{\partial \mathrm{g}}{\partial \ell} = \frac{f_e}{B_0 R_0}\frac{v^2}{v_e^2}\left(1+\frac{v_{\|}^2}{v^2}\right)\left(\sin\theta\frac{\partial \phi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial \phi}{\partial \theta}\right) \quad . \tag{13}$$

Since we are presumably away from gap regions where finite $E_{\|}$ effects are important, these have been ignored. The reader should be aware that similar corrections due to $E_{\|}$ may be important for trapped electron collisional damping, as indicated by Fu and Cheng [7]. A solution to Eq. (13) may be obtained by standard techniques [8]; first, we write the RHS as a Fourier series

$$\frac{\partial \mathrm{g}}{\partial t} + v_{\|}(\theta)\frac{\partial \mathrm{g}}{\partial \ell} = \sum_m Q_m(r,\theta)e^{i(n\varphi - m\theta - \omega t)} \quad , \tag{14}$$

so that a formal solution may be generated by integration along an unperturbed orbit:

$$\mathrm{g} = \int_{-\infty}^{t} dt' \sum_m Q_m(r,\theta')e^{i(n\varphi' - m\theta' - \omega t')} \quad . \tag{15}$$

It is instructive to consider the physical interpretation of the exponential factor, which is written as

$$\begin{aligned} n\varphi' - m\theta' - \omega t' = {} & \underbrace{n\varphi - m\theta - \omega t}_{\text{final coordinates}} + \underbrace{(nq-m)(\theta'-\theta)}_{\text{field line orbit}} \\ & - \underbrace{\omega(t'-t)}_{\text{time osc.}} + \underbrace{(nq-m)\,\omega_t(t'-t)}_{\text{orbit average}} - \underbrace{(nq-m)\,\omega_t(t'-t)}_{-\text{orbit average}} \end{aligned} \tag{16}$$

with $\omega_t = 2\pi/\tau$ the electron transit frequency. Substitution of this decomposition into the orbit integral yields

$$e^{im\theta}\mathrm{g} = e^{i(n\varphi - \omega t)}\sum_m Q_m(r)e^{-ix_m(\theta - \omega_t t)}\int_{-\infty}^{t'} dt' e^{i(t'-t)(x_m\omega_t - \omega)}e^{ix_m(\theta' - \omega_t t')} \quad . \tag{17}$$

An *exact* evaluation of this integral is possible by an expansion of the rightmost exponential term in a series of transit harmonics. For simplicity, however, we include only the lowest order (*i.e.*, transit-averaged) contribution. Since $nq - m \sim 0$ for the present case, we are first able to establish that

$$\frac{1}{\tau}\oint dt' e^{i(nq-m)(\theta' - \omega_t t')} \sim 1 \quad . \tag{18}$$

The resonant contribution from the orbit integral is then found by a simple time integration, and the transit average of g follows by a similar integration (with $\omega \ll \omega_t$):

$$\left\langle \frac{g}{v_\parallel} \right\rangle_m \sim e^{i(n\varphi-\omega t)} Q_m(r) \frac{\tau}{2qR_0} \delta\left[(nq-m)\omega_t - \omega\right] \quad . \tag{19}$$

The energy integral may be evaluated using the delta-function, which becomes

$$\delta\left[(nq-m)\omega_t - \omega\right] = \frac{2\overline{\tau}\mathcal{E}^{1/2}}{|nq-m|}\delta(\mathcal{E}-\mathcal{E}_0) \qquad \text{with} \qquad \tau \equiv \frac{2\pi}{\mathcal{E}^{1/2}}\overline{\tau} \quad . \tag{20}$$

In terms of the dimensionless variables $\zeta \equiv \omega/k_{\parallel m} v_e$, $\hat{\epsilon} \equiv 2\epsilon/(1-\epsilon)$ and $u \equiv z/\hat{\epsilon}$, we find

$$\langle h \rangle_m = -\frac{i\sqrt{\pi} n_e T_e}{2\omega B_0 R_0} \Phi(\zeta, \hat{\epsilon}) \left[\phi'_{m+1} - \phi'_{m-1} + \frac{m+1}{r}\phi_{m+1} + \frac{m-1}{r}\phi_{m-1} \right] \quad , \tag{21}$$

with a prime standing for a radial derivative. In Eq. (21), and in subsequent formulae, we supress the Fourier dependence $\exp[i(n\varphi - \omega t)]$ for compactness. Now, the RHS of Eq. (21) contains the somewhat complicated integral

$$\Phi(\zeta, \hat{\epsilon}) \sim \frac{\zeta^7}{2\sqrt{2}} \int_0^{1/\hat{\epsilon}} du\, (2+u)^2 K(\hat{\epsilon}u)^8 e^{-\zeta^2(1+u)K^2(\hat{\epsilon}u)} + O(\epsilon) \quad . \tag{22}$$

It should be apparent that the radial structure of Φ exhibits spikes with peaks just to the left and right of r_m^*, such that $q(r_m^*) = m/n$. Thus, to lowest order in $\omega R_0/v_e$, we only need to know the radial integral of Φ to calculate the effective jump in potential, and thus the damping. The contribution is twice the integral from the electron resonance surface to the nearest TAE surface (where $nq \sim m + 1/2$):

$$C(\hat{\epsilon}, v_A/v_e) \equiv 2 \int_{nq=m}^{nq=m+1/2} dr\, \Phi(\zeta, \epsilon) = \frac{\omega R_0}{v_e} \frac{r}{ns} G(\epsilon, v_A/v_e) \quad . \tag{23}$$

We remind the reader that local quantities are to be evaluated at the resonant surface r_m^*. The quantity G in Eq. (23) is

$$\begin{aligned} G(\hat{\epsilon}, v_A/v_e) \; &\equiv 2 \int_{v_A/v_e}^{\infty} \frac{d\zeta}{\zeta^2}\, \Phi(\zeta, \hat{\epsilon}) \\ &\sim \frac{1}{\sqrt{2}} \int_0^{1/\hat{\epsilon}} du\, \frac{(2+u)^2}{(1+u)^3} K(\hat{\epsilon}u)^2 \; + \; O\left(\frac{v_A}{v_e}\right)^2 \quad \text{if} \quad \hat{\epsilon} > \left(\frac{v_A}{v_e}\right)^2 . \end{aligned} \tag{24}$$

For the interval $\hat{\epsilon} \in (0.1, 1.0)$, G is nearly constant and to the indicated accuracy, independent of v_A/v_ϵ. The fit $G(\hat{\epsilon}) \simeq 4.470 - 0.424\,\hat{\epsilon} + 0.020\,\hat{\epsilon}^2$ gives a good approximation in this range. Collecting our results, we see that at r^*_{m+1} (where $nq = m+1$) the eigenmode equation is

$$\mathcal{L}_m\left(\frac{\phi_m}{r}\right) = \frac{ir^2}{4}\frac{\sqrt{\pi}\beta_e}{R_0^2}C'\left(\frac{m+1}{r}+\frac{\partial}{\partial r}\right)\left(\frac{m}{r}\phi_m - \frac{\partial\phi_m}{\partial r}\right)\delta(r-r^*_{m+1}) \quad , \tag{25}$$

while at r^*_{m-1} (where $nq = m-1$) it becomes

$$\mathcal{L}_m\left(\frac{\phi_m}{r}\right) = \frac{ir^2}{4}\frac{\sqrt{\pi}\beta_e}{R_0^2}C'\left(\frac{m-1}{r}-\frac{\partial}{\partial r}\right)\left(\frac{m}{r}\phi_m + \frac{\partial\phi_m}{\partial r}\right)\delta(r-r^*_{m-1}) \quad . \tag{26}$$

It is simple task to show that the RHS sides of both Eq. (25) and (26) are self-adjoint. Moreover, the simple form of the δ-function response allows us to obtain jump conditions for the poloidal harmonic ϕ_m. These are calculated in the usual manner by integrating across the resonance regions. At $r = r^*_{m+1}$ we find

$$\Delta\phi_m = -iJ\left[m\phi_m(r^*_{m+1}) - r^*_{m+1}\phi'(r^*_{m+1})\right] \quad , \tag{27}$$

$$\Delta\phi'_m = \frac{m}{r^*_{m+1}}\Delta\phi_m \quad . \tag{28}$$

The corresponding jump at r^*_{m-1} is obtained by replacing $m \to -m$ and $r^*_{m+1} \to r^*_{m-1}$ in Eqs. (27) and (28). In Eq. (27), the coefficient J is

$$J = \frac{\sqrt{\pi}}{4}q^2\beta_e\frac{\omega R_0}{v_\epsilon}\frac{G(\epsilon)}{ns}\frac{1}{1-\omega^2/\omega_A^2} \quad . \tag{29}$$

Unfortunately, the story does not end here. To determine the damping rate, one must have some model with which to calculate a toroidal eigenfunction — with the above jumps included. This is in sharp contrast to the ion Landau damping, which is independent of the shape of the eigenfunction so long as it is radially localized.

3. Weak-Shear Damping

It is possible to obtain an analytic formula for the damping when $\epsilon \ll s \ll 1$ and $m \gg 1$. In this limit, the outer equation reduces to a form containing only the local value of magnetic shear as a parameter:

$$\frac{d}{dx_m}\left(x_m^2-\frac{1}{4}\right)\frac{d\phi_m}{dx_m}-\frac{1}{s^2}\left(x_m^2-\frac{1}{4}\right)\phi_m=0\quad . \tag{30}$$

Concentrating on the region $x_m > 1/2$, we set $x_m \equiv (1+su)/2$. Since the shear is small, the mode is localized within the region $u \sim O(1)$, and decays exponentially for $su \sim O(1)$. To lowest order in s, Eq. (30) becomes

$$u\ddot{F}+(1-u)\dot{F}-\frac{1}{2}F=0 \qquad \text{with} \qquad \phi=e^{-u/2}F(u)\quad , \tag{31}$$

where a dot has been used to indicate a derivative with respect to u, and m-subscripts have been omitted from all quantities. The solutions of Eq. (31) are confluent hypergeometric functions [9], in terms of which the potential has the independent solutions

$$\phi=\begin{cases}\epsilon^{-u/2}U(1/2,1,u)\to e^{-u/2}/\sqrt{u} \ \text{ as } \ u\to\infty\quad ,\\ \epsilon^{-u/2}M(1/2,1,u)\to e^{u/2}/\sqrt{\pi u} \ \text{ as } \ u\to\infty\quad .\end{cases} \tag{32}$$

The jumps at $u = 1/s$ have a particularly simple form in u-space when $m \gg 1$; namely $\Delta\phi = -imJ(\phi - 2\dot{\phi})$ and $\Delta\dot{\phi} = \Delta\phi/2$. Taking $\phi_> = \exp(-u/2)\,U$ and $\phi_< = \exp(-u/2)\,[AU + BM]$, and applying the u-space jump conditions, we find

$$A=1-2imJ\,\frac{(\dot{M}-M)(\dot{U}-U)}{W(U,M)} \qquad \text{and} \qquad B=2imJ\,\frac{(\dot{U}-U)^2}{W(U,M)}\quad , \tag{33}$$

with W the Wronskian derivative. After some additional algebra, it can be shown that as $u \to 0$ the potential exhibits the usual logarithmic divergence plus a residual constant (with $\psi(z)$ the digamma function): $\phi \sim \log u + \psi(1/2) - 2\pi imJ\exp(-1/s)$. What is important in this result is the imaginary term which appears as a consequence of the jump. Indeed, when added to the parameter Δ_∞ of Ref. [2], this term modifies the (algebraic) TAE eigenvalue equation [2] to give:

$$\frac{\hat{\gamma}_e}{\omega}=-\frac{\pi^{3/2}}{6}q^2\beta_e\frac{v_A}{v_e}\left(\frac{5}{2}\epsilon\right)G(\hat{\epsilon})e^{-1/s}\quad , \tag{34}$$

where we have set $\omega = v_A/2qR_0$ on the RHS. That the damping is exponentially small in this case is a direct consequence of the mode itself becoming exponentially small before crossing the electron resonance point. Although this pathology made the analytic calculation possible, it is not necessarily the rule. In general, one can expect some poloidal harmonics to have an appreciable amplitude near the $k_\parallel = 0$ surface. However, by comparison with Eq. (37) of Ref. [1], it is apparent that the electron damping becomes asymptotically *larger* than than the continuum damping as $s \to 0$.

4. Summary

A technique for the calculation of electron Landau damping of TAE modes in reactor-regime plasmas has been derived. Some time ago, the same method was used to estimate the damping in ITER-like plasmas — in conjunction with a boundary layer model of the TAE [10]. More generally, the jump prescription developed herein can be incorporated into a variety of numerical or analytical TAE models, and also to models of the ellipticity- and triangularity-induced Alfvén eigenmodes (EAE, NAE), so long as the aspect ratio is large, the resonance points are sufficiently far from the Alfvén continuum (*i.e.*, $\omega = \omega_A$), and the electron thermal velocity satisfies $v_e/v_A < \sqrt{2\epsilon}$.

Acknowledgements

The author gratefully acknowledges an early proofreading of the manuscript by Dr. S. Sharapov, and also financial support by the Natural Sciences and Engineering Research Council of Canada.

References

[1] Rosenbluth M N, Berk H L, Van Dam J W, and Lindberg D M 1992 Phys. Fluids B **4** 2189

[2] Candy J and Rosenbluth M N 1993 Plasma Phys. and Contr. Fusion **35** 957

[3] Candy J and Rosenbluth M N 1994 Phys. Plasmas **1** 356

[4] Candy J and Rosenbluth M N 1995 Nucl. Fusion **35** 1069

[5] Betti R and Freidberg J P 1992 Phys. Fluids B **4** 1465

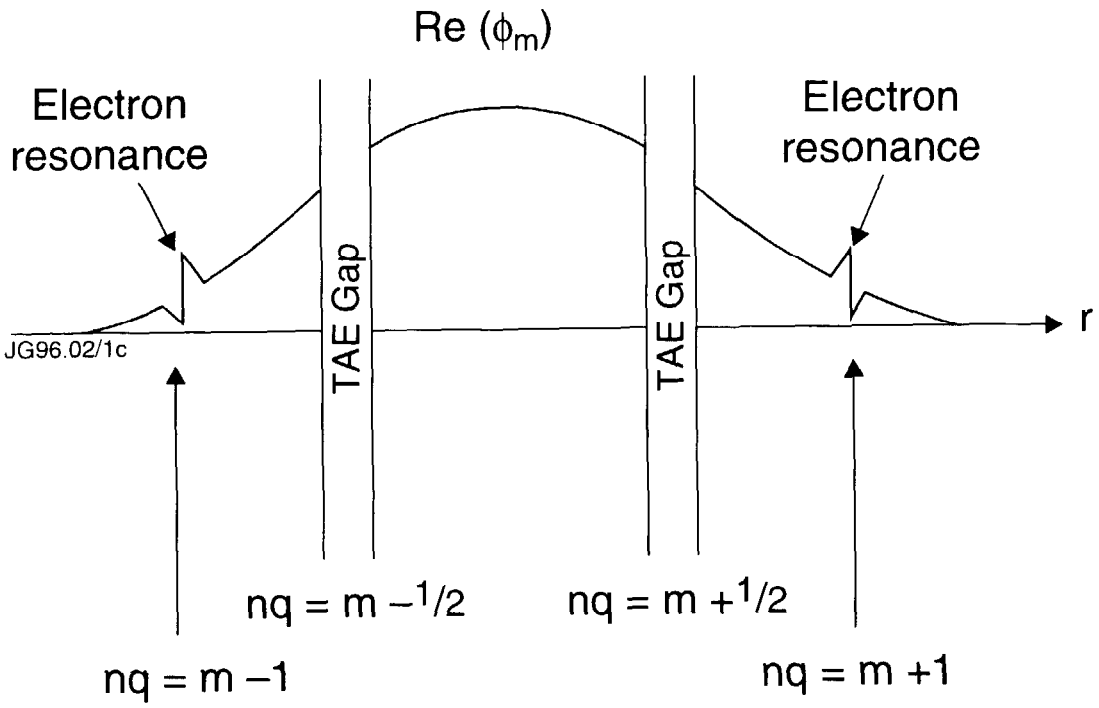
[6] Rosenbluth M N 1994 private communication

[7] Fu G Y and Cheng C Z 1992 Phys. Fluids B **4** 3722

[8] Fu G Y, Cheng C Z and Wong K L 1993 Phys. Fluids B **5** 4040

[9] Slater L. in *Handbook of Mathematical Functions*, edited by M. Abramowitz and I.A. Stegun (U.S. Government Printing Office, Washington, D.C., 1972) Chap. 13

[10] Putvinski S *et al* 1994 *Alpha Particle Physics for ITER* (IAEA/CN/60 E–P–4)



[1] Illustration of eigenmode structure, as well as TAE gap and electron resonance layer locations.