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H. Bindslev

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H. Bindslev

JET-Joint Undertaking, Culham Science Centre, OX14 3DB, Abingdon, UK

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Theoretical models of Thomson scattering in plasmas at frequencies which are not large relative to the plasma frequency rely, among other things, on an accurate description of three wave mixing in plasmas. Thomson scattering experiments making use of waves with frequencies in this range include fast ion diagnostics and various fluctuation and magnetic field diagnostics, but generally exclude conventional electron temperature diagnostics. The theory was originally developed using a traditional fluid approach [AKHIEZER *et al.*, 1962, 1967; SITENKO, 1967]. These results, which have been quoted widely, are, however, at variance with low temperature results obtained by AAMODT and RUSSELL (1992) using a kinetic approach.

In this paper the theory of three wave mixing and Thomson scattering in plasmas, is reexamined in the low temperature limit with a kinetic model, giving a more complete description of Thomson scattering in this limit. Errors in the traditional fluid approach to three wave mixing and scattering are identified and a new corrected fluid approach is outlined. These corrections to the theory may have important consequences for the analysis of collective Thomson scattering experiments, such as fast ion diagnostics, and for the assessment of the feasibility of certain measurements.

In a collisionless plasma the dynamics of the electron momentum distribution, $f(\mathbf{p}, \mathbf{r}, t)$, are governed by the Vlasov equation,

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + q_e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{p}} = 0\,,$$

where $\mathbf{v} = \mathbf{p}/(\gamma m_e)$, $\gamma = \sqrt{1 + p^2/(m_e c)^2}$, $p = |\mathbf{p}|$, and m_e and q_e are the electron rest mass and charge respectively. Three wave mixing and scattering are due to the non-linearity of the dielectric response of plasmas. In a collisionless plasma the non-linearity is represented by the third term in the Vlasov equation. We now seek solutions, $[\mathbf{E}, \mathbf{B}, f]$, to the Maxwell–Vlasov set of equations as perturbation expansions around the equilibrium values, $[0, \mathbf{B}^{(0)}, f^{(0)}]$. The first order equations describe the familiar linear wave propagation. The second order equations account for bilinear interactions between linear waves, which includes three wave mixing and Thomson scattering in the first Born approximation. The scattered wave $[2\sigma] \equiv [\mathbf{E}^{(2\sigma)}, \mathbf{B}^{(2\sigma)}, f^{(2\sigma)}]$ is that part of the second order perturbation which results from the interaction of two linear waves, $[1\mathrm{a}] \equiv [\mathbf{E}^{(1\mathrm{a})}, \mathbf{B}^{(1\mathrm{a})}, f^{(1\mathrm{a})}]$ and $[1\mathrm{b}] \equiv [\mathbf{E}^{(1\mathrm{b})}, \mathbf{B}^{(1\mathrm{b})}, f^{(1\mathrm{b})}]$. Defining the linear operator

$$\mathcal{L} f = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + q_e(\mathbf{v} \times \mathbf{B}^{(0)}) \cdot \frac{\partial f}{\partial \mathbf{p}}\,,$$

the second order Vlasov equation governing $f^{(2\sigma)}$ may be written as

$$\mathcal{L} f^{(2\sigma)} = -\mathbf{F}^{(2\sigma)} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{p}} - \mathbf{F}^{(1\mathrm{a})} \cdot \frac{\partial f^{(1\mathrm{b})}}{\partial \mathbf{p}} - \mathbf{F}^{(1\mathrm{b})} \cdot \frac{\partial f^{(1\mathrm{a})}}{\partial \mathbf{p}}\,, \qquad (1)$$

where $\mathbf{F} = q_e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. The left hand side and the first term on the right hand side of equation (1) are identical to the linearized Vlasov equation. The electromagnetic field of the scattered wave may thus be found from the inhomogeneous wave equation[1]

$$\mathbf{\Lambda}(\mathbf{k}^\sigma,\omega^\sigma)\cdot\mathbf{E}^{(2\sigma)}(\mathbf{k}^\sigma,\omega^\sigma) = \frac{-i}{\omega^\sigma\varepsilon_0}\mathbf{j}^\sigma(\mathbf{k}^\sigma,\omega^\sigma)\ . \tag{3}$$

The source current, $\mathbf{j}^\sigma(\mathbf{k}^\sigma,\omega^\sigma)$, is given by

$$\mathbf{j}^\sigma(\mathbf{k}^\sigma,\omega^\sigma) = q_e\int \mathbf{v} f^\sigma(\mathbf{p},\mathbf{k}^\sigma,\omega^\sigma)\,d\mathbf{p}\ , \tag{4}$$

where $f^\sigma(\mathbf{p},\mathbf{k}^\sigma,\omega^\sigma)$ is the Fourier–Laplace transform of $f^\sigma(\mathbf{p},\mathbf{r},t)$, which satisfies the relation:

$$\mathcal{L}f^\sigma = -\mathbf{F}^{(1a)}\cdot\frac{\partial f^{(1b)}}{\partial\mathbf{p}} - \mathbf{F}^{(1b)}\cdot\frac{\partial f^{(1a)}}{\partial\mathbf{p}}\ . \tag{5}$$

$\Lambda_{ij}(\mathbf{k},\omega) = \varepsilon_{ij}(\mathbf{k},\omega) + N^2\{\hat{k}_i\hat{k}_j - \delta_{ij}\}$ is the wave tensor, $N = |\mathbf{k}|c/\omega$, $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ and $\boldsymbol{\varepsilon}(\mathbf{k},\omega)$ is the dielectric tensor.

Given the source current $\mathbf{j}^\sigma$ the field $\mathbf{E}^{(2\sigma)}$ of the scattered wave can be found from equation (3) (see e.g. BINDSLEV (1991)). To find the source current we first obtain $f^\sigma(\mathbf{p},\mathbf{k}^\sigma,\omega^\sigma)$ from equation (5) by integration along characteristics followed by a Fourier–Laplace transform over space and time. The result is [BINDSLEV, 1993]

$$f^\sigma = \frac{e^{-i\beta\sin\phi}}{\omega_c/\gamma}\sum_{s=-\infty}^{\infty}\frac{1}{i(\alpha-s)}\frac{1}{2\pi}\int_0^{2\pi} e^{i\beta\sin(\phi-\tau)}Q(\phi-\tau)e^{-is\tau}\,d\tau\ , \tag{6}$$

where $\omega_c = -q_eB^{(0)}/m_e$ is the angular electron cyclotron frequency, $\alpha = \gamma(v_\parallel k_\parallel^\sigma - \omega^\sigma)/\omega_c$, $\beta = \gamma v_\perp k_\perp^\sigma/\omega_c$ and

$$Q(\phi) = -\int\frac{d\mathbf{k}\,d\omega}{(2\pi)^4}\left(\mathbf{F}^{(1a)}(\mathbf{p},\mathbf{k}^a,\omega^a)\cdot\frac{\partial f^{(1b)}(\mathbf{p},\mathbf{k}^b,\omega^b)}{\partial\mathbf{p}} + \mathbf{F}^{(1b)}(\mathbf{p},\mathbf{k}^b,\omega^b)\cdot\frac{\partial f^{(1a)}(\mathbf{p},\mathbf{k}^a,\omega^a)}{\partial\mathbf{p}}\right)\ , \tag{7}$$

with $\mathbf{k}^a+\mathbf{k}^b = \mathbf{k}^\sigma$, $\omega^a+\omega^b = \omega^\sigma$, $\mathbf{k}^a-\mathbf{k}^b = \mathbf{k}$ and $\omega^a-\omega^b = \omega$. $v_\parallel$ and $v_\perp$ are the parallel and perpendicular components of $\mathbf{v}$ relative to $\mathbf{B}^{(0)}$, and similarly for other vectors. ϕ is the azimuthal angle of $\mathbf{p}$ in the coordinate system where $\hat{\mathbf{z}} = \mathbf{B}^{(0)}/|\mathbf{B}^{(0)}|$ and $\mathbf{k}^\sigma = k_\perp^\sigma\hat{\mathbf{x}} + k_\parallel^\sigma\hat{\mathbf{z}}$. Expression (4) for the source current with f^σ given by (6) is fully relativistic. We note that in a relativistic treatment of three wave mixing and scattering the full details of the momentum distributions of the interacting waves, [1a] and [1b], are required.

[1] In this paper a spatial Fourier transform and a temporal Laplace transform are used:

$$A(\mathbf{r},t) = \int_{-\infty+i\nu}^{\infty+i\nu}\int_{R^3} A(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\frac{d\mathbf{k}d\omega}{(2\pi)^4}\ ,\quad A(\mathbf{k},\omega) = \int_0^\infty\int_{R^3} A(\mathbf{r},t)e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,d\mathbf{r}dt\ . \tag{2}$$

The Fourier–Laplace transform is defined for values of ω_i large enough for the integral over t to exist. This defines the contours of other integrations when the analytic continuation to real values of ω is required.

In the low temperature limit, or more specifically when f^σ is significant only for values of $\mathbf{v}$ which satisfy the inequalities $v/c \ll 1$, $v_\| k_\|^\sigma/(\omega^\sigma + s\omega_c) \ll 1$, $s \in Z$ and $v_\perp k_\perp^\sigma/\omega_c \ll 1$, we can expand $\exp\{i\beta(\sin(\phi-\tau) - \sin\phi)\}/(\alpha - s)$ in powers of $\mathbf{v}$. This permits the integrations and summations in (4) and (6) to be carried out and we find [BINDSLEV, 1993]

$$j_i^\sigma = \frac{-i\omega^\sigma\varepsilon_0}{n^{(0)}}\int\frac{d\mathbf{k}\,d\omega}{(2\pi)^4}\Big\{\chi_{il}^\sigma\Big[n^{(1b)}E_l^{(1a)} + n^{(1a)}E_l^{(1b)} + \epsilon_{lmn}\left(\overline{v}_m^{(1b)}B_n^{(1a)} + \overline{v}_m^{(1a)}B_n^{(1b)}\right)\Big] + X_{ijl}^\sigma\frac{1}{c}\left(\overline{v}_j^{(1b)}E_l^{(1a)} + \overline{v}_j^{(1a)}E_l^{(1b)}\right)\Big\} \tag{8}$$

where $n^{(a)} = \int f^{(a)}\,d\mathbf{p}$ and $\overline{v}_j^{(a)} = \int v_j f^{(a)}\,d\mathbf{p}$ are the electron density and flux fluctuations associated with the momentum perturbation $f^{(a)}$. $\overline{\mathbf{v}}$ should not be confused with a fluid velocity. χ_{ij}^σ is the cold plasma susceptibility tensor evaluated with the frequency ω^σ, and

$$X_{ijl}^\sigma = -\frac{\omega_p^2}{(\omega^\sigma)^2}(\delta_{ja}\delta_{lb} + \delta_{la}\delta_{jb})\left\{\frac{ik_\perp^\sigma c}{\omega_c}\left[\delta_{a2}T_{ib}^{(1\alpha)} - T_{iab}^{(2)}\right] + \frac{k_\|^\sigma c}{\omega^\sigma}\delta_{a3}T_{ib}^{(1\beta)}\right\}, \tag{9}$$

$$T_{ij}^{(1\alpha)} = \frac{1}{1-\Omega^2}\eta_{ij} - \frac{i\Omega}{1-\Omega^2}\xi_{ij} + \zeta_{ij}, \tag{10}$$

$$T_{ij}^{(1\beta)} = \frac{1+\Omega^2}{(1-\Omega^2)^2}\eta_{ij} - \frac{i2\Omega}{(1-\Omega^2)^2}\xi_{ij} + \zeta_{ij}, \tag{11}$$

$$T_{ijk}^{(2)} = \frac{i\Omega\delta_{k1} + (1-2\Omega^2)\delta_{k2}}{1-4\Omega^2}\eta_{ij} + \frac{2\Omega^2\delta_{k1} - i\Omega\delta_{k2}}{1-4\Omega^2}\xi_{ij} + \frac{i\Omega\delta_{k1} + \delta_{k2}}{1-\Omega^2}\zeta_{ij}. \tag{12}$$

Here $\Omega = \omega_c/\omega^\sigma$ and $\eta_{ij} = \delta_{ij} - \delta_{i3}\delta_{j3}$, $\xi_{ij} = \epsilon_{ij3}$ and $\zeta_{ij} = \delta_{i3}\delta_{j3}$, where ϵ_{ijk} is the standard Levi-Civita symbol ($\epsilon_{ijk}a_jb_k = \{\mathbf{a}\times\mathbf{b}\}_i$). The terms involving $\boldsymbol{\chi}^\sigma$ are in agreement with the result found by AAMODT and RUSSELL (1992). By comparison the traditional fluid approach [AKHIEZER *et al.* (1967); SITENKO (1967)] gives

$$\mathbf{j}^\sigma = \int\frac{d\mathbf{k}\,d\omega}{(2\pi)^4}\Big\{\frac{-i\varepsilon_0}{n^{(0)}}\left(\omega^a\boldsymbol{\chi}^a\cdot\mathbf{E}^{(1a)}n^{(1b)} + \omega^b\boldsymbol{\chi}^b\cdot\mathbf{E}^{(1b)}n^{(1a)}\right) - i\varepsilon_0\omega^\sigma\boldsymbol{\chi}^\sigma\cdot\left(\mathbf{u}^{(1a)}\times\mathbf{B}^{(1b)} + \mathbf{u}^{(1b)}\times\mathbf{B}^{(1a)}\right) - \frac{\varepsilon_0 m_e}{q_e}\omega^\sigma\boldsymbol{\chi}^\sigma\cdot\left[\mathbf{u}^{(1a)}\left(\mathbf{k}^\sigma\cdot\mathbf{u}^{(1b)}\right) + \mathbf{u}^{(1b)}\left(\mathbf{k}^\sigma\cdot\mathbf{u}^{(1a)}\right)\right]\Big\}, \tag{13}$$

where $\mathbf{u}^{(i)} = \mathbf{v}^{(i)}/n$. Expression (13) can be brought to the form presented by AKHIEZER *et al.* and SITEKO by use of the linearized fluid equations and Maxwell's equations.

Comparing expressions (8) and (13) we find that in the first two terms, representing scattering due to the interaction of electric fields with density perturbations ($\mathbf{E}^{(1a)}n^{(1b)}$

and $\mathbf{E}^{(1b)}n^{(1a)}$), the factor $\omega^\sigma \boldsymbol{\chi}^\sigma$ in the kinetic expression (8) is replaced in the first and second terms of the traditional fluid expression (13) by $\omega^a \boldsymbol{\chi}^a$ and $\omega^b \boldsymbol{\chi}^b$ respectively. In the third and fourth terms, representing interaction between particle fluxes and magnetic fields ($\overline{\mathbf{v}}^{(1b)} \times \mathbf{B}^{(1a)}$ and $\overline{\mathbf{v}}^{(1a)} \times \mathbf{B}^{(1b)}$), we find complete agreement between the two results (the difference between $\mathbf{v}/n^{(0)}$ and $\mathbf{u}$ is of higher order and not significant at this point). In the fifth and sixth terms we find considerable differences between the two expressions. While the differences are of minor practical importance to most laser scattering experiments, they are not negligible for the millimetre wave scattering experiments planned at JET [COSTLEY *et al.*, 1988] and TFTR [WOSKOV *et al.*, 1988], and from a theoretical point of view these discrepancies are clearly not satisfactory. The two expressions (8) and (13) for the source current result in different expressions for the dielectric form factors and thus ultimately in different predictions for the scattered power. As an example we note that the ratio of the dielectric form factors, $G_{\text{new}}/G_{\text{old}}$ (or equivalently the ratio of scattering cross sections) associated with the interaction of the incident field, $\mathbf{E}^{(1a)}$, with density fluctuations, $n^{(1b)}$, is given by [BINDSLEV, 1993]

$$\frac{G_{\text{new}}}{G_{\text{old}}} = \left(\frac{\omega^\sigma}{\omega^a}\right)^2 \frac{|\mathbf{e}^{\sigma *} \cdot \boldsymbol{\chi}^\sigma \cdot \mathbf{e}^a|^2}{|\mathbf{e}^{\sigma *} \cdot \boldsymbol{\chi}^a \cdot \mathbf{e}^a|^2} , \tag{14}$$

where $\mathbf{e}$ is the unit electric field vector. To illustrate the significance of these differences we have plotted in Figure 1 the ratio $G_{\text{new}}/G_{\text{old}}$ for X to X mode scattering as a function of ω^σ for a) $\omega_c/\omega^a = 0.68$, b) $\omega_c/\omega^a = 0.80$ and c) $\omega_c/\omega^a = 0.90$. The densities are in all three cases chosen sufficiently low to ensure that both incident [1a] and scattered radiation [2σ] are well away from the cutoff. In the chosen parameter range the present low temperature theory is a good approximation even at the temperatures found in JET. The ratio is not sensitive to density in this parameter range.

The differences between the two results are due to errors in the traditional fluid result. In any fluid approach the source current is derived from the second order fluid equations. In the traditional fluid approach a perturbation expansion of the fluid equations is obtained by assuming perturbation expansions of the density, $n = n^{(0)} + n^{(1)} + n^{(2)} + \ldots$, and the fluid velocity $\mathbf{u} = \mathbf{u}^{(1)} + \mathbf{u}^{(2)} + \ldots$. Thus, in the second order fluid equations terms such as $n^{(1)}\mathbf{u}^{(1)}$ and $n^{(0)}\mathbf{u}^{(2)}$ were retained while the term $n^{(0)}\mathbf{u}^{(1)}$ was assumed to be of first order and hence excluded. However, returning to the kinetic definition of the fluid velocity, $\mathbf{u}^{(i)} = \mathbf{v}^{(i)}/n$, we see that $n^{(0)}\mathbf{u}^{(1)}$ in fact contains terms of second and higher order due to the division by n in the definition of $\mathbf{u}^{(i)}$. These second order terms are lost in the traditional fluid approach. Further second order terms are lost because of the neglect of the pressure term in the second order momentum equation. This is the case even at low temperatures where the pressure term can safely be neglected in linear problems.

The traditional fluid approach can be corrected by deriving the perturbation expansion of the fluid equations using an expansion of the particle flux, $\mathbf{v}$, instead of an expansion of the fluid velocity, $\mathbf{u}$, and by retaining the pressure term in the second order momentum equation [BINDSLEV, 1993]. The latter requires the inclusion of the second order energy equation in the set of fluid equations. With this new fluid approach the kinetic results are recovered.

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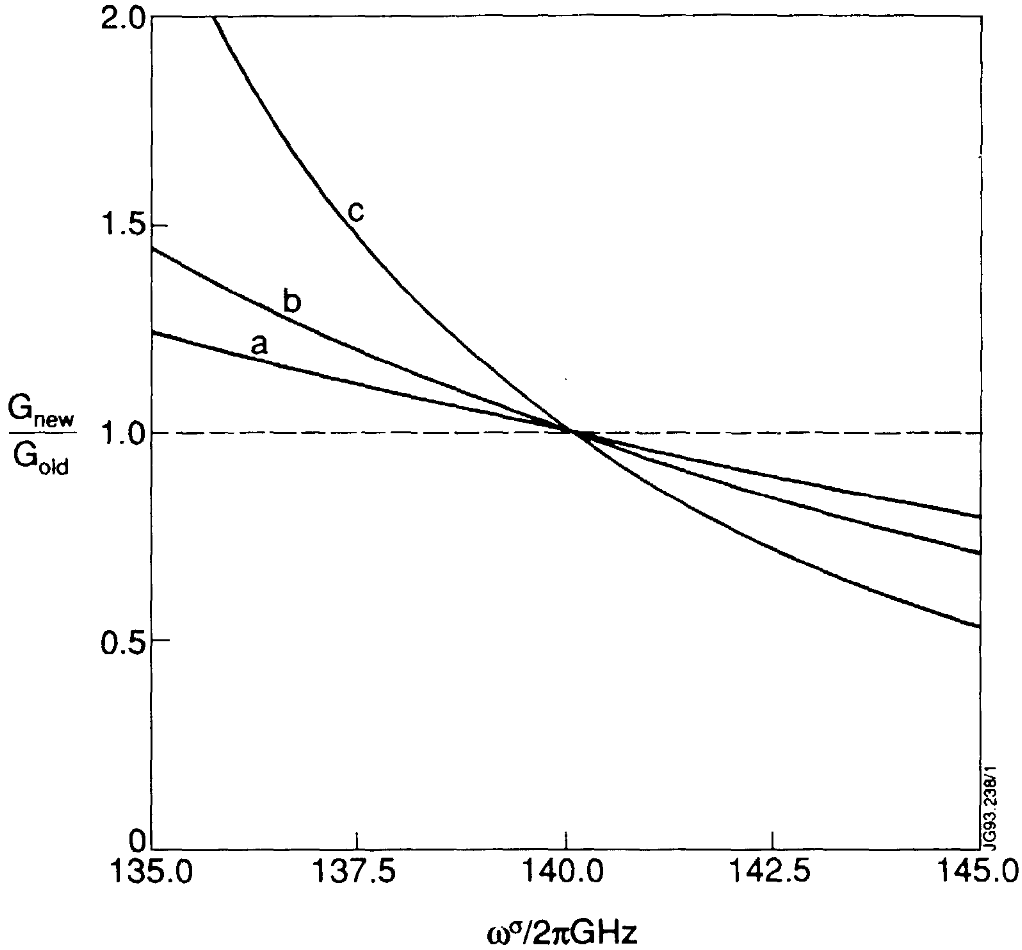

Figure 1: The ratio of dielectric form factors $G_{\text{new}}/G_{\text{old}}$ associated with the interaction of the incident field, $\mathbf{E}^{(1a)}$, with density fluctuations, $n^{(1b)}$, as a function of the frequency of the scattered radiation ω^{σ}.
Parameters: X to X mode scattering, $\angle(\mathbf{k}^{a}, \mathbf{k}^{\sigma}) = 30°$, $\angle(\mathbf{k}^{a}, \mathbf{B}^{(0)}) = \angle(\mathbf{k}^{b}, \mathbf{B}^{(0)}) = 90°$, $\omega^{a}/2\pi = 140$ GHz, a) $B^{(0)} = 3.4T$ ($\omega_c/2\pi = 95$ GHz), $n^{(0)} = 3.0 \times 10^{19} m^{-3}$, b) $B^{(0)} = 4.0T$ ($\omega_c/2\pi = 112$ GHz), $n^{(0)} = 1.0 \times 10^{19} m^{-3}$, c) $B^{(0)} = 4.5T$ ($\omega_c/2\pi = 126$ GHz), $n^{(0)} = 0.5 \times 10^{19} m^{-3}$.