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Low Temperature Models of Non-linear Wave Mixing in Plasmas

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Abstract

The general kinetic description of two wave interaction in plasmas, obtained by perturbation expansion of the kinetic equation, is discussed along with various approximations. The general expression involves the full detail of the distribution function perturbations, f^{a} and f^{b}, of the two interacting waves, a and b, and is at present numerically intractable. Replacing the general expression by a truncated expansion in velocity space the low temperature kinetic (LTK) model is obtained. This model, which involves only the zeroth and first moments of f^{a} and f^{b}, is valid when thermal effects are small for the scattered wave but makes no assumptions about the interacting waves. Alternatively the LTK model may be derived from a truncated sequence of moments equations. The effect of truncating at different moments is discussed. To ensure that the limitations implicit in the truncated moments model apply only to the scattered wave and not to the interacting waves, it is found that the perturbation expansion must be carried out in variables which, like the electron flux, are linear in the distribution function, f. If the perturbation expansion is carried out in fluid velocity, which is non–linear in f, then the limitations of the truncated model apply also to the interacting waves. This is the principal cause of the limited range of applicability of the traditional cold fluid model of scattering. Finally the consequences of linear dependencies and cross correlations between fields and moments are considered.

1 Kinetic description of two wave interaction

The behavior of electromagnetic waves in plasmas are described by Maxwells equations

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \tag{1a}$$

$$\nabla \times \mathbf{B} = \mu_0 (\varepsilon_0 \partial_t \mathbf{E} + \mathbf{j}) \ , \tag{1b}$$

where $\mathbf{j}$ is the plasma current,

$$\mathbf{j} = q_e \int \mathbf{v} f \, d\mathbf{p} \ , \tag{2}$$

and $f = \sum_i \delta(\mathbf{x} - \mathbf{x}_i(t))$ ($\mathbf{x} = (\mathbf{r}, \mathbf{p})$), the microscopic distribution function, is governed by the kinetic equation (Klimontovich, 1982, Section 24)

$$\mathcal{L} f = -\mathbf{F} \cdot \partial_{\mathbf{p}} f \ . \tag{3}$$

Here

$$\mathcal{L} = \partial_t + \mathbf{v} \cdot \partial_{\mathbf{r}} + q_e (\mathbf{v} \times \mathbf{B}^{(0)}) \cdot \partial_{\mathbf{p}} \ , \tag{4}$$

is the linear operator representing the total derivative along unperturbed orbits in phase space, $\mathbf{B}^{(0)}$ is the static magnetic field and

$$\mathbf{F} = q_e \left(\mathbf{E} + \mathbf{v} \times (\mathbf{B} - \mathbf{B}^{(0)}) \right)$$

is the force electromagnetic fields exert on an electron. It is assumed that there is no static electric field. For simplicity we have only included the electron contribution to the plasma current. Where needed the expressions are of course readily generalized to include contributions from ions.

This system of equations is bilinear due to the term on the right hand side of (3), which causes waves in plasmas to interact. The fact that the nonlinearity is limited to second order implies that the interaction of a larger number of waves can be considered as a sequence of two wave interactions. Integrating (3) along the characteristics of $\mathcal{L}$ (the unperturbed orbits) and inserting the resulting right hand side in (2) we find the plasma current $\mathbf{j}$ expressed as a linear integral operator, σ, acting on the bilinear term, $\mathbf{F} \cdot \partial_{\mathbf{p}} f$:

$$j_i = \sigma_i \{ \mathbf{F} \cdot \partial_{\mathbf{p}} f \} \ . \tag{5}$$

Eliminating $\mathbf{B}$ from Maxwell's equations we obtain the wave equation

$$\Lambda_{ij} \mathbf{E}_j = \frac{-i}{\omega \varepsilon_0} j_i^\sigma \tag{6a}$$

$$\Lambda_{ij} = \delta_{ij} + \chi_{ij} + N^2 \left\{ \widehat{k}_i \widehat{k}_j - \delta_{ij} \right\} \ , \quad N = \frac{kc}{w} \ , \tag{6b}$$

where the susceptibility, χ_{ij}, by its relation to the coductivity, σ_{ij}, accounts for the plasma current resulting from the fields acting on the equilibrium distribution, $f^{(0)}$,

$$-i\omega\varepsilon_0 \chi_{ij} \mathbf{E}_j = \sigma_{ij} \mathbf{E}_j = \sigma_i \{ \mathbf{F} \cdot \partial_{\mathbf{p}} f^{(0)} \} \ ,$$

and the *source current*, j^σ, is due to the interaction of the fields with the distribution function perturbations

$$j_i^\sigma = \sigma_i\{\mathbf{F}\cdot\partial_{\mathbf{p}}(f - f^{(0)})\}\,. \tag{7}$$

The inhomogeneous wave equation (6a) together with the expression for the source current (7) and the kinetic equation (3) completely describe wave interactions in plasmas. Assuming now that the bilinear interactions represented by the source current (7) are weak we can usefully approximate the solution with a perturbation expansion. Taking this expansion to second order we describe the bilinear interaction of two waves a and b in the first Born approximation. The field, $\mathbf{E}^{\mathrm{s}}$, of the resulting scattered wave s is then given by

$$\Lambda_{ij}\mathbf{E}_j^{\mathrm{s}} = \frac{-i}{\omega\varepsilon_0}j_i^\sigma \tag{8a}$$

$$j_i^\sigma = \sigma_i\{\mathbf{F}^{\mathrm{a}}\cdot\partial_{\mathbf{p}}f^{\mathrm{b}} + \mathbf{F}^{\mathrm{b}}\cdot\partial_{\mathbf{p}}f^{\mathrm{a}}\}\,. \tag{8b}$$

This set of equations describe the bilinear wave interaction of arbitrary waves a and b and makes no assumptions about s other than that its amplitude must be much smaller than those of a and b. If this is not an acceptable assumption higher order Born approximations may be obtained by considering a sequence of bilinear interactions involving scattered waves. Although it is straight forward to write down the expression for the operator σ (see for instance Sitenko and Kirochkin (1966) and Bindslev (1993)) it remains numerically intractable.

2 Low Temperature Kinetic model

The complexity of the kinetic expression for the source current (8b) is connected with the fact that it involves the full detail of the distribution function perturbations, f^{a} and f^{b}, associated with the interacting waves. Its complexity can, however, be significantly reduced by assuming that thermal effects are small at the frequency and wave vector, $(\mathbf{k}^{\mathrm{s}}, \omega^{\mathrm{s}})$, of the *scattered* wave, or more precisely that $\left\langle \tilde{f}^\sigma \tilde{f}^\sigma \right\rangle$ is significant only for values of $\mathbf{v}$ which satisfy the inequalities

$$\frac{v}{c} \ll 1\,; \qquad \frac{v_\| k_\|^\sigma}{\omega^\sigma + s\omega_c} \ll 1\,, \quad s \in Z\,; \qquad \frac{v_\perp k_\perp^\sigma}{\omega_c} \ll 1\,, \tag{9}$$

where $\omega_c = -q_e B^{(0)}/m_e$ and $\|$ and $\perp$ indicate components parallel and perpendicular to $\mathbf{B}^{(0)}$. With these assumptions the expression for the source current reduces to a numerically tractable expression which involves only the zeroth and the first order moments of the distribution perturbations associated with waves a and b (Bindslev, 1993),

$$j_i^\sigma = \frac{\sigma_{il}^\sigma}{n^{(0)}}\left\{ n^{\mathrm{b}}E_l^{\mathrm{a}} + n^{\mathrm{a}}E_l^{\mathrm{b}} + \epsilon_{lmn}\left(\overline{v}_m^{\mathrm{b}}B_n^{\mathrm{a}} + \overline{v}_m^{\mathrm{a}}B_n^{\mathrm{b}}\right) + Y_{lmn}^\sigma \frac{1}{c}\left(\overline{v}_m^{\mathrm{b}}E_n^{\mathrm{a}} + \overline{v}_m^{\mathrm{a}}E_n^{\mathrm{b}}\right) \right\}\,. \tag{10}$$

Here

$$n^\alpha = \int f^\alpha \, d\mathbf{p}\,, \tag{11a}$$

$$\overline{v}_j^\alpha = \int v_j f^\alpha \, d\mathbf{p}\,, \quad \alpha = \mathrm{a}, \mathrm{b}\,, \tag{11b}$$

and σ^σ is the cold electron conductivity at the frequency of the scattered wave, while

$$Y^\sigma_{imn} = \frac{k^\sigma_j c}{\omega^\sigma} \Gamma^{-1}_{ijab} (\delta_{an}\delta_{bm} + \delta_{am}\delta_{bn}) \,, \tag{12}$$

with Γ^{-1}_{ijab} defined by

$$\begin{aligned} \Gamma^{-1}_{ijab} &= \frac{1}{1-4\Omega^2} \left[-i\Omega \left(\eta_{ia}\xi_{jb} + \xi_{ia}\eta_{jb} \right) - 2\Omega^2 \xi_{ia}\xi_{jb} + (1-2\Omega^2)\eta_{ia}\eta_{jb} \right] \\ &+ \frac{1}{1-\Omega^2} \Big[\eta_{ia}\zeta_{jb} + \zeta_{ia}\eta_{jb} - i\Omega \big(\zeta_{ia}\xi_{jb} + \xi_{ia}\zeta_{jb} \big) \Big] \; + \; \zeta_{ia}\zeta_{jb} \,, \end{aligned} \tag{13}$$

and

$$\Omega = \omega_c / \omega^\sigma \,, \quad \eta_{ij} = \delta_{ij} - \delta_{i3}\delta_{j3} \,, \quad \xi_{ij} = \epsilon_{ij3} \,, \quad \zeta_{ij} = \delta_{i3}\delta_{j3} \,.$$

It is remarkable that although the low temperature kinetic (LTK) expression for the source current (10) involves only the zeroth and first order moments of $f^{\rm a}$ and $f^{\rm b}$ it nonetheless introduces no new assumptions about the nature of the interacting waves a and b relative to the full kinetic expression which involves the full detail of $f^{\rm a}$ and $f^{\rm b}$.

Till this point the source current expression has been symmetrical with respect the interacting waves a and b. We now break this symmetry by assuming that the plasma dielectric response is cold at the frequency and wave vector $(\mathbf{k}^{\rm i}, \omega^{\rm i})$ of one of the interacting waves, hence forth referred to as the incident wave, i, while maintaining the generality for the other wave which we will refer to as the fluctuations, δ.

This permits us to use the cold plasma relations to express $\mathbf{B}^{\rm i}$, $\mathbf{j}^{\rm i}$ and $n^{\rm i}$ all in terms of $\mathbf{E}^{\rm i}$. We then find that the source current is given by (Bindslev, 1993)

$$\begin{aligned} j^\sigma_i &= \frac{\sigma^\sigma_{ij}}{n^{(0)}} E^{\rm i}_l \left\{ \delta_{jl} n^\delta + \frac{1}{c} \left(\epsilon_{jkm}\epsilon_{mnl} N^{\rm i}_n + Y^\sigma_{jkl} \frac{1}{c} \right) \overline{v}^\delta_k \right. \\ & \left. + \frac{\sigma^{\rm i}_{ml}}{q_e} \left[\epsilon_{jmk} B^\delta_k + \frac{1}{c} \left(\delta_{jk} N^{\rm i}_m + Y^\sigma_{jmk} \right) E^\delta_k \right] \right\} , \end{aligned} \tag{14}$$

where $\sigma^{\rm i}$ is the cold electron conductivity at the frequency of the incident wave and $N^{\rm i}_j = k^{\rm i}_j c / \omega^{\rm i}$ is the refractive index of the incident wave.

In a wide range of scattering experiments in laboratory plasmas the frequencies of the incident and the scattered waves are sufficiently high that the plasma can be treated as cold for these waves. The phase velocity of the observed fluctuations are, however, typically of the same order or smaller than the thermal velocities of the electrons and uncorrelated motion may also have to be taken into account, both of which makes it unacceptable to regard the fluctuations as cold collective oscillations. In this case we can make use of the low temperature kinetic model of scattering with the source current given by (14).

3 Moments models

The fact that the low temperature kinetic expressions for the source current, (10) and (14), involve only zeroth and first order moments of the distribution functions of the two interacting waves suggests that the source current may be derived with the same generality from a sequence of moments of the kinetic equation (3). The simplest set of equations we will consider consists of the zeroth and the first moments of (3). In order to truncate the sequence of connected moments equations we assume that the second moment of the distribution function,

$$\overline{v_i v_j} = \int v_i v_j f \, d\mathbf{p} \ , \tag{15}$$

is related to the lower moments as

$$\overline{v_i v_j} = \frac{\overline{v}_i \overline{v}_j}{n} \, . \tag{16}$$

This is in fact the cold fluid model which Sitenko and others used for deriving expressions for the source current (see e.g. Sitenko (1967)). In these derivations the solution was found by a perturbation expansion of the fluid velocity $u_i = \overline{v}_i / n$. The result is

$$\begin{aligned} j_i^\sigma \;&=\; \frac{1}{n^{(0)}} \left(\sigma_{ij}^{\mathrm{a}} E_j^{\mathrm{a}} n^{\mathrm{b}} + \sigma_{ij}^{\mathrm{b}} E_j^{\mathrm{b}} n^{\mathrm{a}} \right) \,+\, \sigma_{il}^{\sigma} \epsilon_{lmn} \left(u_m^{\mathrm{a}} B_n^{\mathrm{b}} + u_m^{\mathrm{b}} B_n^{\mathrm{a}} \right) \\ &\quad - \frac{i m_e}{q_e} \sigma_{il}^{\sigma} \left(u_l^{\mathrm{a}} k_j^{\mathrm{a}} u_j^{\mathrm{b}} + u_l^{\mathrm{b}} k_j^{\mathrm{b}} u_j^{\mathrm{a}} \right) \ . \end{aligned} \tag{17}$$

Replacing a and b by i and δ and then using the linearized continuity equation, the linearized momentum equation, $u_i^{\mathrm{i}} = \sigma_{ij}^{\mathrm{i}} E_j^{\mathrm{i}} / (q_e n^{(0)})$, and Faraday's law, all for i, expression (17) can be written as

$$\begin{aligned} j_i^\sigma \;&=\; \frac{1}{n^{(0)}} \sigma_{ij}^{\mathrm{i}} E_j^{\mathrm{i}} n^{\delta} + \frac{1}{n^{(0)} q_e} \sigma_{ij}^{\sigma} \epsilon_{jkm} \sigma_{kl}^{\mathrm{i}} E_l^{\mathrm{i}} B_m^{\delta} \\ &\quad + \frac{1}{c} \left\{ \sigma_{ij}^{\sigma} \left[N_j^{\mathrm{i}} \delta_{lm} - \delta_{jl} N_m^{\mathrm{i}} - \frac{\omega^{\mathrm{i}}}{c \omega_p^2} \chi_{kl}^{\mathrm{i}} \left(\delta_{jk} k_m^{\mathrm{i}} + \delta_{jm} k_k^{\delta} \right) \right] + \sigma_{kl}^{\mathrm{i}} N_k^{\mathrm{i}} \delta_{im} \right\} E_l^{\mathrm{i}} u_m^{\delta} \ , \end{aligned} \tag{18}$$

where $\omega_p^2 = n^{(0)} q_e^2 / (m_e \varepsilon_0)$. This expression for the source current of the scattered field in a magnetized plasma is identical to expression (20.3) in AKHIEZER *et al.* (1967) and expression (11.5) in SITENKO (1967), except that we have used SI units. This model is accurate in the limit where the two interacting waves (in the case of scattering, the incident wave and the fluctuations) and the resulting scattered wave all are cold collective oscillations (Sitenko, 1995). In this limit the predictions of the cold fluid model coincide with the predictions of the low temperature kinetic model. When one or both the interacting waves are not cold collective oscillations, as in the case of scattering from thermal fluctuations, then the cold fluid expressions (17) and (18) are not guaranteed to be accurate (Bindslev 1993), indeed large errors may occur for collective scattering experiments in fusion plasmas (Bindslev, 1995). For these situations the low temperature kinetic model given by expressions (10) and (14) should be used instead.

In Bindslev (1993) we noted that the failure of the cold fluid model in describing the interaction of arbitrary waves is due to the fact that the set of equations is solved by expansion in electron fluid velocity rater than electron flux and that the sequence of moments equations was truncated too early by assuming (16). We further showed that if the sequence of moments equations is truncated by assuming instead that the third moment of the distribution is zero,

$$\overline{v_i v_j v_k} = \int v_i v_j v_k f \, d\mathbf{p} = 0 \,, \tag{19}$$

so that the second moment $\overline{v_i v_j}$ is given by the second moment equation, and replacing the expansion in fluid velocity with an expansion in flux we recover the low temperature kinetic expressions (10) and (14).

We will now consider in turn the effect of the early truncation (16) and the expansion in fluid velocity. Carrying out the expansion in flux but applying the early truncation (16) we have from Bindslev (1993, equation 96)

$$j_i^\sigma = \frac{\sigma_{il}^\sigma}{n^{(0)}} \left\{ n^{\mathrm{b}} E_l^{\mathrm{a}} + n^{\mathrm{a}} E_l^{\mathrm{b}} + \epsilon_{lmn} \left(\overline{v}_m^{\mathrm{b}} B_n^{\mathrm{a}} + \overline{v}_m^{\mathrm{a}} B_n^{\mathrm{b}} \right) - \frac{i k_j^\sigma m_e}{n^{(0)} q_e} \left(\overline{v}_l^{\mathrm{a}} \overline{v}_j^{\mathrm{b}} + \overline{v}_l^{\mathrm{b}} \overline{v}_j^{\mathrm{a}} \right) \right\} . \tag{20}$$

The derivation of the low temperature kinetic expression for the source current (10) is based on an expansion in powers of velocity of the kernel of the integral operator σ_i (c.f. equation (8b)). The zeroth order terms of this expansion led to the terms $n^{\mathrm{b}} E_l^{\mathrm{a}} + n^{\mathrm{a}} E_l^{\mathrm{b}} + \epsilon_{lmn} \left(\overline{v}_m^{\mathrm{b}} B_n^{\mathrm{a}} + \overline{v}_m^{\mathrm{a}} B_n^{\mathrm{b}} \right)$ in (10), which we note have been recovered in (20). The first order terms in the expansion of the kernel of σ_i led to the term $Y_{lmn}^\sigma \left(\overline{v}_m^{\mathrm{b}} E_n^{\mathrm{a}} + \overline{v}_m^{\mathrm{a}} E_n^{\mathrm{b}} \right) / c$ which is not recovered in (20). We thus find that the difference between (20) and (10) is of first order in the small variables introduced in (9). Therefore, provided the conditions imposed by the inequalities in (9) are sufficiently well satisfied that the first order terms can be neglected the expression (20) like (10) is valid for arbitrary interacting waves a and b. When first order terms cannot be neglected (20) and (10) coincide in the limit where a and b are cold collective oscillations but not for arbitrary a and b.

To see the effect of expanding in fluid velocity rather than flux we turn now to comparing (20) with the cold fluid result (17). First we note that for arbitrary fluctuations the two expressions differ even in the lowest order terms. Making use of the relation $\mathbf{k}^\sigma = \mathbf{k}^{\mathrm{a}} + \mathbf{k}^{\mathrm{b}}$ we can rewrite (17) on the form

$$\begin{aligned} j_i^\sigma = \frac{\sigma_{il}^\sigma}{n^{(0)}} \Bigg\{ & \frac{\Pi_{lj}^\sigma}{i\omega_p^2} \left(\omega^{\mathrm{a}} \chi_{jk}^{\mathrm{a}} E_k^{\mathrm{a}} n^{\mathrm{b}} + \omega^{\mathrm{b}} \chi_{jk}^{\mathrm{b}} E_k^{\mathrm{b}} n^{\mathrm{a}} \right) + \frac{i m_e}{q_e n^{(0)}} \left(\overline{v}_l^{\mathrm{a}} k_j^{\mathrm{b}} v_j^{\mathrm{b}} + \overline{v}_l^{\mathrm{b}} k_j^{\mathrm{a}} v_j^{\mathrm{a}} \right) \\ & + \epsilon_{lmn} \left(\overline{v}_m^{\mathrm{b}} B_n^{\mathrm{a}} + \overline{v}_m^{\mathrm{a}} B_n^{\mathrm{b}} \right) - \frac{i k_j^\sigma m_e}{n^{(0)} q_e} \left(\overline{v}_l^{\mathrm{a}} \overline{v}_j^{\mathrm{b}} + \overline{v}_l^{\mathrm{b}} \overline{v}_j^{\mathrm{a}} \right) \Bigg\} , \end{aligned} \tag{21}$$

where the tensor Π_{ij}^σ has the form

$$\Pi_{ij}^\alpha = -i\omega^\alpha \delta_{ij} + \omega_c \epsilon_{ijk} \hat{B}_k^{(0)} \,, \qquad \hat{\mathbf{B}}^{(0)} = \mathbf{B}^{(0)} / |\mathbf{B}^{(0)}| \,. \tag{22}$$

The tensor $(-i\omega^\alpha/\omega_p^2)\mathbf{\Pi}^\alpha$ is the inverse of the susceptibility tensor associated with the wave α:

$$\frac{\omega^\alpha}{i\omega_p^2} \Pi_{ij}^\alpha \chi_{jk}^\alpha = \delta_{ik} \,. \tag{23}$$

In the limit where both interacting waves a and b are cold collective oscillations their fields and fluid velocities are related as

$$\mathbf{E}_i^{\alpha} = \frac{q_e}{\varepsilon_0 \omega_p^2} \Pi_{ij}^{\alpha} \overline{v}_j^{\alpha} , \quad \alpha = \mathrm{a}, \mathrm{b} , \tag{24}$$

and we have

$$\begin{aligned} \frac{1}{i\omega_p^2} \Pi_{lj}^{\sigma} \omega^{\mathrm{a}} \chi_{jk}^{\mathrm{a}} E_k^{\mathrm{a}} n^{\mathrm{b}} + \frac{i m_e}{q_e n^{(0)}} \overline{v}_l^{\mathrm{a}} k_j^{\mathrm{b}} \overline{v}_j^{\mathrm{b}} &= \frac{q_e}{\varepsilon_0 \omega_p^2} \left(\Pi_{lj}^{\sigma} + i\omega^{\mathrm{b}} \delta_{lj} \right) \overline{v}_j^{\mathrm{a}} n^{\mathrm{b}} \\ &= \frac{q_e}{\varepsilon_0 \omega_p^2} \Pi_{lj}^{\mathrm{a}} \overline{v}_j^{\mathrm{a}} n^{\mathrm{b}} \\ &= E_i^{\mathrm{a}} n^{\mathrm{b}} \end{aligned} \tag{25}$$

and similarly with a and b interchanged. When these relations hold, (17) and (20) coincide. It thus appears that the effect of expanding in fluid velocity rather than flux is to limit the range of validity of even the lowest order terms to the interaction of cold collective oscillations.

To understand where the difference between the two expansions arises we observe that the flux is linearly related to the distribution perturbation f while the fluid velocity is non-linear in f. Thus the fluid velocity perturbation, $\mathbf{u}^{\sigma}$, with frequency and wave vector of the scattered wave involves also the velocity distribution perturbations associated with the interacting waves, a and b:

$$\mathbf{u}^{\sigma} = \frac{\mathbf{v}^{\sigma}}{n^{(0)}} - \frac{\mathbf{v}^{\mathrm{a}} n^{\mathrm{b}} + \mathbf{v}^{\mathrm{b}} n^{\mathrm{a}}}{(n^{(0)})^2} . \tag{26}$$

By perturbation expansion of the first moment equation in $\overline{\mathbf{v}}$ we find that the flux associated with the scattered wave satisfies the relation

$$\begin{aligned} &\partial_t \overline{v}_i^{\sigma} - \frac{q_e}{m_e} (n^{(0)} E_i^{\sigma} + \epsilon_{ijk} \overline{v}_j^{\sigma} B_k^{(0)}) \\ &= -\partial_{r_j} \overline{v_i v_j}^{\sigma} + \frac{q_e}{m_e} \left[n^{\mathrm{a}} E_i^{\mathrm{b}} + n^{\mathrm{b}} E_i^{\mathrm{a}} + \epsilon_{ijk} \left(\overline{v}_j^{\mathrm{a}} B_k^{\mathrm{b}} + \overline{v}_j^{\mathrm{b}} B_k^{\mathrm{a}} \right) \right] \end{aligned} \tag{27}$$

This equation can also be obtained as the first moment of

$$\mathcal{L} f^{\sigma} = -\mathbf{F}^{\sigma} \cdot \partial_{\mathbf{p}} f^{(0)} - \mathbf{F}^{\mathrm{a}} \cdot \partial_{\mathbf{p}} f^{\mathrm{b}} - \mathbf{F}^{\mathrm{b}} \cdot \partial_{\mathbf{p}} f^{\mathrm{a}} \tag{28}$$

which is the second order perturbation of the kinetic equation (3). When solving (27) with (16) for the scattered wave instead of (28) we approximate the detailed behavior of f^{σ} by the behavior of its lowest moments and thereby introduce the low temperature approximation for the scattered wave. This does not prevent us from letting the interacting waves be governed by the first order kinetic equation

$$\mathcal{L} f^{\alpha} = -\mathbf{F}^{\alpha} \cdot \partial_{\mathbf{p}} f^{(0)} , \quad \alpha = \mathrm{a}, \mathrm{b} . \tag{29}$$

In contrast to this, when expanding in $\mathbf{u}$ we find that the fluid velocity perturbation with the frequency and wave vector of the scattered wave is given by

$$\partial_t u_i^{\sigma} - \frac{q_e}{m_e} (E_i^{\sigma} + \epsilon_{ijk} u_j^{\sigma} B_k^{(0)}) = -\frac{1}{n^{(0)}} \left(u_k^{\mathrm{a}} \partial_{r_k} u_i^{\mathrm{b}} + u_k^{\mathrm{b}} \partial_{r_k} u_i^{\mathrm{a}} \right) + \frac{q_e \epsilon_{ijk}}{m_e} \left(\overline{v}_j^{\mathrm{a}} B_k^{\mathrm{b}} + \overline{v}_j^{\mathrm{b}} B_k^{\mathrm{a}} \right) \tag{30}$$

from which the cold fluid source current is derived. Expression (30), unlike (27), does not in general permit the interacting waves to be governed by (29). To see this we multiply (30) by $n^{(0)}$, subtract (27) and make use of (16) to find

$$\begin{aligned} & n^{\mathrm{b}} \left\{ \partial_t \overline{v}_i^{\mathrm{a}} - \frac{q_e}{m_e} \left(n^{(0)} E_i^{\mathrm{a}} + \epsilon_{ijk} \overline{v}_j^{\mathrm{a}} B_k^{(0)} \right) \right\} \\ + \ & n^{\mathrm{a}} \left\{ \partial_t \overline{v}_i^{\mathrm{b}} - \frac{q_e}{m_e} \left(n^{(0)} E_i^{\mathrm{b}} + \epsilon_{ijk} \overline{v}_j^{\mathrm{b}} B_k^{(0)} \right) \right\} = 0 \, . \end{aligned} \tag{31}$$

As n^{a} and n^{b} are independent the identity in (31) generally only holds when the contents of the curly brackets both are identically zero,

$$\partial_t \overline{v}_i^{\alpha} - \frac{q_e}{m_e} \left(n^{(0)} E_i^{\alpha} + \epsilon_{ijk} \overline{v}_j^{\alpha} B_k^{(0)} \right) = 0 \, , \quad \alpha = \mathrm{a}, \, \mathrm{b} \, . \tag{32}$$

(32) is the truncated first moment of (29) which holds only for cold collective oscillations. Hence we see that expanding in $\mathbf{u}$ rather than $\overline{\mathbf{v}}$ has the consequence of imposing the approximation implicit in the truncated sequence of moments equations not only on the scattered wave, but also on the two interacting waves. It is possible to show the more general result that if the expansion is carried out in a set of variables which is non–linear in f and the kinetic equation is approximated e.g. by a truncated sequence of moments then the limitations implicit in this approximation apply not only to the scattered wave but also to the interacting waves, while the limitations apply only to the scattered wave if the expansion is linear in f. It therefore matters which set of variables the expansion is carried out in, except when the complete kinetic model is applied.

4 Fluctuation correlations and linear dependencies

All the low temperature expressions presented here for source currents resulting from the interaction of a cold (high frequency) incident wave with arbitrary (low frequency) fluctuations can be written on the form

$$j_i^{\sigma} = \widehat{\mathrm{C}}_i^{(n)} n^{\delta} + \widehat{\mathrm{C}}_{ij}^{(v)} \overline{v}_j^{\delta} + \widehat{\mathrm{C}}_{ij}^{(B)} B_j^{\delta} + \widehat{\mathrm{C}}_{ij}^{(E)} E_j^{\delta} \, , \tag{33}$$

which appears to suggest that the ten variables included here to characterize the fluctuations are independent. This is not the case. Some of the fluctuations are linearly dependent and all the fluctuations are correlated to some extent. As an example we note that cross correlations between density and magnetic field fluctuations may reduce the predicted scattered power in present microwave scattering experiments in fusion plasmas by more than an order of magnitude (Bindslev, 1994 b).

In general the electron density and flux fluctuations are due to the collective motion and the uncorrelated motion of electrons. The density fluctuations can of course be expressed in terms of the flux fluctuations. The field fluctuations can be expressed in terms of the collective electron flux but not in terms of the flux associated with the uncorrelated motion of electrons because the effect the latter has on fields cannot be distinguished from the effect uncorrelated ion motion has. Consequently we can only attribute the scattering from field fluctuations to scattering from flux fluctuations when distinguishing between scattering due to the collective motion and that due to the uncorrelated motion. The collective motion and the uncorrelated motion are independent as part of the collective motion is influenced by the ion motion while the uncorrelated motion of electrons is not. This leaves us with six degrees of freedom for the set of fluctuations included in (33). Using the continuity equation for the electron flux, expressing the transverse electric field fluctuations in terms of magnetic field fluctuations and noting that there are no longitudinal magnetic fluctuations we find that the source current (33) can be expressed in terms of the six linearly independent fluctuation variables n^δ, $\overline{\mathbf{v}}_\perp$, $\mathbf{B}_\perp$, $E_\parallel$ where $\parallel$ and $\perp$ now refer to components parallel and perpendicular to $\mathbf{k}^\delta$:

$$j_i^\sigma = \widehat{\mathrm{C}}_i^{(n')} n^\delta + \widehat{\mathrm{C}}_{ij}^{(v_\perp)} \overline{v}_{\perp j}^\delta + \widehat{\mathrm{C}}_{ij}^{(B_\perp)} B_j^\delta + \widehat{\mathrm{C}}_{ij}^{(E_\parallel)} E_{\parallel j}^\delta , \tag{34}$$

$$\widehat{\mathrm{C}}_i^{(n')} = \widehat{\mathrm{C}}^{(n)} + \widehat{\mathrm{C}}_{ij}^{(v)} \widehat{k}_j^\delta \omega^\delta / k^\delta , \tag{35a}$$

$$\widehat{\mathrm{C}}_{ij}^{(v_\perp)} = \widehat{\mathrm{C}}_{il}^{(v)} (\delta_{lj} - \widehat{k}_l^\delta \widehat{k}_j^\delta) , \tag{35b}$$

$$\widehat{\mathrm{C}}_{ij}^{(B_\perp)} = \widehat{\mathrm{C}}_{ij}^{(B)} - \widehat{\mathrm{C}}_{il}^{(E)} \epsilon_{lkj} \widehat{k}_k^\delta \omega^\delta / k^\delta , \tag{35c}$$

$$\widehat{\mathrm{C}}_{ij}^{(E_\parallel)} = \widehat{\mathrm{C}}_{il}^{(E)} \widehat{k}_l^\delta \widehat{k}_j^\delta . \tag{35d}$$

It is clear that we cannot unambiguously identify a certain fraction of the scattered field as being due to scattering from fluctuations in a particular quantity. In the cold fluid source current expression (18) it is, nonetheless, common to identify

$$\widehat{\mathrm{C}}_i^{(n)} = (\sigma_{ij}^{\mathrm{i}} / n^{(0)}) E_j^{\mathrm{i}}$$

as the term which accounts for scattering from density fluctuations (Sitenko, 1967, Chap. 11, Sec. 2). Similarly one might identify

$$\widehat{\mathrm{C}}_i^{(n)} = (\sigma_{ij}^{\sigma} / n^{(0)}) E_j^{\mathrm{i}}$$

as the term responsible for scattering from density fluctuations in the low temperature kinetic model (14). We compared the scattered power associated with these terms in Bindslev (1994 a) and found that they could differ quite significantly. The scattering associated with the longitudinal electron flux fluctuations, $\widehat{k}_j \overline{v}_j$, could of course equally well be attributed to scattering from density fluctuations, in which case we would let the coupling term, $\widehat{\mathrm{C}}_i^{(n')}$, defined in (35a), account for scattering by the density fluctuations. Comparing $\widehat{\mathrm{C}}_i^{(n')}$ from the cold fluid model with that from the LTK model we find essentially the same differences in predicted power scattered by density fluctuations as when $\widehat{\mathrm{C}}_i^{(n)}$ were compared in Bindslev (1994 a). To reconcile the two models in the limit of cold collective fluctuations, where both models apply, we note that in this limit the fluctuations have only three degrees of freedom, and hence, we can entirely attribute scattering from field fluctuations to scattering from density and transverse flux fluctuations. As density fluctuations and transverse flux fluctuations are orthogonal (scattering from one cannot be attributed to the other) the two models agree on the scattered power attributable to each. Note that the fact that the quantities are orthogonal does not imply that their fluctuations are not correlated.

5 Summary

From the microscopic kinetic equation and Maxwell's equations a general description of two wave interaction in the first Born approximation may be obtained. This description places no limitations on the interacting waves nor on the scattered wave except that, for the first Born approximation to be useful, it is assumed that the scattered wave is much weaker than the two interacting waves. If this is not acceptable higher order Born approximations can be obtained by considering a sequence of two wave interactions involving scattered waves. In this general kinetic description of scattering the source current, which drives the scattered field, involves the full detail of the distribution function perturbations associated with the two interacting waves. At present this model of wave interaction is numerically intractable.

The source current can be represented as an integral operator acting on the fields and distribution function perturbations of the interacting waves. Expanding the kernel of this operator in powers of velocity and truncating the expansion at second order we obtain the low temperature kinetic (LTK) description of scattering. This model assumes that thermal effects at the frequency and wave vector of the scattered wave are small. It is remarkable that this model involves only zeroth and first order moments of the distribution function perturbations associated with the interacting waves yet does not impose any limitations on the nature of the interacting waves. This description is therefore useful when modelling scattering of high frequency waves by low frequency fluctuations about which nothing is to be assumed. This is the situation in a range of microwave scattering experiments in fusion plasmas.

The LTK model of scattering can also be obtained by perturbation expansion of the sequence of moments of the kinetic equation with the sequence truncated by assuming that the third moment is zero. If the sequence is truncated by assuming a relation for the second moment we recover those terms in the LTK model which resulted from the zeroth order terms in the expansion of the kernel of the source current operator. Such a model is also valid for arbitrary interacting waves but then demands that the thermal effects at the frequency and wave vector of the scattered wave be even smaller.

In the perturbation expansion of the truncated sequence of moment equations it is important that the expansions are carried out in variables which, like the flux $\overline{\mathbf{v}} = \int \mathbf{v} f d\mathbf{p}$, are linear in the distribution function. This ensures that the limitations implicit in the truncated sequence of moments equations is imposed only on the scattered wave and not on the two interacting waves. If, as in the traditional cold fluid approach, the expansion is carried out in fluid velocity, $\mathbf{u} = \overline{\mathbf{v}}/n$, which is non–linear in f, then the limitations of the truncated sequence of moments equations are imposed also on the interacting waves. This is the principal reason why the cold fluid model of scattering is valid only in the limit of scattering from cold collective oscillations.

When attributing parts of the scattered field to fluctuations in individual quantities it is important to take linear dependencies between variables into account and make clear by which set of linearly independent variables the fluctuations are described. It should also be kept in mind that fluctuations in linearly independent quantities can be strongly correlated and that these correlations can significantly influence the total scattered power.

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