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ABSTRACT

The response of a collisionless plasma to global electromagnetic perturbations of an axisymmetric toroidal equilibrium is derived. By adopting a variational formulation for guiding centre motion, the perturbed distribution function is expressed in terms of the linearised guiding centre Lagrangian. Finite orbit widths are retained. In particular, the high particle energy limit where mirror-trapped banana orbits are distorted into "potato-shaped" orbits is considered. In this limit, the time scales associated with the drift and bounce motions of a mirror-trapped orbit become comparable, yielding important consequences on plasma stability. Quadratic forms are constructed in the context of kinetic-magnetohydrodynamic (MHD) models of plasmas composed of a thermal component obeying fluid-like equations and a high energy component described in terms of the collisionless drift-kinetic equation. Relevant applications include improved modelling of energetic ion effects on toroidicity-induced Alfvén gap modes and internal kinks.

1. INTRODUCTION

The influence of energetic particles on the stability of global plasma modes, such as internal kinks and low-m° Toroidal Alfvén Eigenmodes (m° is the poloidal mode number) in magnetically confined plasmas has been demonstrated both theoretically and experimentally (see, e.g., Refs. 1 and 2 and other references therein). High energy particles are produced by external heating methods (such as neutral beam injection or radio frequency heating) or by thermonuclear reactions. The theoretical description of magnetohydrodynamic (MHD) perturbations in such plasmas with a significant non-thermal component is based on hybrid kinetic-MHD models. According to these models, the thermal plasma is assumed to obey macroscopic fluid equations, while the energetic particles are described in terms of the collisionless drift-kinetic equation. Since the energy of these particles can be rather large, i.e. $E_h = 0(1\text{ MeV})$, it becomes necessary to take into account the radial excursion of the fast ion guiding centre trajectories across the magnetic surfaces. In Ref. 3, which considered internal kink modes in the low frequency limit, i.e. $\omega / \langle\omega_D\rangle \rightarrow 0$, the following ordering was assumed:

$$\delta_b \sim \varepsilon^{1/2}\ \rho_\vartheta \sim r. \qquad (1)$$

Here, δ_b is the "banana width" of a mirror-trapped orbit, $\varepsilon = r / R_o$ is the inverse aspect ratio of a toroidal magnetic surface with average minor radius r and major radius R_o, ρ_ϑ is the Larmor radius based on the poloidal magnetic field and $\langle\omega_D\rangle$ is the bounce-averaged magnetic drift frequency. When the ordering (1) is satisfied, the standard banana orbit analysis fails. Trapped particle trajectories are distorted into "potato-shaped" orbits [3] with a characteristic width

$$\delta_p \sim \left(2qv_\perp / \Omega R_o\right)^{2/3} R_o, \tag{2}$$

where q is the magnetic winding index and Ω is the cyclotron frequency. For instance, trapped ions produced near the magnetic axis with an energy $E_h = 1$ MeV in the Joint European Torus (JET) have a potato width $\delta_p \sim 30$ cm, which is about one third of the plasma minor radius (subscript "h" indicates fast particles, h = hot).

As is well known [1], fast particles must be sufficiently energetic in order to decouple from the plasma bulk and exert a stabilising influence, the decoupling condition being $\langle\omega_{Dh}\rangle >> \omega$, with $\langle\omega_{Dh}\rangle \propto E_h$ for the standard banana orbits. However, when the finite radial orbit width of these particles is considered, one finds an upper value of E_h for internal kink stabilisation which occurs when $\delta_p \propto E_h^{1/3}$ becomes comparable to the radius, r_s, of the q = 1 magnetic surface. Hence, it was concluded in Ref. 3 that, for fixed total fast ion energy content, fast particle pressure peaked on the magnetic axis, and a monotonic q profile with on-axis value $q_o < 1$, there exists an optimal value of the mean fast particle energy for stabilisation. Above this optimal value, the fast particles have a weakened stabilising influence, and they even become destabilising with yet larger δ_p. It was also pointed out in Ref. 3 that the extrapolation of the zero-orbit-width approximation would overestimate the stabilising influence if one allowed the radial width of the fast particle distribution function to become narrower than the average potato orbit width.

Fast particles can also destabilise other classes of modes, like the Toroidal Alfvén Eigenmode (TAE). It can be expected that instabilities driven by energetic particles are less easily excited when the orbit width becomes comparable with the radial extent of the mode. In fact, it was shown in Ref. 4 that the orbit width has the effect of reducing the power transfer between the fast particles and the electromagnetic perturbation by a factor Δ_m / Δ_b from that predicted by the narrow orbit theory. Here, Δ_b is the radial excursion of a passing fast ion and

$\Delta_m \sim r_m^2 / m^\circ R_0$ is the radial width of a mode with poloidal mode number m° peaked around the surface of radius r_m. This result was obtained analytically in the limit $k_\vartheta^{-1} > \Delta_b > \Delta_m$ and $\Delta_b > \rho_h$. In Ref. 5, it was also shown numerically that the growth rate of toroidal Alfvén modes destabilised by fast ions is further reduced when $k_\vartheta \rho_h$ becomes larger than unity. This result was based on the gyrokinetic approximation for the fast particle response.

In this paper, we present a general formalism applicable to the solution of the linearised drift-kinetic equation for global, low-frequency perturbations of an axisymmetric toroidal plasma when the orbit width becomes comparable with the plasma equilibrium scale length, i.e. when the potato ordering (1) is satisfied. We observe that the latter ordering implies

$$\delta \equiv \frac{\rho_h}{R_0} \sim \left(\frac{r}{R}\right)^{3/2}. \tag{3}$$

The applicability of the guiding centre description of particle orbits requires that δ be a small parameter. Hence, the potato ordering can be assumed provided the local inverse aspect ratio is also considered to be small and higher order terms in ε are consistently neglected. Concerning the radial mode structure, our analysis is limited to poloidal mode numbers m° such that

$$k_\vartheta \rho_h = m^\circ \rho_h / r < 1. \tag{4}$$

We point out that in many applications (including internal kinks and low-m° toroidal Alfvén modes) the radial structure of the MHD mode is characterised by an "outer" region, where $k_r \sim k_\theta$ (k_r is the effective radial wave number), and an "inner" layer across which the mode amplitude varies rapidly, i.e. $k_r >> k_\vartheta$. In these cases, asymptotic matching techniques can be used whereby fast ion effects are considered in the outer region but may be neglected in the layer owing to the relatively little time they spend there. Even though $k_r \rho_h$ may become large in the layer, the effect of fast particles whose orbits cross the layer can still be described in the outer region on the basis of the drift-kinetic approximation provided Eq. (4) is satisfied and, in addition, the particle time of flight across the layer is short compared with the mode period, i.e. $\omega^{-1} >> (\rho_h + \Delta_m) / v_{Dh}$, with Δ_m now representing the width of the layer and v_{Dh} the magnitude of the drift velocity. This inequality corresponds to the condition for which the guiding centre orbit as a whole is only slightly perturbed when crossing the layer of rapid amplitude variation [6].

The analysis of global plasma modes implies that, in general, the eikonal representation for perturbed quantities (see, e.g. Ref. 7) cannot be adopted. Solutions of the drift-kinetic equation for global plasma modes obtained without using to the eikonal representation have appeared in the literature, under the assumption of neglibible orbit widths [8] or in restricted frequency domains [3, 9, 10]. For instance, Antonsen and Lee [8] derived an energy principle for perturbations that grow on the time scale associated with the drifting of particles across field lines. Their elegant solution of the drift-kinetic equation considers a Lagrangian frame of reference moving with the field lines and is based on the adiabatic conservation of both the guiding centre magnetic moment and the longitudinal invariant for particles with negligible orbit widths. The aim of the present paper is to extend Antonsen and Lee's solution to the case where finite orbit widths are important. To do this we will adopt an Eulerian rather than a Lagrangian frame of reference as originally used by Antonsen and Lee. The final solution for the perturbed distribution function will be conveniently expressed in compact form in terms of Littlejohn's Lagrangian for guiding centre motion [11]. The advantages of adopting a Lagrangian formalism in the study of thermal plasma/fast ion systems have also been discussed in a recent paper by Edery et al [12].

This paper is organised as follows. In Section 2 we introduce the guiding centre model for particle orbits. A solution of the linearised drift-kinetic equation for global perturbations of an axisymmetric equilibrium is derived in Section 3. In Section 4 we specialise to the case of perturbations obeying the ideal MHD constraint. In Section 5, quadratic forms are obtained which are pertinent to the analysis of fast particle effects on global modes. An application to internal kink stabilisation is presented in Section 6, followed by our conclusions in Section 7.

2. GUIDING CENTRE MODEL

In this section, we outline the guiding centre equations of motion correct to relevant order in the parameter δ. We are interested in perturbations which vary slowly on the cyclotron time scale, specifically we order (the subscript "h" is dropped in this section)

$$\Omega^{-1}\ \partial / \partial t = 0(\delta). \qquad (5)$$

In addition, we assume that the $\vec{E}\times\vec{B}$ drifts are of the same order of magnitude in δ as the ∇B and curvature drifts. This is accomplished by setting

$$E / B = 0(\delta). \tag{6}$$

The discussion of the case where the $\vec{E}\times\vec{B}$ drift is 0(1) is obviously more involved. However, the ordering we are presenting is consistent with the assumption that perturbations are slowly varying over the gyroradius scale length.

At this point, following Northrop [13], we can write at once the equations for the guiding centre velocity and acceleration along field lines correct to first order in δ. Let $\vec{R}$ denote the guiding centre position. The guiding centre velocity is

$$\dot{\vec{R}} = v_{\|}\vec{b} + \frac{1}{m\Omega}\vec{b}\times\left\{\mu\nabla B + mv_{\|}^2\vec{\kappa} - Ze\vec{E}\right\} = v_{\|}\vec{b} + \vec{v}_D + \vec{v}_{E\times B}. \tag{7}$$

In this expression, the dot indicates time derivative, $v_{\|}$ is the parallel velocity, $\vec{b}\equiv\vec{B}/B$ is the unit vector along field lines, $\mu\equiv mv_{\perp}^2/2B$ is the magnetic moment, $\vec{\kappa}\equiv\left(\vec{b}\cdot\nabla\right)\vec{b}$ is the curvature vector and Z is the particle charge number. The parallel velocity is $v_{\|}\vec{b}=0(1)$, while, as a consequence of the ordering (6), $\vec{v}_D \sim \vec{v}_{E\times B} = 0(\delta)$. Terms of $0\left(\delta^2\right)$ which have been neglected include the polarisation drift and the drift $\left(v_{\|}/\Omega\right)\,\vec{b}\times\left(\partial\vec{b}/\partial t\right)$. The parallel acceleration to order δ is

$$m\dot{v}_{\|} = -\mu\vec{b}\cdot\nabla B + Ze\ \vec{b}\cdot\vec{E} + mv_{\|}\ \vec{\kappa}\cdot\dot{\vec{R}}. \tag{8}$$

Equations (7) and (8) are consistent with Littlejohn's guiding centre Lagrangian [11]

$$L = \left(\frac{Ze}{c}\vec{A} + mv_{\|}\vec{b}\right)\cdot\dot{\vec{R}} + \frac{1}{\Omega}y\dot{\alpha} - \tfrac{1}{2}mv_{\|}^2 - y - Ze\phi, \tag{9}$$

where $y\equiv\mu B$ has the meaning of "perpendicular energy" and α is the gyro-angle in velocity space. The Lagrangian (9) is regarded as a function of the variables

$$L = L\left(\vec{R}, v_{\|}, y, \alpha;\ \dot{\vec{R}}, \dot{v}_{\|}, \dot{y}, \dot{\alpha};\ t\right), \tag{10}$$

in which $\alpha, \dot{v}_{\parallel}$ and $\dot{y}$ happen not to appear. Let Z_i, i = 1, ..., 6, represent the six guiding centre variables $(\vec{R}, v_{\parallel}, y, \alpha)$. It can be easily verified that the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Z}_i}\right) = \frac{\partial L}{\partial Z_i}, \tag{11}$$

correct to order δ yield Eqs. (8) and (9), and, in addition

$$\dot{\mu} = 0, \quad \dot{\alpha} = \Omega(\vec{R}), \tag{12}$$

conditions that also follow from Northrop's guiding centre theory. The first of Eqs. (12) shows that μ is a (formal) constant of motion, at least to relevant order of accuracy in δ. The real motion of a particle is approximated in guiding centre theory by $\vec{r}(t) = \vec{R}(t) + \vec{\rho}(t)$, where $\vec{\rho}(t) = \Omega^{-1}\ \vec{v}_{\perp}(t) \times \vec{b}$, $\vec{v}_{\perp} = v_{\perp}(\vec{e}_1 \sin\alpha + \vec{e}_2 \cos\alpha)$, with $\vec{e}_1$, $\vec{e}_2$ orthogonal unit vectors in the plane perpendicular to $\vec{b}$ and $\rho \equiv |\vec{\rho}|$ the Larmor radius.

Next, we turn our attention to the relevant kinetic equation. The Vlasov equation in the Cartesian coordinates $\vec{z} = (\vec{x}, \vec{v})$ can be written as

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{6} \dot{z}_i \frac{\partial f}{\partial z_i} = 0. \tag{13}$$

Now, this equation is covariant with respect to arbitrary changes of coordinates, and therefore is valid in any coordinate systems. In particular, it takes exactly the same form if the guiding centre coordinates Z_i are used instead of z_i. Hence, we are led to consider the Vlasov equation

$$\frac{\partial f}{\partial t} + \dot{\vec{R}} \cdot \nabla f + \dot{v}_{\parallel} \frac{\partial f}{\partial v_{\parallel}} + \dot{y} \frac{\partial f}{\partial y} + \dot{\alpha} \frac{\partial f}{\partial \alpha} = 0, \tag{14}$$

where $f = f\left(\vec{R}, v_{\parallel}, y, \alpha;\ t\right)$. Note that this equation is exact provided $\dot{\vec{R}}, \dot{v}_{\parallel}, \dot{y}$ and $\dot{\alpha}$ are known to arbitrary order in the gyroradius. However, for our purposes, we only need the equations of motion to order δ, as given by Eqs. (7), (8) and (12). Then Eq. (14) can be simplified by neglecting corrections of order δ. First, we recognise that the term proportional to $\partial f / \partial \alpha$ in this equation is the largest of all. Expanding f in powers of δ, $f = f_0 + f_1 + \ldots$, the leading term satisfies $\dot{\alpha}\ \partial f_0 / \partial \alpha = 0$. The first order equation is averaged over α, so that $\oint d\alpha\ \dot{\alpha}\ \partial f_1 / \partial \alpha = 0$ because of periodicity ($\dot{\alpha}$ can be taken out of the integral). Since $\dot{\vec{R}}, \dot{v}_{\parallel}$ and $\dot{y}$ as given by Eqs. (7), (8) and (12) do not depend on α to first order in δ, the α-averaging of the first order Eq. (14) is trivial and we are led to consider the collisionless drift-kinetic equation consistent with the orderings (5) and (6)

$$\frac{\partial f_o}{\partial t} + \dot{\vec{R}} \cdot \nabla f_o + \dot{v}_{\|} \frac{\partial f_o}{\partial v_{\|}} + \dot{y} \frac{\partial f_o}{\partial y} = 0. \tag{15}$$

We can drop the subscript "o" in the following.

We shall restrict ourselves to perturbations of axisymmetric equilibria, where the toroidal angle φ is an ignorable coordinate. Thus, at equilibrium we set $\partial / \partial t = \partial / \partial \varphi = 0$ and f = F. The equilibrium distribution function is in general a function of the three invariants P_φ, E, μ and of the index σ (defined below Eq. (19)):

$$F = F\left(P_\varphi, E, \mu;\ \sigma\right). \tag{16}$$

The invariants are expressed in terms of the guiding centre variables $\vec{Z}$. The toroidal canonical momentum is

$$P_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \frac{Ze}{c} \psi + mRv_{\|} \frac{B_\varphi}{B}, \tag{17}$$

where ψ is the poloidal magnetic flux function (equal to the covariant toroidal component of the magnetic potential vector) and $R = |\vec{R}|$ is the distance of the guiding centre from the toroidal axis of symmetry. The particle energy is

$$E = \frac{1}{2} m v_{\|}^2 + y + Ze\phi\left(\vec{R}\right), \tag{18}$$

and the magnetic moment is

$$\mu = y / B\left(\vec{R}\right). \tag{19}$$

For various choices of $\left(P_\varphi, E, \mu\right)$, one or two orbits may exist. If two orbits exist, F may have two different values for the same $\left(P_\varphi, E, \mu\right)$. The additional dependence on the index σ becomes necessary in order to distinguish between the two orbits. For instance, we may set $\sigma = \mathrm{sgn}\ \left(\dot{\vartheta}\right)$ (we observe that the choice $\sigma = \mathrm{sgn}\ \left(v_{\|}\right)$ is no longer adequate for the non-standard orbits satisfying the ordering (1); a detailed discussion of the possible orbit types can be found, e.g., in Ref. 14).

The ratio of the two terms defining P_φ is

$$\frac{mcRv_\parallel}{Ze\psi} = 0\left(\frac{\delta_b}{r}\right), \tag{20}$$

which we consider to be of order unity. In the limit $\delta_b / r \to 0$, where we can neglect the term proportional to $v_\parallel$ in Eq. (17), $F = F(\psi, E, \mu; \sigma)$ is constant along the magnetic field lines.

3. LINEAR PERTURBATION THEORY

The perturbations of interest lead to the violation of the conservation of E and P_φ. Let us denote perturbed quantities by a superscript "(1)". Then, with $f = F + f^{(1)}$, the linearised drift-kinetic equation takes the form

$$\frac{df^{(1)}}{dt} + \dot{\vec{R}}^{(1)} \cdot \nabla F + \dot{v}_\parallel^{(1)} \frac{\partial F}{\partial v_\parallel} + \dot{y}^{(1)} \frac{\partial F}{\partial y} = 0. \tag{21}$$

Note that, in general, for a generic quantity X, $(d/dt)\, X^{(1)} = (dX/dt)^{(1)} - \dot{\vec{R}}^{(1)} \cdot \nabla X$. In order to avoid confusion, we specify that in our notations $\dot{X}^{(1)} \equiv (dX/dt)^{(1)}$. We observe that, having chosen $\vec{R}$, $v_\parallel$ and y as variables of our Eulerian coordinate system, perturbed quantities such as $\vec{R}^{(1)}, v_\parallel^{(1)}$ and $y^{(1)}$ will not appear. With the help of Eqs. (17)-(19), we can rewrite Eq. (21) as

$$\frac{df^{(1)}}{dt} + \left(\dot{\vec{R}}^{(1)} \cdot \nabla P_\varphi + \dot{v}_\parallel^{(1)} \frac{\partial P_\varphi}{\partial v_\parallel}\right) \frac{\partial F}{\partial P_\varphi} + \left(Ze\dot{\vec{R}}^{(1)} \cdot \nabla\phi + mv_\parallel \dot{v}_\parallel^{(1)} + \dot{y}^{(1)}\right) \frac{\partial F}{\partial E}$$
$$+ \left(\frac{\dot{y}^{(1)}}{B} - \frac{y}{B} \dot{\vec{R}}^{(1)} \cdot \nabla B\right) \frac{\partial F}{\partial \mu} = 0. \tag{22}$$

We now use the results of the previous section to calculate the quantities inside the brackets. First, we observe that the Euler-Lagrange Eq. (11) corresponding to $Z_i = R_i$, i = 1, 2, 3, can be cast in the form

$$Ze\left(\vec{E}^* + \frac{1}{c} \dot{\vec{R}} \times \vec{B}^*\right) = m\dot{v}_\parallel \vec{b} + \mu \nabla B, \tag{23}$$

where

$$\vec{B}^* = \nabla \times \vec{A}^* = \vec{B} + \frac{B}{\Omega} v_\parallel \nabla \times \vec{b}, \tag{24}$$

$$\vec{E}^* = -\frac{1}{c} \vec{A}_t^* - \nabla\phi, \tag{25}$$

and

$$\vec{A}^* = \vec{A} + \frac{B}{\Omega} v_\parallel \vec{b}, \tag{26}$$

is a "modified vector potential" [11]. We have introduced the notation $X_t \equiv \partial X / \partial t$. Dotting Eq. (23) with $\dot{\vec{R}}$ we get

$$m v_\parallel \dot{v}_\parallel = -\left(\frac{Ze}{c} \vec{A}_t + m v_\parallel \vec{b}_t + Ze \nabla \phi + \mu \nabla B \right) \cdot \dot{\vec{R}}. \tag{27}$$

From $\dot{\mu} = 0$ and $y = \mu B$ follows

$$\dot{y} = \left(B_t + \dot{\vec{R}} \cdot \nabla B \right) \mu. \tag{28}$$

From (27) and (28) we can construct the total time derivative of the particle energy, $\dot{E} = m v_\parallel \dot{v}_\parallel + \dot{y} + Ze\dot{\phi}$. It is easy to show that

$$\dot{E} = -L_t = -\left(\frac{Ze}{c} \vec{A}_t + m v_\parallel \vec{b}_t \right) \cdot \dot{\vec{R}} + \mu B_t + Ze \phi_t. \tag{29}$$

Equation (29) shows that E is a constant of motion when the potentials $\vec{A}$ and ϕ are time independent. Clearly, $L_t = 0$ at equilibrium. Thus, the linearised version of Eq. (29) yields

$$m v_\parallel \dot{v}_\parallel^{(1)} + \dot{y}^{(1)} + Ze \dot{\vec{R}}^{(1)} \cdot \nabla \phi = -L_t^{(1)} - Ze \ d\phi^{(1)} / dt. \tag{30}$$

The left-hand-side of Eq. (30) corresponds to the term proportional to $\partial F / \partial E$ in Eq. (22). The term proportional to $\partial F / \partial \mu$ can be easily obtained from the linearised version of Eq. (28):

$$\frac{1}{B} \dot{y}^{(1)} - \frac{1}{B^2} y \dot{\vec{R}}^{(1)} \cdot \nabla B = \mu \frac{d}{dt} \left(\frac{B^{(1)}}{B} \right). \tag{31}$$

The Euler-Lagrange equation for the toroidal coordinate φ gives an expression for the time derivative of P_φ:

$$\frac{Ze}{c} \dot{P}_\varphi = \frac{\partial L}{\partial \varphi} = \left(\frac{Ze}{c} \frac{\partial \vec{A}}{\partial \varphi} + m v_\parallel \frac{\partial \vec{b}}{\partial \varphi} \right) \cdot \dot{\vec{R}} - Ze \frac{\partial \phi}{\partial \varphi} - \mu \frac{\partial B}{\partial \varphi}. \tag{32}$$

Since $\partial L / \partial \varphi = 0$ at equilibrium, the linearised version of (32) can be cast in the form

$$\dot{\vec{R}}^{(1)} \cdot \nabla P_\varphi + \dot{v}_\parallel{}^{(1)} \frac{\partial P_\varphi}{\partial v_\parallel} = \frac{c}{Ze} \frac{\partial L^{(1)}}{\partial \varphi} - \frac{d}{dt} P_\varphi{}^{(1)}. \tag{33}$$

The left-hand side of Eq. (33) corresponds to the term proportional to $\partial F / \partial P_\varphi$ in Eq. (22). The perturbed distribution function can be written as

$$f^{(1)} = P_\varphi{}^{(1)} \frac{\partial F}{\partial P_\varphi} + Ze\phi^{(1)} \frac{\partial F}{\partial E} - \mu \frac{B^{(1)}}{B} \frac{\partial F}{\partial \mu} + h^{(1)}, \tag{34}$$

where $h^{(1)}$ corresponds to the "non-adiabatic part" of $f^{(1)}$. Then with the help of Eqs. (30), (31) and (33), we find that $h^{(1)}$ satisfies the equation

$$\frac{dh^{(1)}}{dt} = \frac{\partial F}{\partial E} \frac{\partial L^{(1)}}{\partial t} - \frac{c}{Ze} \frac{\partial F}{\partial P_\varphi} \frac{\partial L^{(1)}}{\partial \varphi}. \tag{35}$$

Equation (35) can be simplified further. If we consider the product $\vec{A}^{*(1)} \cdot \dot{\vec{R}} = \vec{A}^{(1)} \cdot \dot{\vec{R}} + (mc / Ze) v_\parallel \vec{b}^{(1)} \cdot \dot{\vec{R}}$, it can be easily recognised that the last term is $0(\delta^2)$, i.e. it has the same order of magnitude as terms in $\dot{\vec{R}}$ that we have neglected. Therefore, the term proportional to $\vec{b}^{(1)} \cdot \dot{\vec{R}}$ cannot be consistently retained. This implies that in Eq. (35) we need only use the perturbed Lagrangian to leading order in δ,

$$L^{(1)} = \frac{Ze}{c} \vec{A}^{(1)} \cdot \dot{\vec{R}} - Ze\phi^{(1)} - \mu B^{(1)} + 0(\delta). \tag{36}$$

Equation (35) can be integrated along unperturbed orbits using standard techniques (see, e.g. Ref. 15). Let us introduce the coordinates $\vec{R} = (\psi, \vartheta, \varphi)$, where ψ labels the equilibrium magnetic surfaces and ϑ is a generalised poloidal angle. The guiding centre equations of motion in these coordinates can be written as

$$\dot{\psi} = \dot{\vec{R}} \cdot \nabla\psi; \qquad \dot{\vartheta} = \dot{\vec{R}} \cdot \nabla\vartheta; \qquad \dot{\varphi} = \dot{\vec{R}} \cdot \nabla\varphi, \tag{37}$$

with $\dot{\vec{R}}$ given by Eq. (7). At equilibrium, the projection of the orbit on the poloidal cross-section is a closed curve. For either mirror-trapped or passing orbits, we define the bounce time

$$\tau_b = \oint d\tau = \oint d\psi / \dot{\psi} = \oint d\vartheta / \dot{\vartheta}, \tag{38}$$

as the time it takes to close an equilibrium orbit on the poloidal plane. We assume that perturbations have the form

$$X^{(1)} = \hat{X}^{(1)}(\psi, \vartheta) \; \exp(- \, i\omega t - in^\circ \varphi), \tag{39}$$

where n° is the toroidal mode number. Equation (35) can be written as

$$\frac{dh^{(1)}}{dt} = -\, i(\omega - n^\circ \omega_*) \frac{\partial F}{\partial E} L^{(1)}, \tag{40}$$

where we have defined

$$\omega_* \equiv \frac{c}{Ze} \frac{\partial F / \partial P_\varphi}{\partial F / \partial E}. \tag{41}$$

Note that ω_* is a constant of the unperturbed particle motion. This definition of ω_* is entirely determined by the equilibrium distribution function and is not expressed through gradients operating on perturbed quantities, as it sometimes appears in the literature.

The formal solution of Eq. (40) is

$$h^{(1)} = -\, i(\omega - n^\circ \omega_*) \frac{\partial F}{\partial E} \int_{-\infty}^{t} L^{(1)}(\tau) d\tau, \tag{42}$$

where $L^{(1)}(\tau) = \hat{L}^{(1)}[\psi(\tau), \vartheta(\tau)] \exp\left[-\, i\omega\tau - in^\circ \varphi(\tau)\right]$ and the τ dependence is through Eq. (37). We have used the causality prescription to set the lower limit of integration. Let us separate $\varphi(\tau)$ into its secular and oscillating parts,

$$\varphi(\tau) = \langle \dot{\varphi} \rangle \tau + \varphi^{(\sim)}(\tau), \tag{43}$$

where the brackets indicate bounce averaging:

$$\langle X \rangle = \frac{1}{\tau_b} \oint X d\tau. \tag{44}$$

The quantity $\tilde{L}^{(1)} = \hat{L}^{(1)} \exp\left(-\, in^\circ \varphi^{(\sim)}\right)$ is a periodic function of τ which can be expanded in Fourier series,

$$\tilde{L}^{(1)}(\tau) = \sum_{-\infty}^{\infty}{}_p\, Y_p\left(\varepsilon, \mu, P_\varphi;\ \sigma\right) \exp\left(-\, ip\omega_b \tau\right), \tag{45}$$

where $\omega_b \equiv 2\pi / \tau_b$. The Fourier coefficients are defined as

$$Y_p\left(E, \mu, P_\varphi, \sigma\right) = \oint \frac{d\tau}{\tau_b} \tilde{L}^{(1)} \exp\left(ip\omega_b \tau\right). \tag{46}$$

Inserting (45) in (42) and carrying out the time integration, we find

$$h^{(1)} = (\omega - n^\circ\omega_*)\frac{\partial F}{\partial E}\sum_{-\infty}^{\infty}{}_p\, Y_p(E,\mu,P_\varphi,\sigma)\frac{\exp\left[-\ i(\omega + n^\circ\langle\dot\varphi\rangle + p\omega_b)t\right]}{\omega + n^\circ\langle\dot\varphi\rangle + p\omega_b}. \tag{47}$$

An alternative expression for $h^{(1)}$ can be obtained by setting

$$h^{(1)} = -\ i(\omega - n^\circ\omega_*)\frac{\partial F}{\partial E}g^{(1)}$$

and

$$g^{(1)}(t) = g^{(1)}(0) + \int_0^t L^{(1)}(\tau)d\tau. \tag{48}$$

Using the periodicity condition for the function $\tilde g^{(1)}(\tau) = \hat g^{(1)}(\psi,\vartheta)\ \exp\left(-\ in\varphi^{(\sim)}\right)$, we can determine the constant of integration in (48) to be

$$g^{(1)}(0) = \frac{\int_0^{\tau_b} L^{(1)}d\tau}{\exp\left[-\ i(\omega + n\langle\dot\varphi\rangle)\tau_b\right] - 1}. \tag{49}$$

Both expressions (47) and (49) for $h^{(1)}$ exhibit the mode-particle resonance condition

$$\omega + n^\circ\langle\dot\varphi\rangle + p\omega_b = 0, \qquad p = 0,\ \pm1,\ \pm2, \tag{50}$$

which applies to either mirror-trapped or passing orbits. In order to make contact with standard results in the zero orbit width limit [15], let us write $\langle\dot\varphi\rangle$ as

$$\langle\dot\varphi\rangle = \frac{1}{\tau_b}\int_{\tau_o}^{\tau_o+\tau_b} d\tau\left(\frac{d\beta}{d\tau}\right) + \frac{1}{\tau_b}\int_{\tau_o}^{\tau_o+\tau_b} d\tau\left(\frac{d}{d\tau}q\vartheta\right), \tag{51}$$

where $\beta = \varphi - q\vartheta$ and τ_0 corresponds to an arbitrary starting point along the orbit. For $\delta_b / r \to 0$, the first term on the right-hand side reduces to the bounce-averaged magnetic precession frequency, $\omega_D \equiv \left\langle\dot{\vec R}\cdot\nabla\beta\right\rangle$. The second term vanishes for trapped particles. For passing particles, since their trajectory remains on a ψ = const surface under the assumption $\delta_b \to 0$, $q(\psi)$ can be taken out of the integral and the second term reduces to $q\omega_b$. Thus, the resonance condition for passing particles in the limit $\delta_b / r \to 0$ takes the standard form

$$\omega + n^\circ\omega_D + (n^\circ q + p)\omega_b = 0, \qquad p = 0, \pm1,\ \pm2. \tag{52}$$

If we now consider finite values of δ_b / r, the splitting of $\langle\dot{\varphi}\rangle$ in two terms as in Eq. (51) is no longer convenient. In fact, both terms on the right-hand side of Eq. (51) will now depend on E, P_φ, μ and σ through the initial time τ_0 in a complicated manner. In this case, we prefer to use the expression (50) for the resonance condition. This expression also has the advantage of treating passing and trapped particle resonances in a unified way.

Considering large orbit widths, $\delta_b \sim r$, one of the main departures from the standard orbit theory is the disappearance of an adiabatic invariant. In fact, in the limit $\delta_b / r << 1$, one finds $\langle\omega_D\rangle / \omega_b \sim \delta_b / r$, so that the timescales associated with the periodic bounce and drift motions are associated with two adiabatically conserved action integrals [13]: the longitudinal invariant, $J = m\oint_b v_\parallel^2 d\tau$, where at equilibrium the loop integral corresponds to a closed orbit in the poloidal plane, and the so-called "flux" or third adiabatic invariant, $\Phi = \oint_D \psi d\beta$, where at equilibrium the loop integral correspond to a toroidal revolution of the banana centre. However, in the large orbit limit, $\langle\dot{\varphi}\rangle \sim \omega_b$ and the two timescales become comparable [3]. Thus, it is no longer possible to speak in terms of two adiabatic invariants (in addition to μ) that are independently conserved when $\delta_b \sim r$.

4. IDEAL MHD PERTURBATIONS

In this section, we specialise to the case of ideal MHD perturbations, i.e. perturbations which satisfy the constraint

$$\vec{B}^{(1)} = \nabla \times \left(\vec{\xi}_\perp \times \vec{B}\right). \tag{53}$$

By using Eq. (1), $\vec{\xi}_\perp$ can be identified as the displacement of a field line. This constraint excludes the possibility of modes that give rise to magnetic reconnection. We point out, however, that magnetic reconnection can be allowed in a narrow layer where non-ideal effects are taken into account but the fast ion effects are assumed to be unimportant. Solutions for the perturbed quantities in the reconnection layer can be matched asymptotically to the eigenfunctions in the outer region, which is a way to obtain an approximation to the global mode structure. Then, the analysis of this section can be applied to determine the outer mode structure of a perturbation that gives rise to magnetic reconnection.

Equation (53) suggests a gauge where the perturbed vector potential is perpendicular to the equilibrium magnetic field, i.e.

$$\vec{A}^{(1)} = \vec{\xi}_\perp \times \vec{B}. \tag{54}$$

With the vector potential in this form, the perturbed Lagrangian is

$$L^{(1)} = -mv_\parallel^2 \vec{\xi}_\perp \cdot \vec{\kappa} - \left(B^{(1)} + \xi_\perp \cdot \nabla B\right)\mu - \left(\phi^{(1)} + \xi_\perp \cdot \nabla\phi\right)Ze + 0(\delta). \tag{55}$$

Note that the perturbed longitudinal invariant at fixed energy in a frame moving with the field lines is proportional to the bounce averaged perturbed Lagrangian to leading order in δ,

$$J^{(1)} = \oint L^{(1)} d\tau, \tag{56}$$

as in Eq. (41) of Ref. 8. If we now assume that $\phi = 0$ at equilibrium, it follows from Eq. (54) and $E_\parallel^{(1)} = 0$ that also $\phi^{(1)} = 0$ and that

$$\vec{E}^{(1)} = (i\omega / c)\vec{\xi}_\perp \times \vec{B}. \tag{57}$$

In this case, the perturbed Lagrangian reduces to

$$L^{(1)} = -\left(mv_\parallel^2 - \mu B\right)\, \vec{\xi} \cdot \vec{\kappa} + \mu B \nabla \cdot \vec{\xi}_\perp, \tag{58}$$

where $B^{(1)} = B_\parallel^{(1)}$ has been eliminated through Eq. (53).

The equilibrium field can be represented by

$$\vec{B} = \nabla\psi \times \nabla\varphi + I(\psi)\nabla\varphi. \tag{59}$$

The toroidal component of the perturbed canonical momentum is

$$P_\varphi^{(1)} = \frac{Ze}{c} R^2 A_\varphi^{(1)} |\nabla\varphi| + mv_\parallel R^2 \vec{b}^{(1)} \cdot \nabla\varphi = -\frac{Ze}{c}\vec{\xi}_\perp \cdot \nabla\psi + 0(\delta). \tag{60}$$

Using Eqs. (34), (42), and (60), the perturbed distribution function can be written as

$$f^{(1)} = -\frac{Ze}{c}\vec{\xi}_\perp \cdot \nabla\psi \frac{\partial F}{\partial P_\varphi} - \mu \frac{B_\parallel{}^{(1)}}{B}\frac{\partial F}{\partial \mu} - \; i(\omega - n^\circ \omega_*)\frac{\partial F}{\partial E}\int_{-\infty}^{t} \; L^{(1)} d\tau. \tag{61}$$

This equation agrees with the result by Antonsen and Lee [8] in the limit $\delta_b / r \rightarrow 0$, where $P_\varphi \approx \psi$ and the adiabatic term $(Ze/c)\left(\vec{\xi}_\perp \cdot \nabla\psi\right)\partial F / \partial P_\varphi$ reduces to $\vec{\xi}_\perp \cdot \nabla F$. In addition to this change in the adiabatic part of $f^{(1)}$, the main difference with Ref. 8 is in the evaluation of the time integral when the finite size of the unperturbed orbits becomes important, as discussed in the previous section.

5. QUADRATIC FORMS

Let us consider the plasma momentum balance equation (sum over all plasma species). For motions that develop on the slow timescale associated with the particle drift velocity, inertial terms may be neglected and the linearised momentum balance equation reduces to

$$0 = -\nabla p_c{}^{(1)} - \nabla \cdot \vec{P}_h{}^{(1)} + \frac{1}{c}\left(\vec{J} \times \vec{B}\right)^{(1)}. \tag{63}$$

In (63), the subscript "c" refers to the core plasma, whose pressure is assumed to be isotropic, and $\vec{J}$ includes the contributions of both the core plasma and the high energy particles. We can construct a quadratic form by taking the scalar product of Eq. (63) with the adjoint displacement, $\vec{\xi}_\perp^\dagger$:

$$\Delta W = \delta W_{MHD} + \delta W_{hot}, \tag{64}$$

where

$$\delta W_{MHD} = -\frac{1}{2}\int d^3x \left[\frac{1}{c}\vec{\xi}_\perp^\dagger \cdot \left(\vec{J} \times \vec{B}\right)^{(1)} - \vec{\xi}_\perp^\dagger \cdot \nabla p_c{}^{(1)}\right] \tag{65}$$

and

$$\delta W_{hot} = \frac{1}{2}\int d^3x \; \vec{\xi}_\perp^\dagger \cdot \nabla \cdot \vec{P}_h{}^{(1)}. \tag{66}$$

In (64), δW_{MHD} is formally identical to the MHD energy functional for an isotropic plasma. However, it is important to observe that the self-adjoint property of the conventional δW_{MHD} is spoiled by the inclusion of the fast particle current density in Eq. (65), or, to put it differently, by the exclusion of the adiabatic part of the fast particle pressure response in (65). The latter response, which is included in our expression for δW_{hot}, is modified by the finite orbit width theory, as was pointed out below Eq. (61). The non-adiabatic fast pressure response is frequency dependent and, for predominantly real values of ω, it includes the contribution of mode-particle resonances leading to an imaginary part of δW_{hot}.

Thus, in general, ΔW is a complex, non-variational form. This lack of self-adjointness implies that necessary and sufficient criteria for stability cannot be obtained solely on the basis of the sign of the real part of ΔW, so that a normal mode analysis is required. In practice, ΔW becomes a useful form if it can be established that it consists of large and small terms, with the hot particles contributing to the small terms. Further, the large terms can be shown to constitute a self-adjoint form. By minimising the self-adjoint part, a leading order eigenfunction $\vec{\xi}_o$ is determined, with $\vec{\xi}_o^{\dagger} = \vec{\xi}_o^*$. In this way, only the leading order eigenfunction enters our expression for δW_{hot}. In the next section, the use of δW_{hot} in the theory of internal kink stabilisation will be considered as a specific example. δW_{hot} also represents the predominant fast particle contribution to the dispersion relation of Toroidal Alfvén Eigenmodes, whose detailed analysis can be found, e.g., in Ref. 16 for fast particles with negligible orbit width.

In the remainder of this section, we provide the relevant analysis for the reduction of δW_{hot}, Eq. (66), where we set $\vec{\xi}_o^{\dagger} = \vec{\xi}_o^*$. Following Ref. 8, the perturbed stress tensor can be written as

$$\vec{P}^{(1)} = p_\perp^{(1)}\vec{I} + \left(p_\parallel^{(1)} - p_\perp^{(1)}\right)\vec{b}\vec{b} + \left(p_\parallel - p_\perp\right)\left(\vec{b}\vec{b}^{(1)} + \vec{b}^{(1)}\vec{b}\right) + B_\parallel^{(1)}\frac{\partial}{\partial B}\vec{P}, \quad (67)$$

where $p_\parallel$ and $p_\perp$ are the components of the equilibrium pressure tensor, $\vec{P} = p_\perp\vec{I} + (p_\parallel - p_\perp)\vec{b}\vec{b}$, $\vec{I}$ is the identity tensor and

$$\begin{pmatrix} p_\perp^{(1)} \\ p_\parallel^{(1)} \end{pmatrix} = \int d^3v \begin{pmatrix} \mu B \\ m v_\parallel^2 \end{pmatrix} f^{(1)}. \quad (68)$$

We have neglected in Eq. (67) contributions relating to the parallel diamagnetic flow which, at equilibrium, arise from taking F to be a function of P_φ. In fact, it can be shown that these contributions are at most of order δ_b / R_o. Now, after straightforward manipulations, considering perturbations that vanish at the plasma edge, δW_{hot} can be written as

$$\delta W_{hot} = \frac{1}{2}\int d^3x \left[\frac{(\sigma - 1)\left|\vec{B}_\perp^{(1)}\right|^2 + (\tau - 1)\left|B_\parallel^{(1)}\right|^2}{4\pi} + \frac{(\sigma - 1)J_\parallel \vec{b}\cdot\vec{\xi}^* \times \vec{B}_\perp^{(1)}}{c} \right.$$

$$-\frac{1}{2}\int d^3x d^3v L^{(1)*} f^{(1)}, \tag{69}$$

where $\sigma - 1 = 4\pi(p_\perp - p_\parallel)_h / B^2$, $\tau - 1 = (4\pi / B)\partial p_{\perp h} / \partial B$ and $L^{(1)*}$ is the complex conjugate linearised Lagrangian (58). The last integral in Eq. (69) can be split in two parts:

$$-\frac{1}{2}\int d^3x d^3v \; L^{(1)*} f^{(1)} = \; \delta W_1 + \delta W_2(\omega), \tag{70}$$

where

$$\delta W_1 = \frac{1}{2}\int d^3x d^3v \; L^{(1)*}\left(\vec{\xi}_\perp \cdot \nabla\psi \frac{Ze}{c}\frac{\partial F}{\partial P_\varphi} + \mu \frac{B_\parallel^{(1)}}{B}\frac{\partial F}{\partial \mu} \right) \tag{71}$$

and

$$\delta W_2 = \frac{1}{2}\int d^3x d^3v (\omega - n^\circ \omega_*) \frac{\partial F}{\partial E} \; L^{(1)*} \int_{-\infty}^{t} L^{(1)} d\tau. \tag{72}$$

δW_1 represents the generalisation of the fluid-like pressure response for finite orbit widths. In fact, for $\delta_b / r \to 0$, δW_1 reduces to

$$\delta W_1 \approx -\int d^3x \left[\left(\vec{\xi}_\perp \cdot \hat{\nabla} p_\parallel\right)\left(\vec{\xi}_\perp^* \cdot \vec{\kappa}\right) + B^{-1}\left(B_\parallel^{(1)*} + \vec{\xi}_\perp^* \cdot \nabla B\right)\left(\vec{\xi}_\perp \cdot \hat{\nabla} p_\perp\right) + \left(B_\parallel^{(1)} / B\right)\left(\vec{\xi}_\perp^* \cdot \hat{\nabla} p_\perp\right) \right], \tag{73}$$

where $\hat{\nabla} = \nabla - (\nabla B)\partial / \partial B$, which corresponds to the conventional result (see, e.g., Eq. (49) of Ref. 8). δW_2 depends on the frequency and has an intrinsically kinetic origin. The fluid approximation can be recovered when both limits $\delta_b / r \to 0$ and $\omega \to \infty$ are taken, where δW_2 (ω) vanishes. For modes that grow on the Alfvén time scale of the plasma bulk, ω tends to be comparable with or below the time scales associated with the fast particle orbits and the fluid approximation becomes inadequate.

The techniques outlined at the end of Sec. 3 help to reduce further the kinetic integral δW_2. Using the expansion of $L^{(1)}$ in harmonics of the orbit periodicity, as given by Eq. (45), and carrying out the time integration in Eq. (72), we obtain

$$\delta W_2 = -\frac{i}{2}\int d^3x d^3v (\omega - n^\circ \omega_*) \frac{\partial F}{\partial E} \sum_{\ell=-\infty}^{\infty} Y_\ell^* e^{i\ell\omega_b \tau} \sum_{p=-\infty}^{\infty} \frac{Y_p e^{-ip\omega_b \tau}}{\omega + n^\circ \langle \dot{\varphi} \rangle + p\omega_b}. \tag{74}$$

It is convenient to introduce new phase-space variables through the transformation $(\vec{x},\vec{v}) \rightarrow (P_\varphi, \varphi, E, \tau, \mu, \alpha)$, where τ is the time along the orbit. The Jacobian of this transformation is a constant:

$$d^3x d^3v = \left(c / Zem^2\right) \sum_\sigma dP_\varphi dE d\mu d\tau d\varphi d\alpha. \tag{75}$$

Carrying out the integrations over τ, φ and α, δW_2 reduces to

$$\delta W_2 = -\frac{2\pi^2 c}{Zem^2} \sum_\sigma \int dP_\varphi dE d\mu \tau_b (\omega - n^\circ \omega_*) \frac{\partial F}{\partial E} \sum_{-\infty}^{\infty} p \frac{|Y_p|^2}{\omega + n^\circ \langle \dot{\varphi} \rangle + p\omega_b}. \tag{76}$$

As was pointed out at the end of Sec. 3, the ratio $\langle \dot{\varphi} \rangle / \omega_b$ becomes of order unity for large trapped orbit widths. Thus, with the potato ordering (1), several terms of the series in Eq. (76) must be retained to obtain convergence. In this case, the alternative form of $f^{(1)}$ as given by Eqs. (34), (48) and (49) may be more practical. Using this alternative form, δW_2 can be expressed as

$$\delta W_2 = \frac{2\pi^2 c}{Zem^2} i \sum_\sigma \int dP_\varphi dE d\mu \ (\omega - n^\circ \omega_*) \frac{\partial F}{\partial E} \left\{ \frac{|Y(\tau_b)|^2}{\exp\left[-i(\omega + n^\circ \langle \dot{\varphi} \rangle)\tau_b\right] - 1} + W(\tau_b) \right\}, \tag{77}$$

where, for any given choice of $(P_\varphi, E, \mu, \sigma)$ and ω, $Y(\tau_b)$ and $W(\tau_b)$ can be found by integrating numerically the system of equations:

$$\begin{aligned} \dot{\psi} &= \dot{\vec{R}} \cdot \nabla\psi, \\ \dot{Y} &= L^{(1)}, \\ \dot{\vartheta} &= \dot{\vec{R}} \cdot \nabla\vartheta, \\ \dot{W} &= L^{(1)*} Y. \\ \dot{\varphi} &= \dot{\vec{R}} \cdot \nabla\varphi, \end{aligned} \tag{78}$$

The integration is to be carried out from an initial arbitrary point $(\psi_o, \vartheta_o, \varphi_o)$ and $Y_o = 0$ at $\tau = 0$ to the time $\tau = \tau_b$ at which $\psi(\tau_b) = \psi_o$ and $\vartheta(\tau_b) = \vartheta_o$. Likewise, δW_1 can be written as

$$\delta W_1 = \frac{2\pi^2 c}{Zem^2} \sum_\sigma \int dP_\varphi dE d\mu Z(\tau_b), \tag{79}$$

where

$$\dot{Z} = \left[\left(\vec{\xi}_\perp \cdot \nabla\psi\right) \frac{Ze}{c} \frac{\partial F}{\partial P_\varphi} + \mu \frac{B_\parallel{}^{(1)}}{B} \frac{\partial F}{\partial \mu} \right] L^{(1)*}. \tag{80}$$

6. INTERNAL KINK STABILISATION

As a specific application of the theory presented in this paper, we shall discuss the problem of internal kink stabilisation in the presence of high energy ions with non-negligible orbit widths [3]. This problem in the limit of zero fast particle orbit widths has been studied by several authors, and we refer to Ref. 1 for a short review of the subject and the essential bibliography.

A derivation of the relevant dispersion relation can be obtained following the normal mode analysis approach outlined, e.g., in Ref. 17. Here, we follow a different approach based on the use of the quadratic form ΔW. As pointed out in the previous section, although ΔW is not in general a self-adjoint form, it can be used if a perturbative procedure can be established with the leading order terms constituting a self-adjoint form. For this purpose, we introduce the inverse aspect ratio $\varepsilon \equiv r/R$ as the relevant expansion parameter. We also consider a standard low-β (= kinetic/magnetic pressure) ordering, $\beta \sim \varepsilon^2$, where the equilibrium nested flux surfaces have cross-sections which depart from circles by shaping terms at most $0(\varepsilon)$. The q profile is taken to be a monotomic function of r with an on-axis value $q_0 < 1$ and finite magnetic shear in the $q \sim 1$ region. Finally, we consider a collection of high energy ions with mirror-trapped orbits and normalised pressure $\beta_h \sim \beta_c$. We point out that, with this ordering for β_h the fast ions will carry a diamagnetic current along the equilibrium field lines, associated with the trapped orbits and the P_φ dependence of the distribution function. This current density is of order ε compared with the total parallel current density, i.e.

$$J_{\parallel h} \sim \varepsilon J_{\parallel}. \tag{81}$$

The poloidal modulation of $J_{\parallel h}$ is in general of order unity.

After standard manipulations, the MHD functional δW_{MHD} can be written in the form

$$\delta W_{MHD} = \frac{1}{2}\int d^3x \left\{ \frac{\left|B_{\parallel}^{(1)}\right|^2}{4\pi} + \left(\frac{\left|B_{\perp}^{(1)}\right|^2}{4\pi} + \frac{1}{c} J_{\parallel} \vec{b} \cdot \vec{\xi}_{\perp}^{\dagger} \times \vec{B}_{\perp}^{(1)} \right) + \frac{1}{c} B_{\parallel}^{(1)} \vec{b} \cdot \vec{J}_{\perp} \times \vec{\xi}_{\perp}^{\dagger} \right.$$

$$\left. + \left(\vec{\xi}_{\perp} \cdot \nabla p_c\right) \nabla \cdot \vec{\xi}_{\perp}^{\dagger} + \Gamma p_c \left|\nabla \cdot \vec{\xi}\right|^2 \right\}, \tag{82}$$

where we have used the equation of state $p_c^{(1)} = -\vec{\xi}_\perp \cdot \nabla p_c - \Gamma p_c(\nabla \cdot \vec{\xi})$ for the isotropic core plasma pressure, with $\Gamma = 5/3$ the ratio of specific heats. We have also assumed that perturbations vanish at the plasma edge. The parallel displacement $\xi_\parallel$ enters only the positive-definite term involving $|\nabla \cdot \vec{\xi}|^2$.

Formally, the largest of all terms in (82) is the one involving $|B_\parallel^{(1)}|^2$. In order to minimise this term we set

$$\nabla \cdot \vec{\xi}_\perp + 2\vec{\xi}_\perp \cdot \vec{\kappa} = 0(\varepsilon \xi_\perp / R), \tag{83}$$

from which it follows that $|B_\parallel^{(1)} / B|^2 = 0(\varepsilon^2 \xi^2 / R^2)$ [cf. Eq. (53)]. The term involving $|\nabla \cdot \vec{\xi}|^2$ can be minimised separately with an appropriate choice of $\xi_\parallel$. After these initial positions, ΔW up to terms of order ε^3 becomes:

$$\Delta W = \Delta W_2 + \Delta W_3 + 0(\varepsilon^4), \tag{84}$$

where

$$\Delta W_2 = \frac{1}{2}\int d^3x \left[\frac{|\vec{B}_\perp^{(1)}|^2}{4\pi} + \frac{1}{c}\left(\frac{\overline{J_\parallel}}{B}\right)\vec{B} \cdot \vec{\xi}_\perp^\dagger \times \vec{B}_\perp^{(1)}\right] = 0(\varepsilon^2), \tag{85}$$

$$\Delta W_3 = \frac{1}{2}\int d^3x \frac{1}{c}\left[\left(\frac{J_\parallel}{B}\right) - \left(\frac{\overline{J_\parallel}}{B}\right)\right] \vec{B} \cdot \vec{\xi}_\perp^\dagger \times \vec{B}_\perp^{(1)} + \delta W_{hot} = 0(\varepsilon^3), \tag{86}$$

and an overbar indicates flux surface averaging.

It can be easily shown that ΔW_2 is a self-adjoint form for perturbations which vanish at the plasma edge. The corresponding Euler-Lagrange equation for the radial displacement is

$$\frac{d}{dr} r^3 (\vec{k} \cdot \vec{B})^2 \frac{d\xi_r}{dr} = 0(\varepsilon), \tag{87}$$

where $\vec{k} \cdot \vec{B} = (B_\vartheta / r)(1 - q)$. Note that the flux surface average of the fast particle current is included in ΔW_2 and gives a contribution $0(\varepsilon)$ to the equilibrium q profile. Thus, ΔW_2 is minimised by the cylindrical displacement

$$\vec{\xi}_{\perp o} = \xi_o[(\vec{e}_r + i\vec{e}_\vartheta)H(r_s - r) - ir\delta(r - r_s)\vec{e}_\vartheta] \exp[i(\vartheta - \varphi)], \tag{88}$$

with $\xi_o = \text{constant}$, H(x) is the unit step function, and $\vec{\xi}_{\perp o}^{\dagger} = \vec{\xi}_{\perp o}^{*}$. When $\vec{\xi}_{\perp o}$ is inserted in ΔW_2, corrections $0(\varepsilon)$ vanish upon poloidal integration and ΔW_2 drops to $0(\varepsilon^4)$. Also, the first integral in Eq. (86) vanishes to leading order when $\vec{\xi}_{\perp} = \vec{\xi}_{\perp o}$ is used. Thus, we are left with the leading order result

$$\Delta W \approx \frac{1}{2}\int d^3x \vec{\xi}_{\perp o}^{*} \cdot \left(\nabla \cdot \vec{P}_h^{(1)}\right) = 0\left(\varepsilon^3\right), \tag{89}$$

where $\vec{P}_h^{(1)}$ is computed to leading order in ε using the cylindrical displacement (88). Hence, we conclude that, when $\beta_h \sim \beta \sim \varepsilon^2$, the minimum values of ΔW is $0\left(\varepsilon^3\right)$ and stability (apart from dissipative effects) is entirely determined by the sign of δW_{hot}. If we wish to consider standard internal kink modes, which correspond to $\Delta W = 0\left(\varepsilon^4\right)$, we need to depress the ordering of β_h to $0\left(\varepsilon^3\right)$. Then, $\delta W_{hot} = 0\left(\varepsilon^4\right)$ competes with the usual MHD result. For instance, one can adopt Bussac's model for the MHD functional $\delta\hat{W}_{MHD} = \delta W_{MHD} / \left[(\xi_o / R_o)^2 V_s B^2\right] \approx -c_o \varepsilon_s^2 \left[\beta_p^2 - \left(\beta_p^{MHD}\right)^2\right]$, where c_o is a numerical factor of order unity that depends on the q profile, $V_s = 2\pi^2 R_o r_s^2$, $\varepsilon_s \equiv r_s / R_o$, β_p is the plasma bulk poloidal beta properly defined [18] within the q = 1 volume, and $\beta_p^{MHD} \le 0.3$ depends on q and on plasma shaping. By comparison, $\delta\hat{W}_{hot} \equiv \delta W_{hot} / \left[(\xi_o / R)^2 V_s B^2\right] \approx c_1 \varepsilon_s \beta_{ph}$ in the thin banana limit, with β_{ph} the fast ion poloidal beta. Thus, $\delta W_{MHD} \sim \delta W_{hot}$ when $\beta_{ph} \sim \varepsilon_s \beta_p^2$.

We point out that the fast ion parallel current density plays a role, through its flux surface average value, by contributing to the field line bending energy term, ΔW_2, but it does not contribute explicitly to δW_{hot} to relevant leading order in ε. Thus, we disagree with a statement in Ref. 12, where the part of ΔW_2 involving $J_{\parallel h}$ is used to nearly cancel δW_{hot}. As a result, it is argued in Ref. 12 that the trapped fast particle contributions to ΔW scale as $(1-q)^2 \varepsilon_s \beta_{ph}$. Instead, we have shown that the part of ΔW_2 involving $J_{\parallel h}$ is annihilated, together with the other terms in ΔW_2, by the cylindrical displacement (88). As a result, we find that δW_{hot} scales as a single power of (1 - q). This result remains valid when finite particle orbit widths are taken into account [3].

A dispersion relation can be obtained by including the kinetic energy contribution,

$$\delta I = -\frac{1}{2}\int d^3x \ \omega(\omega - \omega_{*i}) m_i n_i |\vec{\xi}|^2, \tag{90}$$

which is important in an inertial layer of width $\Delta / r_s \sim |\omega(\omega - \omega_{*i})|^{1/2} / \omega_A$ around the q = 1 surface, with $\omega_{*i} = n^{o}(c / Z_i e B_\vartheta R n_i)(dp_i / dr)$ the thermal ion diamagnetic frequency, $\omega_A = v_A / (3\hat{s}R_o)$ the relevant Alfvén frequency and $\hat{s} \equiv r_s q'(r_s)$. The Euler-Lagrange equation for ξ_r within the layer is $(d/dx)\left\{\left[\omega(\omega - \omega_{*i}) - \omega_A^2 x^2\right] d\xi_r / dx\right\} = 0$, where $x \equiv (r - r_s)/r_s$. This equation can be solved readily and matched to the solution of the outer Eq. (87). It is then straightforward to show that $\delta\hat{I} \equiv \delta I / \left[(\xi_o / R_o)^2 V_s B^2\right] \approx -\, i\left[\omega(\omega - \omega_{*i}) / \omega_A^2\right]^{1/2}$. Combining $\delta\hat{I} + \Delta\hat{W} = 0$ yields the dispersion relation

$$\left[\omega(\omega - \omega_{*i})\right]^{1/2} = -i\omega_A\left(\delta\hat{W}_{MHD} + \delta\hat{W}_{hot}\right), \tag{91}$$

where all terms are formally of the same order when $\beta_h \sim \varepsilon\beta_c \sim \varepsilon^3$. For $\beta_h \sim \varepsilon^2$, $\delta\hat{W}_{MHD} / \delta\hat{W}_{hot} \sim \varepsilon$ and the MHD integral drops out of Eq. (91)

The evaluation of $\delta\hat{W}_{hot}$ for finite orbit widths in the limit $\omega / \langle\dot{\varphi}\rangle \to 0$ has been reported in Ref. 3. For the sake of completeness, we summarise here the main results. For "potato" orbits, $(v_\parallel / v_\perp)^2 \sim \delta_p / R \sim \varepsilon$, and using Eq. (83) the perturbed Lagrangian reduces to

$$L^{(1)} = -\mu B\left(\vec{\xi}_\perp \cdot \vec{\kappa}\right) + 0(\varepsilon). \tag{92}$$

Also, Eq. (83) implies that $B_\parallel^{(1)} / B_\perp^{(1)} \sim \varepsilon$. Thus, the term involving $B_\parallel^{(1)}$ can be neglected in Eq. (71) for δW_1. Furthermore, $\sigma - 1 = 0\left(\varepsilon^2\right)$ and $\tau - 1 = 0(\varepsilon)$, so that also the first integral in Eq. (69) can be neglected. Taking the limit $\omega / \langle\dot{\varphi}\rangle \to 0$, straightforward algebra leads to the leading order result

$$\delta W_{hot} \approx 2\pi^2 \xi_0^2 m^{-3} \int dP_\varphi dE d\Lambda \tau_b \frac{\partial F}{\partial P_\varphi} \frac{E^3 \Lambda^2}{R^2 \Omega\langle\dot{\varphi}\rangle} (A - S), \tag{93}$$

where $\Lambda = \mu B_o / E$,

$$A = \frac{R^2 \Omega\langle\dot{\varphi}\rangle}{m E \Lambda}\left\langle\left|\frac{\xi_r}{\xi_o}\right|^2 \frac{\varepsilon \cos\vartheta}{q}\right\rangle \tag{94}$$

$$S = \sum_{-\infty}^{\infty} p \frac{\langle\dot{\varphi}\rangle}{\langle\dot{\varphi}\rangle + p\omega_b}\left|\frac{R_o Y_p}{\xi_o \mu B_o}\right|^2 \tag{95}$$

are dimensionless quantities, and Y_p is defined in Eq. (46).

Analytical progress is possible with the following choice for the trapped particle distribution function:

$$F\left(P_\varphi, E, \Lambda\right) = C\, H\left(P_\varphi\right) H\left(P_{\varphi *} - P_\varphi\right) \delta(E - E_h)\, \delta(\Lambda - 1), \qquad (96)$$

where $P_{\varphi *}$ and E_h are constant values and C is a normalization constant. With this choice, all fast ions have an energy E_h and are mirror-trapped with their orbit tips distributed along a vertical layer up to a distance r_* from the magnetic axis, corresponding to $P_{\varphi *} = (Ze/c)\psi(r_*)$ (>0). This choice mocks up the case where the fast ions are produced by ion cyclotron radio frequency heating absorbed along a vertical resonant layer passing through the magnetic axis and focused within a distance r_* from this axis. High-energy ions populate the low-field side of the Tokamak up to a distance $r \le r_* + \delta_{ph}$ from the axis, where $\delta_{ph} = \left(2qv_h / \Omega R_o\right)^{2/3} R_o$ and $mv_h^2 / 2 \equiv E_h$. When $r_* \le \delta_{ph}$, most fast ions have potato-shaped orbits and our new analysis is required. Let us define the orbit tip radius, r_h, such that $P_\varphi = (Ze/c)\psi(r_h)$. In the limit $r_h << \delta_{ph}$, small shear $\left(\hat{s} << 1 \text{ for } r \le \delta_{ph}\right)$, and for $\Lambda = 1$ the bounce frequency is

$$\omega_b \approx \alpha_o \Omega_h q_o^{-2} \left(\delta_{ph} / R_o\right)^2 \propto E_h^{2/3}, \qquad (97)$$

where $\alpha_o = 3\sqrt{\pi}\Gamma(2/3)/4\Gamma(1/6)$ and $\Omega_h = Z_h eB / m_h c$. The precession frequency is

$$\langle \dot{\varphi} \rangle \approx \alpha_1 \Omega_h q_o^{-1} \left(\delta_{ph} / R_o\right)^2 \propto E_h^{2/3}, \qquad (98)$$

where $\alpha_1 = \sqrt{\pi}\Gamma(2/3)/2\Gamma(1/6)$. The ratio of the two frequencies,

$$\langle \dot{\varphi} \rangle / \omega_b \approx 2q_o / 3, \qquad (99)$$

is of order unity, as anticipated at the end of Sec. 3.

For a parabolic q profile, the integration of $\delta\hat{W}_{hot}$ leads formally to the result [3]

$$\delta\hat{W}_{hot} = \frac{r_s}{R_o} g\left(\frac{r_*}{r_s}, \frac{\delta_{ph}}{r_s}, q_o\right), \qquad (100)$$

where the function g is determined numerically. In the limit r_*, $\delta_{ph} << r_s$, shear effects are negligible and the function $g \to \hat{g}\left(r_* / \delta_{ph}, q_o\right)$. An example of the behaviour of $\hat{g}$ versus r_* / δ_{ph} for $q_o = 0.7$ is shown in Fig. 1. The thin banana approximation is recovered for $r_* / \delta_{ph} > 1$. For $r_* / \delta_{ph} \leq 1$, the finite orbit theory leads to a value of $\delta\hat{W}_{hot}$ which is smaller than that obtained by extrapolation of the thin banana theory, corresponding to the dashed curve in Fig. 1.

From these results it may be concluded that the thin banana approximation yields roughly the correct numerical value of $\delta\hat{W}_{hot}$, provided one does not allow the radial width of the fast ion distribution function to become smaller than the average orbit width. The situation, however, becomes more complex if the q profile varies significantly along the particle orbit, for instance if δ_{ph} approaches r_s. In this case, the full dependence of the function g in Eq. (100) must be investigated. Results are summarised in Fig. 2 where a parabolic q profile has been assumed. We can conclude from this figure that fast particle stabilisation is hindered as $\delta\hat{W}_{hot}$ changes sign when δ_{ph} (or r_*, whichever is larger) approaches r_s. Thus, it is important to keep the fast ions well contained within the $q = 1$ surface in order to exploit their stabilising potential.

Considering parameters of high-power ($P_{RF} \geq 10$ MW), ICRF heated experiments in the Joint European Torus (JET), we find that r_s ~30 - 50 cm, δ_p ~ 20 - 30 cm, while the width of the power deposition profile can be as narrow as r_* ~ 15 - 20 cm. During sawtooth-free periods, the value of q on axis is believed to drop significantly below unity. In this case, because of q variation, finite orbit effects can alter significantly the stabilisation properties as compared to the prediction of the thin banana orbit theory. JET experimental parameters lie in the range δ_p/r_s ~ 0.4 → 1, and r_*/r_s ~ 0.3 → 0.7. It follows that there is indeed a significant quantative difference between the prediction of the thin banana theory and that of the full orbit theory, the latter being more pessimistic. One interesting consequence that may be checked experimentally is the existence of an optimum ICRF heating power to produce maximum stabilisation. Exceedingly large values of P_{RF} may result in orbits whose size is too large to produce effective stabilisation.

7. CONCLUSIONS

In this paper, we have studied the response of a collisionless guiding centre plasma component to global perturbations of an axisymmetric toroidal magnetic configuration. The solution of the linearised drift-kinetic equation has been expressed in compact form in terms of the guiding centre Lagrangian, as given, e.g., in Ref. [11].

In the limit where the radial excursion of particle orbits across magnetic surfaces is negligible, the perturbed distribution function reduces to the one obtained by Antonsen and Lee [8], which applies to arbitrary poloidal mode numbers. Thus our work generalises that of Ref. 8 by taking into account finite orbit widths. In particular, the present analysis is capable of handling the "potato-orbit" limit [3] where $\delta_b \sim r$, with $\delta_b \sim \varepsilon^{1/2}\rho_\vartheta$ the width of a "banana" orbit and $\varepsilon \equiv r / R_0$ the inverse aspect ratio of the toroidal plasma. Since the guiding centre approximation relies on the ratio between the Larmor radius and the magnetic curvature radius, $\delta \equiv \rho / R_0$, as being small, the potato ordering requires for consistency that ε is also a small expansion parameter. Indeed, $\delta_b \sim r$ implies that $\varepsilon \sim \delta^{2/3}$ in a Tokamak. In this respect, we note that the analysis of Ref. 8 assumed $\delta \to 0$ and $\varepsilon \sim 1$. Also, we observe that the time scales associated with the drift and bounce motions of a mirror-trapped particle become comparable in the potato limit. This may have important consequences, as the constraints on the particle dynamics imposed by the conservation of the longitudinal and flux invariants [13] are simultaneously satisfied or violated when $\delta_b \sim r$, depending on the relevant space and time scales of variation of the electromagnetic field.

Quadratic forms have been constructed by taking the scalar product of the momentum balance equation with the adjoint displacement for the case of perturbations satisfying the MHD constraint, $\vec{B}^{(1)} = \nabla \times \left(\vec{\xi}_\perp \times \vec{B}\right)$. Because of the presence of the high energy particles whose dynamics is intrinsically kinetic, these quadratic forms are in general non-Hermitian, therefore necessary and sufficient criteria for stability cannot be obtained solely on the basis of their sign. However, these quadratic forms become useful in those cases where the mode structure to leading order is determined by the bulk plasma while the kinetic fast particle response can be treated perturbatively. Examples are internal kink modes [1, 3] and the Toroidicity-induced Alfvén Eigenmodes [4-6, 9, 16], which can be either stabilised or driven unstable by the high energy ions.

As a specific application of the present theory, we have discussed the problem of internal kink stabilisation [1, 3]. We have shown that the stabilising influence of the high energy ions is weakened when the potato width becomes important, and they may even become destabilizing as δ_p approaches the radius of the $q = 1$ surface.

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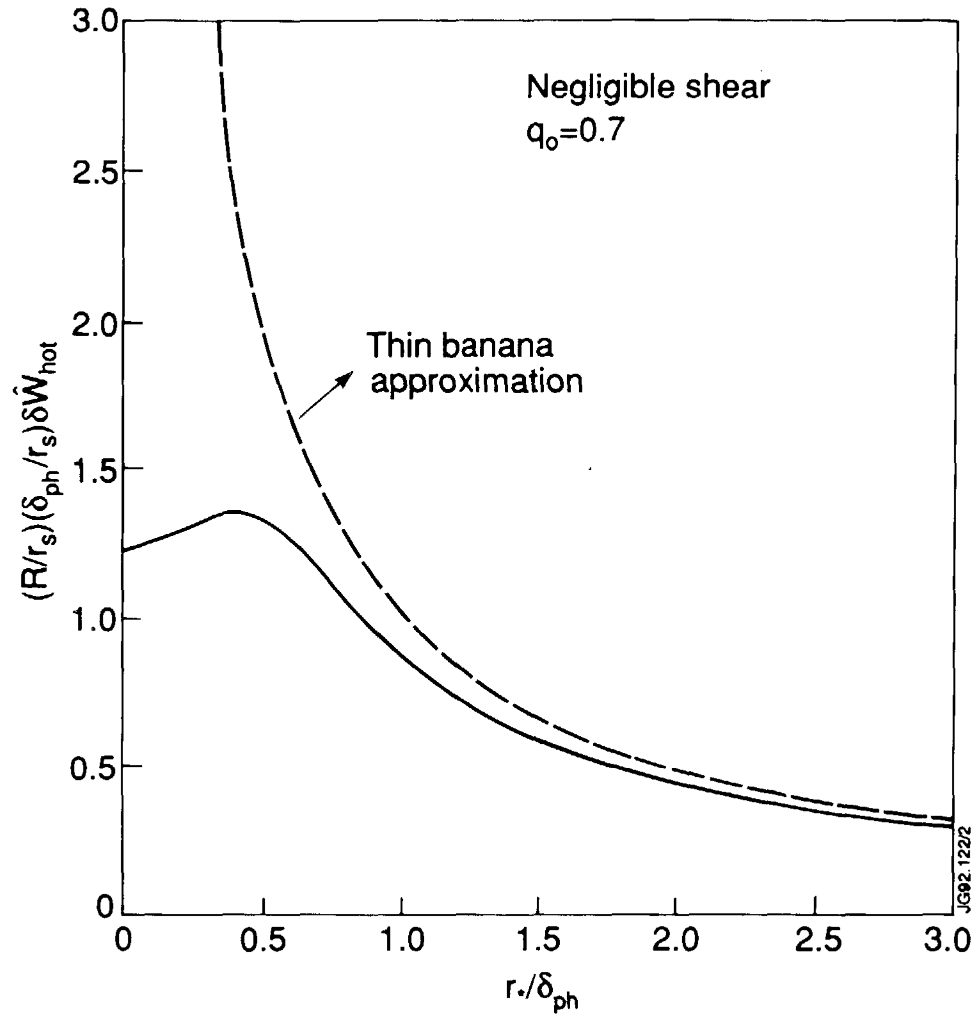

Fig. 1 Graphs of $(R/r_s)\left(\delta_{ph}/r_s\right)\delta\hat{W}_{hot}$ as a function of r_*/δ_{ph} for a flat q profile in the region $0 \le r \le r_*$ and $q_o = 0.7$. Solid curve: full orbit theory. Dashed curve: thin banana approximation. The dashed curve is the hyperbola $(R/r_s)\left(\delta_{ph}/r_s\right)\delta\hat{W}_{hot} = \delta_{ph}/r_*$.

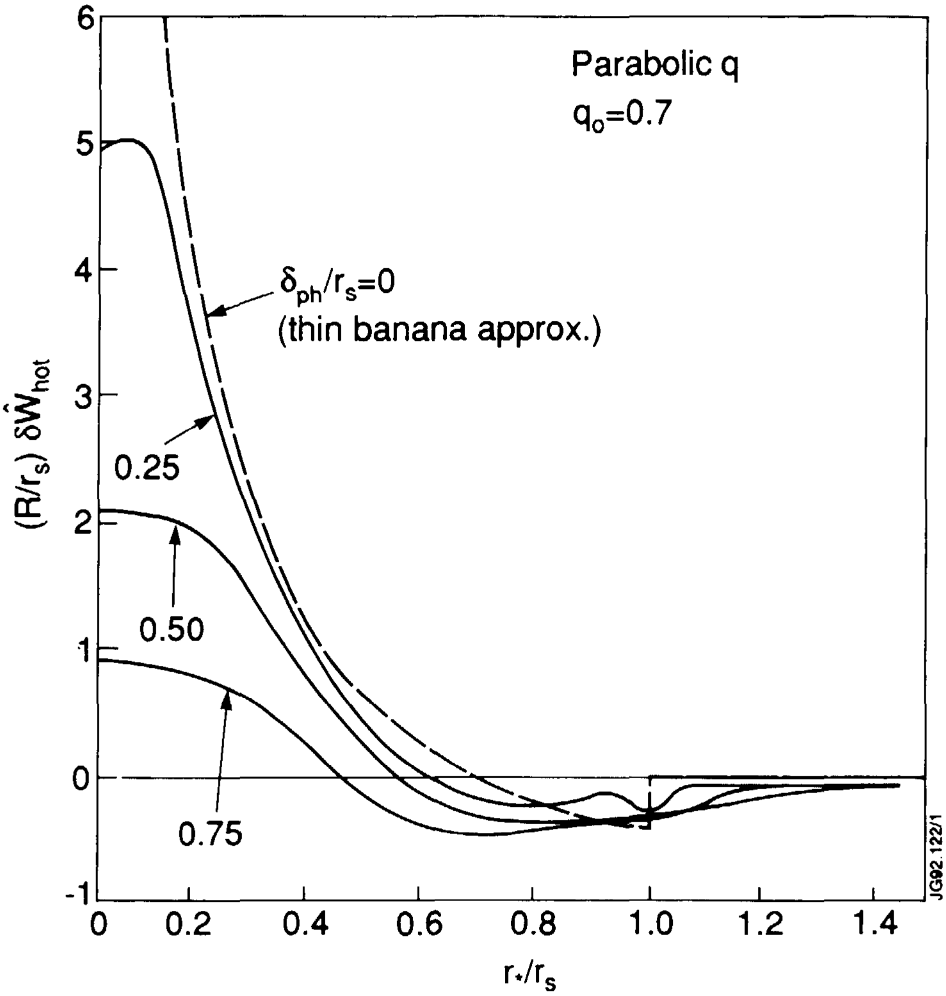

Fig. 2 Graphs of $(R/r_s)\ \delta\hat{W}_{hot}$ as a function r_*/r_s and of $\left(\delta_{ph}/r_s\right)$, as indicated near each curve, for a parabolic q profile with $q_o = 0.7$.