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Three Wave Mixing and Thomson Scattering in Plasmas

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ABSTRACT

The theory of three wave mixing and Thomson scattering in plasmas, is reexamined in the low temperature limit with a kinetic model, giving a more complete description of Thomson scattering in this limit. Errors in the traditional fluid approach to three wave mixing and scattering are identified and a new corrected fluid approach is given. These corrections to the theory may have important consequences for the analysis of collective Thomson scattering experiments, such as fast ion diagnostics, and for the assessment of the feasibility of certain measurements.

1 INTRODUCTION

Theoretical models of Thomson scattering in plasmas at frequencies which are not large relative to the plasma frequency rely, among other things, on an accurate description of three wave mixing in plasmas. Thomson scattering experiments making use of waves with frequencies in this range include fast ion diagnostics and various fluctuation and magnetic field diagnostics, but generally exclude traditional electron temperature diagnostics.

In this paper the theory of three wave mixing and Thomson scattering in plasmas, both unmagnetized and magnetized, is reexamined in the low temperature limit, giving a more complete description of scattering in this limit and reconciling the results derived from fluid and from kinetic descriptions of the plasma.

In many applications the dielectric response of plasmas is considered in the linear approximation. There is, however, a small non–linear part to the dielectric response, which gives rise to interaction between linearly independent waves. To lowest order in the non–linear terms this interaction results in three wave mixing which includes Thomson scattering. In Thomson scattering an incident wave interacts either with another macroscopic wave or with microscopic (e.g. thermal) fluctuations in the plasma, and thereby sets up an additional current, the source current, which drives a third wave, the scattered wave.

Given the source current, expressions for the scattered field and for the power received by a receiving antenna located outside the plasma have been obtained by a number of authors including SIMONICH and YEH (1972), BRETZ (1987), HUGHES and SMITH (1989) and BINDSLEV (1991). This part of the theory appears to be adequately developed for the Thomson scattering experiments envisaged at present.

Expressions for the source current were traditionally derived using a fluid description of the plasma [AKHIEZER *et al.*, 1962 and 1967, and SITENKO, 1967]. Recently AAMODT and RUSSELL (1992) derived the leading terms and some of the secondary terms entering the expression for the source current in the low temperature limit using a kinetic description of the plasma. Their expression, including the leading terms, is at variance with the traditional fluid result. Here we present the complete expression for the source current in the low temperature limit derived with a kinetic plasma model, confirming the leading terms found by AAMODT and RUSSELL. Our kinetic result is thus also at variance with the traditional fluid result. It is shown that the discrepancy is due to fundamental errors in the nonlinear terms of the traditional fluid description. It is further demonstrated that when these errors are corrected the fluid approach yields the same expression for the source current as the kinetic approach, confirming the correctness of the new expression for the source current in the low temperature limit.

The changes to the theory of three wave mixing and Thomson scattering resulting from the new expression for the source current can have important consequences for analyzing results from scattering experiments and indeed for judging the feasibility of certain measurements. As demonstrated elsewhere [BINDSLEV,1991] a modelling appropriate for JET's collective Thomson scattering diagnostic [COSTLEY *et al.*, 1988] will often require that relativistic effects be included. A tractable relativistic theory of three wave mixing is, however, still under development.

In Section 2 of this paper we give a detailed kinetic treatment of three wave mixing of macroscopic waves and derive the source current resulting from the interaction of macroscopic waves. Our approach is entirely different from that adopted by AAMODT and RUSSELL (1992) and thus complements and extends their kinetic treatment. In Sections 3 and 4 we establish where the traditional fluid approach breaks down and how it may be remedied. In Section 5 the expression for the source current is generalized to account for the interaction between a macroscopic wave and microscopic fluctuations. The source current expressions are combined with previous work in Section 6 to give the equation of transfer for a scattering system, together with the scattering cross section. The findings of this paper are summed up in Section 7.

2 KINETIC MODEL

In this section we investigate three wave mixing of macroscopic waves in a collisionless plasma using a kinetic model. Both unmagnetized and magnetized plasmas are considered.

In a collisionless plasma the dynamics of the electron momentum distribution, $f(\mathbf{p}, \mathbf{r}, t)$, are governed by the Vlasov equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + q_e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{p}} = 0 \, , \tag{1}$$

where $\mathbf{v} = \mathbf{p}/(\gamma m_e)$, $\gamma = \sqrt{1 + p^2/(m_e c)^2}$, $p = |\mathbf{p}|$, and m_e and q_e are the electron rest mass and charge respectively. The macroscopic electromagnetic fields, $\mathbf{E}(\mathbf{r}, t)$ and

$\mathbf{B}(\mathbf{r}, t)$ are governed by Maxwell's equations,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2a}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t} \right) , \tag{2b}$$

with the plasma current, $\partial \mathbf{P}/\partial t$, given by

$$\frac{\partial \mathbf{P}}{\partial t} = q_e \int \mathbf{v} f \, d\mathbf{p} . \tag{2c}$$

We note that the expressions (2a), (2b) and (2c) are all linear. The non–linearity of the dielectric response of a collisionless plasma is entirely due to the third term in the Vlasov equation (1). The set of equations (1) to (2c) has a set of time and space independent solutions, $f^{(0)}(\mathbf{p})$, $\mathbf{E}^{(0)}$ and $\mathbf{B}^{(0)}$, corresponding to a stationary and homogeneous plasma. For each of these solutions there is a frame of reference in which $\mathbf{E}^{(0)} = 0$. In this frame we seek new solutions as small perturbations to the time and space independent solutions:

$$\begin{aligned} f(\mathbf{p},\mathbf{r},t) &= f^{(0)}(\mathbf{p}) + f^{(1)}(\mathbf{p},\mathbf{r},t) + f^{(2)}(\mathbf{p},\mathbf{r},t) + \ldots \\ \mathbf{E}(\mathbf{r},t) &= \mathbf{E}^{(1)}(\mathbf{r},t) + \mathbf{E}^{(2)}(\mathbf{r},t) + \ldots \\ \mathbf{B}(\mathbf{r},t) &= \mathbf{B}^{(0)} + \mathbf{B}^{(1)}(\mathbf{r},t) + \mathbf{B}^{(2)}(\mathbf{r},t) + \ldots . \end{aligned} \tag{3}$$

Here the upper index indicates the order of the perturbation, so on introducing δ as the perturbation variable we have $f^{(1)} \propto \delta$, $f^{(2)} \propto \delta^2$, etc..

The fact that Maxwell's equations, (2a) and (2b), and the expression for the plasma current, (2c), are linear and that they hold for any value of δ implies that they hold for each perturbation order, n, individually:

$$\nabla \times \mathbf{E}^{(n)} = -\frac{\partial \mathbf{B}^{(n)}}{\partial t} , \tag{4a}$$

$$\nabla \times \mathbf{B}^{(n)} = \mu_0 \left(\varepsilon_0 \frac{\partial \mathbf{E}^{(n)}}{\partial t} + \frac{\partial \mathbf{P}^{(n)}}{\partial t} \right) , \tag{4b}$$

$$\frac{\partial \mathbf{P}^{(n)}}{\partial t} = q_e \int \mathbf{v} f^{(n)} \, d\mathbf{p} . \tag{4c}$$

To zeroth order in δ the Vlasov equation reads

$$(\mathbf{p} \times \mathbf{B}^{(0)}) \cdot \frac{\partial f^{(0)}}{\partial \mathbf{p}} = 0 \tag{5}$$

which has solutions of the form

$$f^{(0)} = f^{(0)}(p_\perp, p_\parallel) \tag{6}$$

corresponding to gyrotropic distributions. The first order equation is the familiar linearized Vlasov equation, which describes linear wave propagation:

$$\mathcal{L} f^{(1)} = -\mathbf{F}^{(1)} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{p}} \, . \tag{7}$$

Here we have introduced the linear operator, $\mathcal{L}$,

$$\mathcal{L} f = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + q(\mathbf{v} \times \mathbf{B}^{(0)}) \cdot \frac{\partial f}{\partial \mathbf{p}} \, , \tag{8}$$

and the shorthand notation, $\mathbf{F}^{(n)}$, for the force the fields $\mathbf{E}^{(n)}$ and $\mathbf{B}^{(n)}$ exert on an electron moving with velocity $\mathbf{v}$,

$$\mathbf{F}^{(n)} = q_e \left(\mathbf{E}^{(n)} + \mathbf{v} \times \mathbf{B}^{(n)} \right) \, . \tag{9}$$

Let the waves $[f^{(1a)}, \mathbf{E}^{(1a)}, \mathbf{B}^{(1a)}]$ and $[f^{(1b)}, \mathbf{E}^{(1b)}, \mathbf{B}^{(1b)}]$ be two independent solutions to Maxwell's equations and the linearized Vlasov equation. For brevity we will refer to these solutions as [1a] and [1b], and similarly for other solutions. We note that the sum is also a solution to this set of equations and that within this set there is no coupling between the two solutions. Bilinear coupling between linear waves and thus scattering in the first Born approximation is accounted for by the second order equation:

$$\mathcal{L} f^{(2)} = -\mathbf{F}^{(2)} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{p}} - \mathbf{F}^{(1)} \cdot \frac{\partial f^{(1)}}{\partial \mathbf{p}} \, . \tag{10}$$

Assume that in the linear approximation the two waves [1a] and [1b] coexist in the plasma. Noting that equation (10) is linear in second order quantities we can construct a second order solution as a sum of three elements, $[2a] \equiv [f^{(2a)}, \mathbf{E}^{(2a)}, \mathbf{B}^{(2a)}]$,

$[2b] \equiv [f^{(2b)}, \mathbf{E}^{(2b)}, \mathbf{B}^{(2b)}]$ and $[2\sigma] \equiv [f^{(2\sigma)}, \mathbf{E}^{(2\sigma)}, \mathbf{B}^{(2\sigma)}]$, where each element satisfies Maxwell's equations, (4a) and (4b), with the associated plasma current given by (4c) with n = 2a, 2b, 2c. Elements [2a] and [2b] satisfy the second order Vlasov equation with no cross terms in the linear wave quantities:

$$\mathcal{L} f^{(2\alpha)} = -\mathbf{F}^{(2\alpha)} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{p}} - \mathbf{F}^{(1\alpha)} \cdot \frac{\partial f^{(1\alpha)}}{\partial \mathbf{p}} , \qquad \alpha = a,\, b\, ; \tag{11}$$

while the equation for element $[2\sigma]$ includes only cross terms in the linear wave quantities:

$$\mathcal{L} f^{(2\sigma)} = -\mathbf{F}^{(2\sigma)} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{p}} - \mathbf{F}^{(1\mathrm{a})} \cdot \frac{\partial f^{(1\mathrm{b})}}{\partial \mathbf{p}} - \mathbf{F}^{(1\mathrm{b})} \cdot \frac{\partial f^{(1\mathrm{a})}}{\partial \mathbf{p}} . \tag{12}$$

Elements [2a] and [2b] are the second order modifications to the linear waves [1a] and [1b] respectively while $[2\sigma]$ is a new wave resulting from the bilinear interaction between the linear waves [1a] and [1b]. This is the first Born approximation to the scattered wave.

Noting that the left hand side and the first term on the right hand side of equation (12) are identical to the linearized Vlasov equation, the electromagnetic field of the scattered wave may be found from the inhomogeneous wave equation[1]

$$\mathbf{\Lambda}(\mathbf{k}^\sigma, \omega^\sigma) \cdot \mathbf{E}^{(2\sigma)}(\mathbf{k}^\sigma, \omega^\sigma) = \frac{-i}{\omega^\sigma \varepsilon_0} \mathbf{j}^\sigma(\mathbf{k}^\sigma, \omega^\sigma) . \tag{14}$$

The source current, $\mathbf{j}^\sigma(\mathbf{k}^\sigma, \omega^\sigma)$, is given by

$$\mathbf{j}^\sigma(\mathbf{k}^\sigma, \omega^\sigma) = q_e \int \mathbf{v} f^\sigma(\mathbf{p}, \mathbf{k}^\sigma, \omega^\sigma)\, d\mathbf{p} , \tag{15}$$

and $f^\sigma(\mathbf{p}, \mathbf{k}^\sigma, \omega^\sigma)$ is the Fourier–Laplace transform of $f^\sigma(\mathbf{p}, \mathbf{r}, t)$, which satisfies the relation:

[1] In this paper a spatial Fourier transform and a temporal Laplace transform are used:

$$A(\mathbf{r}, t) = \frac{1}{(2\pi)^4} \int_{-\infty+i\nu}^{\infty+i\nu} \int_{R^3} A(\mathbf{k}, \omega) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\, d\mathbf{k} d\omega , \tag{13a}$$

$$A(\mathbf{k}, \omega) = \int_0^\infty \int_{R^3} A(\mathbf{r}, t) e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\, d\mathbf{r} dt . \tag{13b}$$

The Fourier–Laplace transform, $A(\mathbf{k}, \omega)$, is defined for complex ω with a sufficiently large imaginary part, ω_{Im}, that the integral over t in (13b) exists. This defines the contours to be taken in any other integrations in the definition of the transform when the analytic continuation of $A(\mathbf{k}, \omega)$ to the rest of the domain of ω is required.

$$\mathcal{L}f^{\sigma} = -\mathbf{F}^{(1a)} \cdot \frac{\partial f^{(1b)}}{\partial \mathbf{p}} - \mathbf{F}^{(1b)} \cdot \frac{\partial f^{(1a)}}{\partial \mathbf{p}} \,. \tag{16}$$

$\mathbf{\Lambda}(\mathbf{k},\omega)$ is the wave tensor,

$$\Lambda_{ij}(\mathbf{k},\omega) = \varepsilon_{ij}(\mathbf{k},\omega) + N^2\{\widehat{k}_i\widehat{k}_j - \delta_{ij}\} \,, \tag{17}$$

where $N = |\mathbf{k}|c/\omega$ is the refractive index, $\widehat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ is the unit wave vector and $\boldsymbol{\varepsilon}(\mathbf{k},\omega)$ is the dielectric tensor.

Given the source current, $\mathbf{j}^{\sigma}$, the field $\mathbf{E}^{(2\sigma)}$ of the scattered wave can be found from equation (14) (see e.g. BINDSLEV (1991)). To find the source current we first obtain $f^{\sigma}(\mathbf{p},\mathbf{k}^{\sigma},\omega^{\sigma})$ from equation (16). This can, for instance, be done by integrating equation (16) along characteristics (see e.g. CARRIER and PEARSON, 1976 for a general discussion of this method) followed by a Fourier–Laplace transform over space and time. Alternatively, the Fourier–Laplace transformation may be carried out first and followed by an integration over the azimuthal angle of $\mathbf{p}$. Details of this approach as applied to the present problem may be found in BINDSLEV (1992) starting at equation (7.25) on page 112. The result is

$$f^{\sigma}(\mathbf{p},\mathbf{k}^{\sigma},\omega^{\sigma}) = \frac{e^{-i\beta\sin\phi}}{\omega_c/\gamma}\int_0^{\infty} e^{-i\alpha\tau}\left\{e^{i\beta\sin(\phi-\tau)}Q(p_{\|},p_{\perp},\phi-\tau,\mathbf{k}^{\sigma},\omega^{\sigma})\right\}d\tau \,, \tag{18}$$

where $\omega_c = -q_e B^{(0)}/m_e$ is the angular electron cyclotron frequency,

$$\alpha = \frac{v_{\|}k_{\|}^{\sigma} - \omega^{\sigma}}{\omega_c/\gamma} \,, \tag{19}$$

$$\beta = \frac{v_{\perp}k_{\perp}^{\sigma}}{\omega_c/\gamma} \,, \tag{20}$$

and

$$\begin{aligned} Q(p_{\|},p_{\perp},\phi,\mathbf{k}^{\sigma},\omega^{\sigma}) = & -\int\left(\mathbf{F}^{(1a)}(\mathbf{p},\mathbf{k}^{a},\omega^{a})\cdot\frac{\partial f^{(1b)}(\mathbf{p},\mathbf{k}^{b},\omega^{b})}{\partial\mathbf{p}}\right. \\ & \left. + \mathbf{F}^{(1b)}(\mathbf{p},\mathbf{k}^{b},\omega^{b})\cdot\frac{\partial f^{(1a)}(\mathbf{p},\mathbf{k}^{a},\omega^{a})}{\partial\mathbf{p}}\right)\frac{d\mathbf{k}\,d\omega}{(2\pi)^4} \,, \end{aligned} \tag{21}$$

with

$$\mathbf{k}^{\mathrm{a}} + \mathbf{k}^{\mathrm{b}} = \mathbf{k}^{\sigma} \quad , \quad \omega^{\mathrm{a}} + \omega^{\mathrm{b}} = \omega^{\sigma} \; , \tag{22}$$
$$\mathbf{k}^{\mathrm{a}} - \mathbf{k}^{\mathrm{b}} = \mathbf{k} \quad , \quad \omega^{\mathrm{a}} - \omega^{\mathrm{b}} = \omega \; .$$

$p_{\parallel}$ and $p_{\perp}$ are the parallel and perpendicular components of $\mathbf{p}$ relative to $\mathbf{B}^{(0)}$, and similarly for other vectors. ϕ is the azimuthal angle of $\mathbf{p}$ in the coordinate system where $\hat{\mathbf{z}} = \mathbf{B}^{(0)}/|\mathbf{B}^{(0)}|$ and $\mathbf{k}^{\sigma} = k_{\perp}^{\sigma}\hat{\mathbf{x}} + k_{\parallel}^{\sigma}\hat{\mathbf{z}}$.

In the unmagnetized case f^{σ} is given by

$$f^{\sigma}(\mathbf{p}, \mathbf{k}^{\sigma}, \omega^{\sigma}) = \frac{-iq_e}{\omega^{\sigma} - \mathbf{v} \cdot \mathbf{k}^{\sigma}} Q(\mathbf{p}, \mathbf{k}^{\sigma}, \omega^{\sigma}) \; . \tag{23}$$

For both the magnetized and the unmagnetized case the definition of the Fourier–Laplace transform, (13a) and (13b), ensures that the integration over $\mathbf{p}$ in expression (15) follows a contour corresponding to the causal solution.

Expression (15) for the source current with f^{σ} given by (18) or (23) is fully relativistic. We note that in a relativistic treatment of three wave mixing and scattering the full details of the momentum distributions of the interacting waves, [1a] and [1b], are required.

In the subsequent analysis in this paper we will assume certain limitations on the momentum distribution $f^{(2\sigma)}$ which allow the source current, $\mathbf{j}^{\sigma}$, to be expressed solely in terms of the zeroth and first order moments of $f^{(1\mathrm{a})}$ and $f^{(1\mathrm{b})}$. These assumptions effectively amount to limiting the scope of the theory to a low temperature plasma.

We include a separate derivation for unmagnetized plasmas because of the insight this affords without the extensive algebra required for the magnetized case.

2.1 Unmagnetized Plasma

In this subsection the plasma is assumed to be unmagnetized, i.e. $\mathbf{B}^{(0)} = 0$. We further assume that f^{σ} is significant only for values of $\mathbf{v}$ and $v = |\mathbf{v}|$ which satisfy the inequalities

$$\frac{|\mathbf{v} \cdot \mathbf{k}^{\sigma}|}{\omega^{\sigma}} \ll 1 \; , \tag{24}$$

$$\frac{v}{c} \ll 1 \,. \tag{25}$$

Although these limitations are only directly imposed on f^σ, they will in most cases only be satisfied if similar conditions are imposed on $f^{(1a)}$, $f^{(1b)}$ and $f^{(0)}$, implying a cold plasma. With (24) and (25) satisfied we can make use of the following expansions:

$$\frac{1}{\omega^\sigma - \mathbf{v}\cdot\mathbf{k}^\sigma} = \frac{1}{\omega^\sigma}\left(1 + \frac{\mathbf{v}\cdot\mathbf{k}^\sigma}{\omega^\sigma}\right) + O\left\{\left(\frac{\mathbf{v}\cdot\mathbf{k}^\sigma}{\omega^\sigma}\right)^2\right\}, \tag{26}$$

$$\mathbf{v} = \frac{\mathbf{p}}{m_e} + O\left\{(v/c)^2\right\} . \tag{27}$$

Retaining only terms which after partial integration are of zeroth and first order in $\mathbf{v}$, the source current integral (15) contains the following terms:

$$q_e^2 \int \mathbf{v}\mathbf{E}^{(i)} \cdot \frac{\partial f^{(j)}}{\partial \mathbf{p}}\, d\mathbf{p} = -\frac{q_e^2}{m_e}\mathbf{E}^{(i)} n^{(j)} , \tag{28a}$$

$$q_e^2 \int \mathbf{v}\,\frac{\mathbf{v}\cdot\mathbf{k}^\sigma}{\omega^\sigma}\mathbf{E}^{(i)} \cdot \frac{\partial f^{(j)}}{\partial \mathbf{p}}\, d\mathbf{p} = -\frac{q_e}{m_e}\left(\mathbf{E}^{(i)}\frac{\mathbf{j}^{(j)}\cdot\mathbf{k}^\sigma}{\omega^\sigma} + \mathbf{j}^{(j)}\frac{\mathbf{E}^{(i)}\cdot\mathbf{k}^\sigma}{\omega^\sigma}\right) , \tag{28b}$$

$$q_e^2 \int \mathbf{v}\left(\mathbf{v} \times \mathbf{B}^{(i)}\right) \cdot \frac{\partial f^{(j)}}{\partial \mathbf{p}}\, d\mathbf{p} = -\frac{q_e}{m_e}\mathbf{j}^{(j)} \times \mathbf{B}^{(i)} , \tag{28c}$$

where

$$n^{(i)} = \int f^{(i)}\, d\mathbf{p} , \tag{29}$$

$$\mathbf{j}^{(i)} = q_e \int \mathbf{v} f^{(i)}\, d\mathbf{p} , \tag{30}$$

and $(i,j) = (1\mathrm{a},1\mathrm{b}), (1\mathrm{b},1\mathrm{a})$.

Inserting (28a) to (28c) in the expression for the source current we find:

$$
\begin{aligned}
\mathbf{j}^{\sigma} \; &= \; \frac{iq_e}{m_e\omega^{\sigma}} \int \Big\{ q_e \left(\mathbf{E}^{(1a)} n^{(1b)} + \mathbf{E}^{(1b)} n^{(1a)} \right) \\
&+ \mathbf{E}^{(1a)} \frac{\mathbf{j}^{(1b)} \cdot \mathbf{k}^{\sigma}}{\omega^{\sigma}} + \mathbf{j}^{(1b)} \frac{\mathbf{E}^{(1a)} \cdot \mathbf{k}^{\sigma}}{\omega^{\sigma}} + \mathbf{E}^{(1b)} \frac{\mathbf{j}^{(1a)} \cdot \mathbf{k}^{\sigma}}{\omega^{\sigma}} + \mathbf{j}^{(1a)} \frac{\mathbf{E}^{(1b)} \cdot \mathbf{k}^{\sigma}}{\omega^{\sigma}} \\
&+ \mathbf{j}^{(1b)} \times \mathbf{B}^{(1a)} + \mathbf{j}^{(1a)} \times \mathbf{B}^{(1b)} \Big\} \frac{d\mathbf{k}\, d\omega}{(2\pi)^4} \, .
\end{aligned} \tag{31}
$$

If one of the interacting waves, e.g. [1a], is monochromatic then the convolution integral in the expression (31) for the source current can be eliminated. Assume that all quantities relating to wave [1a] have the form:

$$
A^{(1a)}(\mathbf{r}, t) = A^{(1a')} \exp\left\{ i \left(\mathbf{k}^{a} \cdot \mathbf{r} - \omega^{a} t \right) \right\} + c.c. \, . \tag{32}
$$

The expression for the source current then takes the form $\mathbf{j}^{\sigma} = \mathbf{j}^{\sigma^-} + \mathbf{j}^{\sigma^+}$ where

$$
\begin{aligned}
\mathbf{j}^{\sigma^-} \; &= \; \frac{iq_e}{m_e\omega^{\sigma}} \Big\{ q_e \left(\mathbf{E}^{(1a')} n^{(1b)} + \mathbf{E}^{(1b)} n^{(1a')} \right) \\
&+ \mathbf{E}^{(1a')} \frac{\mathbf{j}^{(1b)} \cdot \mathbf{k}^{\sigma}}{\omega^{\sigma}} + \mathbf{j}^{(1b)} \frac{\mathbf{E}^{(1a')} \cdot \mathbf{k}^{\sigma}}{\omega^{\sigma}} + \mathbf{E}^{(1b)} \frac{\mathbf{j}^{(1a')} \cdot \mathbf{k}^{\sigma}}{\omega^{\sigma}} + \mathbf{j}^{(1a')} \frac{\mathbf{E}^{(1b)} \cdot \mathbf{k}^{\sigma}}{\omega^{\sigma}} \\
&+ \mathbf{j}^{(1b)} \times \mathbf{B}^{(1a')} + \mathbf{j}^{(1a')} \times \mathbf{B}^{(1b)} \Big\} \, ,
\end{aligned} \tag{33}
$$

and the quantities associated with wave [1b] are evaluated at the wavevector $\mathbf{k}^{b} = \mathbf{k}^{\sigma} - \mathbf{k}^{a}$ and frequency $\omega^{b} = \omega^{\sigma} - \omega^{a}$. $\mathbf{j}^{\sigma^+}$ is similar in form to $\mathbf{j}^{\sigma^-}$, the main difference being that the quantities associated with [1b] are evaluated at the sum wave vector $\mathbf{k}^{b} = \mathbf{k}^{\sigma} + \mathbf{k}^{a}$ and sum frequency $\omega^{b} = \omega^{\sigma} + \omega^{a}$. Terms of the type $\mathbf{j}^{\sigma^+}$ are usually neglected when analysing scattering. Whether this is permissible clearly depends on the spectrum of the wave [1b].

2.2 Magnetized Plasma

We now allow the plasma to be magnetized, i.e. $\mathbf{B}^{(0)} \neq 0$. Here we assume that f^{σ} is significant only for values of $\mathbf{v}$ which satisfy the inequalities

$$
\frac{v}{c} \; \ll \; 1 \, , \tag{34}
$$

$$\frac{v_{\|}k_{\|}^{\sigma}}{\omega^{\sigma}+s\omega_c} \ll 1\,; \qquad s \in Z\,, \tag{35}$$

$$\frac{v_{\perp}k_{\perp}^{\sigma}}{\omega_c} \ll 1\,. \tag{36}$$

s takes on all integer values (Z is the set of all integers, positive and negative). Although these limitations are only directly imposed on f^{σ}, they will in most cases only be satisfied if similar conditions are imposed on $f^{(1a)}$, $f^{(1b)}$ and $f^{(0)}$, implying a cold plasma. We will refer to the terms on the left hand sides of equations (34), (35) and (36) as the small terms.

These assumptions are similar to those made by AAMODT and RUSSELL (1992) in their derivation of the expression for the source current. The approach adopted here is, however, entirely different to that taken by AAMODT and RUSSELL, and thus complements their derivation. Care is taken to make explicit all important steps in the derivation and expose the points in the derivation where the assumptions (34) to (36) are required.

With the assumptions made above, the expression for the source current, $\mathbf{j}^{\sigma}$, can be expanded in velocity. As in the unmagnetized case we retain only terms of zeroth and first orders in velocity in the final result.

To expand the expression for $\mathbf{j}^{\sigma}$ in $\mathbf{v}$ we begin by expanding equation (18) in powers of velocity. To this end it is convenient to cast (18) in a slightly different form. We note that the part enclosed in braces in equation (18) is periodic in τ over 2π. This part can therefore be expanded in a Fourier series

$$e^{i\beta\sin(\phi-\tau)}Q(\phi-\tau) = \sum_{s=-\infty}^{\infty} e^{is\tau}\frac{1}{2\pi}\int_0^{2\pi} e^{i\beta\sin(\phi-\tau')}Q(\phi-\tau')e^{-is\tau'}\,d\tau'\,. \tag{37}$$

To reduce the length of equations we have omitted showing explicitly that Q depends on $p_{\|}$, $p_{\perp}$, $\mathbf{k}^{\sigma}$ and ω^{σ}. Inserting (37) in (18), integrating over τ and replacing τ' with τ, gives

$$f^{\sigma} = \frac{e^{-i\beta\sin\phi}}{\omega_c/\gamma}\sum_{s=-\infty}^{\infty}\frac{1}{i(\alpha-s)}\frac{1}{2\pi}\int_0^{2\pi} e^{i\beta\sin(\phi-\tau)}Q(\phi-\tau)e^{-is\tau}\,d\tau\,. \tag{38}$$

Here we have made use of the fact that $\alpha_{\mathrm{Im}} < 0$, from which it follows that

$$\int_0^{\infty} e^{-i(\alpha-s)\tau}\,d\tau = \left[\frac{e^{-i(\alpha-s)\tau}}{-i(\alpha-s)}\right]_0^{\infty} = \frac{1}{i(\alpha-s)}\,.$$

When conditions (34) and (35) are satisfied we may expand $-i/(\alpha-s)$ in $v_{\|}$,

$$\frac{1}{i(\alpha - s)} \approx \frac{i\omega_c}{\omega^\sigma + s\omega_c}\left(1 + \frac{v_\| k_\|^\sigma}{\omega^\sigma + s\omega_c} + \ldots\right) , \tag{39}$$

and when conditions (34) and (36) are satisfied we have

$$e^{i\beta\sin(\phi-\tau)} \approx 1 + \frac{iv_\perp k_\perp^\sigma}{\omega_c}\sin(\phi - \tau) + \ldots . \tag{40}$$

Inserting (39) and (40) in expression (38) for f^σ, retaining only first order terms in $\mathbf{v}$, and inserting the result in expression (15) we find that to first order in the small terms the source current is given by

$$\begin{aligned} \mathbf{j}^\sigma &= \frac{q_e}{2\pi}\sum_{s=-\infty}^{\infty}\frac{i}{\omega^\sigma + s\omega_c}\int d\mathbf{p}\int_0^{2\pi} d\tau\, \mathbf{v} \\ &\left(1 + \frac{k_\|^\sigma v_\|}{\omega^\sigma + s\omega_c} + \frac{ik_\perp^\sigma v_\perp}{\omega_c}\left[\sin(\phi - \tau) - \sin\phi\right]\right) Q(\phi - \tau)e^{-is\tau} . \end{aligned} \tag{41}$$

Here it was assumed that $\hat{\mathbf{z}} = \mathbf{B}^{(0)}/|\mathbf{B}^{(0)}|$ and $\mathbf{k}^\sigma = k_\perp^\sigma\hat{\mathbf{x}} + k_\|^\sigma\hat{\mathbf{z}}$.

Since the integrand in (41) is periodic in ϕ over 2π, the interval over which the ϕ–integration runs, it follows that ϕ may be replaced by $\phi + \tau$. Expression (41) then takes the form

$$\begin{aligned} \mathbf{j}^\sigma &= \frac{q_e}{2\pi}\sum_{s=-\infty}^{\infty}\frac{i}{\omega^\sigma + s\omega_c}\int d\mathbf{p}\int_0^{2\pi} d\tau\, \mathbf{R}(\tau)\cdot\mathbf{v} \\ &\left(1 + \frac{k_\|^\sigma v_\|}{\omega^\sigma + s\omega_c} + \frac{ik_\perp^\sigma v_\perp}{\omega_c}\left[\sin(\phi) - \sin(\phi + \tau)\right]\right) Q(\phi)e^{-is\tau} , \end{aligned} \tag{42}$$

where

$$\mathbf{R}(\tau) = \left\{\begin{array}{ccc} \cos\tau & -\sin\tau & 0 \\ \sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{array}\right\} . \tag{43}$$

For convenience we introduce the following set of pseudo tensors:

$$\eta_{ij} = \delta_{ij} - \delta_{i3}\delta_{j3} , \quad \xi_{ij} = \epsilon_{ij3} , \quad \zeta_{ij} = \delta_{i3}\delta_{j3} , \tag{44}$$

which are given explicitly by

$$\boldsymbol{\eta} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Bmatrix}, \quad \boldsymbol{\xi} = \begin{Bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}, \quad \boldsymbol{\zeta} = \begin{Bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{Bmatrix}. \tag{45}$$

This permits $\mathbf{R}(\tau)$ to be written as $\mathbf{R}(\tau) = \cos\tau\boldsymbol{\eta} - \sin\tau\boldsymbol{\xi} + \boldsymbol{\zeta}$. ϵ_{ijk} is the standard Levi–Civita symbol ($\epsilon_{ijk}a_j b_k = \{\mathbf{a}\times\mathbf{b}\}_i$).

On integrating (42) over τ we find

$$\begin{aligned} \mathbf{j}^\sigma &= q_e \sum_{s=-\infty}^{\infty} \frac{i\omega_c}{\omega^\sigma + s\omega_c} \int d\mathbf{p}\, Q(\phi) \\ &\quad \left(\left[1 + \frac{k_\parallel^\sigma v_\parallel}{\omega^\sigma + s\omega_c} + \frac{ik_\perp^\sigma v_\perp}{\omega_c}\sin(\phi) \right] \mathbf{R}^{(1)} - \frac{ik_\perp^\sigma v_\perp}{\omega_c}\mathbf{R}^{(2)}(\phi) \right) \cdot \mathbf{v} \end{aligned} \tag{46}$$

where

$$\begin{aligned} \mathbf{R}^{(1)} &= \frac{1}{2\pi}\int_0^{2\pi} \mathbf{R}(\tau) e^{-is\tau}\, d\tau \\ &= \frac{\delta_{1s} + \delta_{-1s}}{2}\boldsymbol{\eta} - \frac{\delta_{1s} - \delta_{-1s}}{2i}\boldsymbol{\xi} + \delta_{0s}\boldsymbol{\zeta} \end{aligned}$$

and

$$\begin{aligned} \mathbf{R}^{(2)} &= \frac{1}{2\pi}\int_0^{2\pi} \mathbf{R}(\tau)\sin(\phi+\tau) e^{-is\tau}\, d\tau \\ &= \left\{ \frac{\delta_{0s}\sin\phi}{2} + \frac{\delta_{2s}e^{i\phi} - \delta_{-2s}e^{-i\phi}}{4i} \right\}\boldsymbol{\eta} \\ &\quad + \left\{ \frac{-\delta_{0s}\cos\phi}{2} + \frac{\delta_{2s}e^{i\phi} + \delta_{-2s}e^{-i\phi}}{4} \right\}\boldsymbol{\xi} + \frac{\delta_{1s}e^{i\phi} - \delta_{-1s}e^{-i\phi}}{2i}\boldsymbol{\zeta}. \end{aligned}$$

Summing over s in (46) and writing the result in component form with implicit summation over repeated indices we find

$$j_i^\sigma = \frac{q_e\omega_c}{\omega^\sigma}\int d\mathbf{p}\, Q(\phi)\left(\left[1+\frac{ik_\perp^\sigma v_y}{\omega_c}\right]T_{ij}^{(1\alpha)}+\frac{k_\parallel^\sigma v_z}{\omega^\sigma}\,T_{ij}^{(1\beta)}-\frac{ik_\perp^\sigma v_k}{\omega_c}\,T_{ijk}^{(2)}\right)v_j \qquad (47)$$

where

$$\begin{aligned} T_{ij}^{(1\alpha)} &= \sum_{s=-\infty}^{\infty}\frac{1}{1+s\Omega}R_{ij}^{(1)} \\ &= \frac{1}{1-\Omega^2}\,\eta_{ij}-\frac{i\Omega}{1-\Omega^2}\,\xi_{ij}+\zeta_{ij}\ ; \end{aligned} \qquad (48)$$

$$\begin{aligned} T_{ij}^{(1\beta)} &= \sum_{s=-\infty}^{\infty}\frac{1}{(1+s\Omega)^2}R_{ij}^{(1)} \\ &= \frac{1+\Omega^2}{(1-\Omega^2)^2}\,\eta_{ij}-\frac{i2\Omega}{(1-\Omega^2)^2}\,\xi_{ij}+\zeta_{ij}\ ; \end{aligned} \qquad (49)$$

$$\begin{aligned} T_{ijk}^{(2)}v_j &= v_\perp\sum_{s=-\infty}^{\infty}\frac{1}{1+s\Omega}\mathbf{R}^{(2)}\ , \\ T_{ijk}^{(2)} &= \frac{i\Omega\delta_{k1}+(1-2\Omega^2)\delta_{k2}}{1-4\Omega^2}\,\eta_{ij}+\frac{2\Omega^2\delta_{k1}-i\Omega\delta_{k2}}{1-4\Omega^2}\,\xi_{ij}+\frac{i\Omega\delta_{k1}+\delta_{k2}}{1-\Omega^2}\,\zeta_{ij}\ , \end{aligned} \qquad (50)$$

and $\Omega = \omega_c/\omega^\sigma$, $v_x = v_\perp\cos\phi$, $v_y = v_\perp\sin\phi$ and $v_z = v_\parallel$. Equation (47) contains integrals of the form

$$I = \int gF_i\frac{\partial f}{\partial p_i}\,d\mathbf{p}\ .$$

Integrating by parts with respect to p_i and noting that $\partial F_i/\partial p_i = 0$ we find

$$I = -\int\frac{\partial g}{\partial p_i}F_i f\,d\mathbf{p}\ .$$

These integrals contain terms of the form

$$\begin{aligned}
\frac{\partial}{\partial p_l} T_{ij}^{(1\alpha)} v_j &= \frac{1}{m_e} T_{il}^{(1\alpha)} \\
\frac{\partial}{\partial p_l} v_y T_{ij}^{(1\alpha)} v_j &= \frac{1}{m_e} \left(\delta_{l2} T_{ij}^{(1\alpha)} v_j + v_y T_{il}^{(1\alpha)} \right) \\
\frac{\partial}{\partial p_l} v_z T_{ij}^{(1\beta)} v_j &= \frac{1}{m_e} \left(\delta_{l3} T_{ij}^{(1\beta)} v_j + v_z T_{il}^{(1\beta)} \right) \\
\frac{\partial}{\partial p_l} T_{ijk}^{(2)} v_j v_k &= \frac{1}{m_e} \left(T_{ilj}^{(2)} + T_{ijl}^{(2)} \right) v_j \; .
\end{aligned}$$

Here, making use of condition (34), we have neglected terms of second and higher order in v/c. Partial integration of equation (47) thus gives

$$\begin{aligned}
j_i^\sigma &= \frac{i q_e^2}{m_e \omega^\sigma} \int d\mathbf{p} \int \frac{d\mathbf{k}\, d\omega}{(2\pi)^4} \left(F_l^{(1\mathrm{a})} f^{(1\mathrm{b})} + F_l^{(1\mathrm{b})} f^{(1\mathrm{a})} \right) \\
&\quad \left(T_{il}^{(1\alpha)} + \left\{ \frac{i k_\perp^\sigma}{\omega_c} \left[\delta_{j2} T_{il}^{(1\alpha)} + \delta_{l2} T_{ij}^{(1\alpha)} \right] \right.\right. \\
&\quad \left.\left. + \frac{k_\parallel^\sigma}{\omega^\sigma} \left[\delta_{j3} T_{il}^{(1\beta)} + \delta_{l3} T_{ij}^{(1\beta)} \right] - \frac{i k_\perp^\sigma}{\omega_c} \left[T_{ijl}^{(2)} + T_{ilj}^{(2)} \right] \right\} v_j \right) \; .
\end{aligned} \tag{51}$$

Noting that $\mathbf{T}^{(1\alpha)}$ is simply related to the cold plasma susceptibility tensor, $\boldsymbol{\chi}^\sigma = \boldsymbol{\chi}(\omega^\sigma)$, associated with the scattered wave, [2σ],

$$\chi_{ij}^\sigma = -\frac{\omega_p^2}{(\omega^\sigma)^2} T_{ij}^{(1\alpha)} \, , \qquad \omega_p^2 = \frac{n^{(0)} q_e^2}{m_e \varepsilon_0} \; , \tag{52}$$

and introducing the rank three pseudo tensor

$$X_{ijl}^\sigma = -\frac{\omega_p^2}{(\omega^\sigma)^2} (\delta_{ja}\delta_{lb} + \delta_{la}\delta_{jb}) \left\{ \frac{i k_\perp^\sigma c}{\omega_c} \left[\delta_{a2} T_{ib}^{(1\alpha)} - T_{iab}^{(2)} \right] + \frac{k_\parallel^\sigma c}{\omega^\sigma} \delta_{a3} T_{bl}^{(1\beta)} \right\} \tag{53}$$

equation (51) can be written

$$j_i^\sigma = \frac{-i\omega^\sigma \varepsilon_0}{n^{(0)}} \int \frac{d\mathbf{k}\, d\omega}{(2\pi)^4} \int d\mathbf{p} \left[\left(E_l^{(1a)} + \epsilon_{lmn} v_m B_n^{(1a)} \right) f^{(1b)} + \left(E_l^{(1b)} + \epsilon_{lmn} v_m B_n^{(1b)} \right) f^{(1a)} \right] \left(\chi_{il}^\sigma + X_{ijl}^\sigma \frac{v_j}{c} \right) . \tag{54}$$

Integrating over $\mathbf{p}$ and retaining only terms up to first order in v, we find

$$j_i^\sigma = \frac{-i\omega^\sigma \varepsilon_0}{n^{(0)}} \int \frac{d\mathbf{k}\, d\omega}{(2\pi)^4} \left\{ \chi_{il}^\sigma \left[n^{(1b)} E_l^{(1a)} + n^{(1a)} E_l^{(1b)} + \epsilon_{lmn} \left(\overline{v}_m^{(1b)} B_n^{(1a)} + \overline{v}_m^{(1a)} B_n^{(1b)} \right) \right] + X_{ijl}^\sigma \frac{1}{c} \left(\overline{v}_j^{(1b)} E_l^{(1a)} + \overline{v}_j^{(1a)} E_l^{(1b)} \right) \right\} \tag{55}$$

where

$$n^{(a)} = \int f^{(a)}\, d\mathbf{p} , \tag{56a}$$

$$\overline{v}_j^{(a)} = \int v_j f^{(a)}\, d\mathbf{p} . \tag{56b}$$

$\overline{\mathbf{v}}^{(a)}$ is the electron flux associated with the momentum perturbation $f^{(a)}$ and should not be confused with a fluid velocity. It is important to note that both the tensors χ_{ij}^σ and X_{ijk}^σ are evaluated with the frequency, ω^σ and, in the case of X_{ijk}^σ, the wave vector, $\mathbf{k}^\sigma$, of the scattered wave.

If one of the interacting waves, e.g. [1a], is monochromatic then the convolution integral in the expression (55) for the source current can be eliminated. Assuming that all quantities relating to wave [1a] have the form (32) (see also discussion of expression (33)), the expression for the source current then takes the form

$$j_i^{\sigma-} = \frac{-i\omega^\sigma \varepsilon_0}{n^{(0)}} \left\{ \chi_{il}^\sigma \left[n^{(1b)} E_l^{(1a')} + n^{(1a')} E_l^{(1b)} + \epsilon_{lmn} \left(\overline{v}_m^{(1b)} B_n^{(1a')} + \overline{v}_m^{(1a')} B_n^{(1b)} \right) \right] + X_{ijl}^\sigma \frac{1}{c} \left(\overline{v}_j^{(1b)} E_l^{(1a')} + \overline{v}_j^{(1a')} E_l^{(1b)} \right) \right\} . \tag{57}$$

Equations (55) and (57) are the expressions for the source current in the low temperature limit, derived on the basis of the kinetic plasma model. The terms in expression (57) involving the susceptibility tensor χ^σ are identical to those found by AAMODT and RUSSELL (1992). For the remainder of the terms, AAMODT and RUSSELL introduce a number of additional simplifications, making it difficult to compare the terms in (57) involving X^σ_{ijk} with their result.

3 TRADITIONAL FLUID APPROACH

The expression for the source current used most widely, explicitly or implicitly, is that derived by AKHIEZER *et al.* (1962 and 1967) and SITENKO (1967) on the basis of the cold fluid equations. Their expression for the source current of the scattered field in a magnetized plasma is

$$\begin{aligned} j_i^{\sigma-} &= -\frac{i\omega^{\mathrm{a}}}{4\pi}E_j^{(1\mathrm{a})}\left\{\chi_{ij}^{(1\mathrm{a})}\frac{n^{(1\mathrm{b})}}{n_0} - i\frac{q}{m_e c}\frac{\omega^\sigma}{\omega_p^2}\chi^\sigma_{il}\chi^{(1\mathrm{a})}_{kj}\epsilon_{lkm}B_m^{(1\mathrm{b})}\right. \\ &\quad + \frac{u_l^{(1\mathrm{b})}}{\omega^{\mathrm{a}}}\left(\frac{\omega^\sigma}{\omega^{\mathrm{a}}}\chi^\sigma_{ik}\left[k_k^{\mathrm{a}}\delta_{jl} - \delta_{kj}k_l^{\mathrm{a}}\right.\right. \\ &\quad \left.\left.\left. - \frac{(\omega^{\mathrm{a}})^2}{\omega_p^2}\chi_{mj}^{(1\mathrm{a})}\left((k_m^\sigma - k_m^{\mathrm{a}})\delta_{kl} + \delta_{mk}k_l^{\mathrm{a}}\right)\right] + \chi_{kj}^{(1\mathrm{a})}k_k^{\mathrm{a}}\delta_{il}\right)\right\}. \end{aligned} \tag{58}$$

Here $\mathbf{u}$ is the fluid velocity

$$\mathbf{u}^{(a)} = \frac{\overline{\mathbf{v}}^{(a)}}{n}, \tag{59a}$$

$$n = n^{(0)} + n^{(1\mathrm{a})} + n^{(1\mathrm{b})} + n^{(2\sigma)} + \ldots, \tag{59b}$$

where $n^{(a)}$ and $\overline{\mathbf{v}}^{(a)}$ are defined by expressions (56a) and (56b) respectively. Note that Gaussian units are used in (58). The rectangular brackets in (58) are missing in SITENKO's expression (11.5): this is clearly a misprint, but it has nevertheless been reproduced in a number of more recent articles.

In the derivation of (58) AKHIEZER *et al.* and SITENKO ignore the difference between $\overline{\mathbf{v}}^a/n$ and $\overline{\mathbf{v}}^a/(n^0 + n^a)$ and consequently lose second order terms. Since scattering is due to second order terms this is not acceptable. The momentum equation from which they start (see SITENKO (1967) equation 11.1) also leaves out second order terms. For these reasons (58) is not the correct expression for the source current in the cold plasma limit.[2] In particular we note that the first term in (58), which represents scattering

[2] AKHIEZER *et al.* (1967) also gave a derivation based on a kinetic model of an unmagnetized plasma.

due to density fluctuations, corresponds to the first term in equation (57) with the important substitution of $\omega^a\chi^a$ for $\omega^\sigma\chi^\sigma$, which means that in this term AKHIEZER *et al.* and SITENKO incorrectly use the plasma conductivity associated with wave [1a] rather than that associated with the scattered wave [2σ].

Their result has nevertheless been quoted widely and used for predicting the performances of diagnostics and for analyzing results, and the methods they employed may have been used for a range of other non-linear plasma phenomena. We shall therefore investigate their approach in detail to clarify where it goes wrong and how a fluid approach may be applied correctly to the problem of three wave mixing and scattering in a plasma. To this end we need the details of the derivation of their result. This does not appear to be available in the literature, so we give it here.

In the traditional fluid approach the expression for the source current of the scattered field is derived as follows. The total current associated with the electrons is given by

$$\mathbf{j} = q_e n \mathbf{u} \; . \tag{60}$$

In the cold fluid approximation which neglects pressure terms and collisions the density, n, and fluid velocity, $\mathbf{u}$, are governed by the continuity equation,

$$\frac{\partial n}{\partial t} + \frac{\partial\{u_i n\}}{\partial r_i} = 0 \; , \tag{61}$$

and the momentum equation,

$$\frac{\partial \mathbf{u}}{\partial t} + u_i \frac{\partial \mathbf{u}}{\partial r_i} = \frac{q_e}{m_e}(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \; . \tag{62}$$

As in Section 2, solutions are sought to this set of equations as perturbations to a time and space independent solution, i.e.

$$\begin{array}{rcccccccl} n(\mathbf{r},t) & = & n^{(0)} & + & n^{(1)}(\mathbf{r},t) & + & n^{(2)}(\mathbf{r},t) & + & \ldots \\ \mathbf{u}(\mathbf{r},t) & = & & & \mathbf{u}^{(1)}(\mathbf{r},t) & + & \mathbf{u}^{(2)}(\mathbf{r},t) & + & \ldots \\ \mathbf{E}(\mathbf{r},t) & = & & & \mathbf{E}^{(1)}(\mathbf{r},t) & + & \mathbf{E}^{(2)}(\mathbf{r},t) & + & \ldots \\ \mathbf{B}(\mathbf{r},t) & = & \mathbf{B}^{(0)} & + & \mathbf{B}^{(1)}(\mathbf{r},t) & + & \mathbf{B}^{(2)}(\mathbf{r},t) & + & \ldots , \end{array} \tag{63}$$

This derivation unfortunately has some mistakes which bring the result into agreement with their incorrect fluid result.

corresponding to a homogeneous and stationary plasma where the equilibrium values of $\mathbf{u}$ and $\mathbf{E}$ both are identically zero.

As in the kinetic treament in Section 2, inserting (63) in Maxwell's equations and equations (60), (61) and (62) results in a set of equations for each perturbation order which is linear in quantities of that order. We may therefore represent the first and second order solutions as a sum of the elements [1a], [1b] and [2a], [2b], [2σ] respectively, where the significance of each element is the same as in Section 2, and hence [2σ] is the scattered wave which is found from the second order equations including only cross terms in first order quantities.

The equations for [2σ] are Maxwell's equations, the current equation,

$$\mathbf{j}^{(2\sigma)} = q_e \left(n^{(0)} \mathbf{u}^{(2\sigma)} + n^{(1\mathrm{a})} \mathbf{u}^{(1\mathrm{b})} + n^{(1\mathrm{b})} \mathbf{u}^{(1\mathrm{a})} \right) , \tag{64}$$

and the second order momentum equation for $\mathbf{u}^{(2\sigma)}$.

3.1 Unmagnetized plasma

First we consider an unmagnetized plasma, i.e. $\mathbf{B}^{(0)} = 0$. The momentum equation for $\mathbf{u}^{(2\sigma)}$ then takes the form

$$\begin{aligned} \frac{\partial \mathbf{u}^{(2\sigma)}}{\partial t} + u_i^{(1\mathrm{a})} \frac{\partial \mathbf{u}^{(1\mathrm{b})}}{\partial r_i} + u_i^{(1\mathrm{b})} \frac{\partial \mathbf{u}^{(1\mathrm{a})}}{\partial r_i} & \\ = \frac{q_e}{m_e} \left(\mathbf{E}^{(2\sigma)} + \mathbf{u}^{(1\mathrm{a})} \times \mathbf{B}^{(1\mathrm{b})} + \mathbf{u}^{(1\mathrm{b})} \times \mathbf{B}^{(1\mathrm{a})} \right) . & \end{aligned} \tag{65}$$

Note that because $\mathbf{u}^{(0)} = 0$ we do not need a separate equation for $n^{(2\sigma)}$.

Fourier transforming (64) and (65), assuming that wave [1a] is monochromatic and of the form (32), and ignoring terms involving the sum frequency, one finds

$$\begin{aligned} \mathbf{j}^{(2\sigma^-)} &= \frac{i q_e^2 n^{(0)}}{\omega^\sigma m_e} \left[\mathbf{E}^{(2\sigma)} + \mathbf{u}^{(1\mathrm{a}')} \times \mathbf{B}^{(1\mathrm{b})} + \mathbf{u}^{(1\mathrm{b})} \times \mathbf{B}^{(1\mathrm{a}')} \right] \\ &+ \frac{q_e n^{(0)}}{\omega^\sigma} \left[\left(\mathbf{u}^{(1\mathrm{a}')} \cdot \mathbf{k}^{\mathrm{b}} \right) \mathbf{u}^{(1\mathrm{b})} + \left(\mathbf{u}^{(1\mathrm{b})} \cdot \mathbf{k}^{\mathrm{a}} \right) \mathbf{u}^{(1\mathrm{a}')} \right] \\ &+ q_e \left[n^{(1\mathrm{a}')} \mathbf{u}^{(1\mathrm{b})} + n^{(1\mathrm{b})} \mathbf{u}^{(1\mathrm{a}')} \right] . \end{aligned} \tag{66}$$

The first term in this expression is the current driven by the scattered field itself. The rest of the terms make up the source current, $\mathbf{j}^{\sigma}$, which drives the scattered wave. At this point AKHIEZER *et al.* consider the interaction of a monochromatic wave, [1a], with plasma fluctuations, [1b]. They further assume that the fluctuations are electrostatic, implying that $\mathbf{B}^{(1b)} = 0$ and hence that the second term in (66) is identically zero. A further consequence of the electrostatic approximation is that $\mathbf{u}^{(1b)} \parallel \mathbf{k}^{b}$ which makes it possible to solve for $\mathbf{u}^{(1b)}$ in the linearized continuity equation for wave [1b]: $\mathbf{u}^{(1b)} = \mathbf{k}^{b}\omega^{b}n^{(1b)}/\left(k^{b}\right)^{2} n^{(0)}$. Using this relation, Faraday's law, and the linearised momentum equation (the two latter for wave [1a]), the third, fourth and fifth terms in (66) take the form

$$\frac{iq_e^2 n^{(0)}}{\omega^{\sigma} m_e}\mathbf{u}^{(1b)} \times \mathbf{B}^{(1a')} + \frac{q_e n^{(0)}}{\omega^{\sigma}}\left[\left(\mathbf{u}^{(1a')} \cdot \mathbf{k}^{b}\right)\mathbf{u}^{(1b)} + \left(\mathbf{u}^{(1b)} \cdot \mathbf{k}^{a}\right)\mathbf{u}^{(1a')}\right]$$

$$= \frac{iq_e^2}{m_e\omega^{a}}\frac{\omega^{b}}{\omega^{\sigma}\left(k^{b}\right)^{2}}\left(\mathbf{E}^{(1a')} \cdot \mathbf{k}^{b}\right)\mathbf{k}^{\sigma} n^{(1b)} . \qquad (67)$$

Making use of the continuity and momentum equations for wave [1a], the sixth term in (66) takes the form

$$q_e n^{(1a')}\mathbf{u}^{(1b)} = \frac{iq_e^2}{m_e\omega^{a}}\frac{\omega^{b}}{\omega^{a}\left(k^{b}\right)^{2}}\mathbf{k}^{b}\left(\mathbf{k}^{a} \cdot \mathbf{E}^{(1a')}\right) n^{(1b)} . \qquad (68)$$

while the seventh term can be written

$$q_e n^{(1b)}\mathbf{u}^{(1a')} = \frac{iq_e^2}{m_e\omega^{a}}\mathbf{E}^{(1a')} n^{(1b)} . \qquad (69)$$

Collecting expressions (67), (68) and (69) the source current takes the form

$$\mathbf{j}^{\sigma-} = \frac{iq_e^2}{m_e\omega^{a}}\left[\mathbf{E}^{(1a')} + \frac{\omega^{b}}{\left(k^{b}\right)}\left(\frac{\mathbf{k}^{b} \cdot \mathbf{E}^{(1a')}}{\omega^{\sigma}}\mathbf{k}^{\sigma} + \frac{\mathbf{k}^{a} \cdot \mathbf{E}^{(1a')}}{\omega^{a}}\mathbf{k}^{b}\right)\right] n^{(1b)} . \qquad (70)$$

This expression for the source current of the scattered field is identical to that given by AKHIEZER *et al.* (1967), page 144.

In an isotropic medium transverse and longitudinal oscillations decouple. In the cold plasma approximation, although longitudinal oscillations exist, only transverse oscillations propagate. In expression (70) one can isolate that part, $\mathbf{j}_{\perp}^{\sigma}$, of the source current which is responsible for the scattering of a transverse wave [1a] into a transverse wave [2σ]:

$$\mathbf{j}_\perp^{\sigma^-} = \frac{iq_e^2}{m_e\omega^{\mathrm{a}}}\mathbf{E}_\perp^{(1\mathrm{a}')}n^{(1\mathrm{b})} . \tag{71}$$

Here $\mathbf{E}_\perp^{(1\mathrm{a}')}$ is the component of $\mathbf{E}^{(1\mathrm{a}')}$ which is perpendicular to $\mathbf{k}^\sigma$. Expression (71) is identical to that given in SITENKO (1967), equation (10.13).

For comparison with the result derived by the kinetic approach in the previous section we also give the expression for $\mathbf{j}^{\sigma^-}$ without assuming that wave [1b] is purely electrostatic and retaining the symmetry between waves [1a] and [1b]:

$$\begin{aligned}
\mathbf{j}^{\sigma^-} \quad = \quad & \frac{iq_e^2}{m_e}\left(\frac{1}{\omega^{\mathrm{a}}}\mathbf{E}^{(1\mathrm{a}')}n^{(1\mathrm{b})} + \frac{1}{\omega^{\mathrm{b}}}\mathbf{E}^{(1\mathrm{b})}n^{(1\mathrm{a}')}\right) \\
& + \frac{iq_e}{m_e\omega^\sigma}\left(\mathbf{E}^{(1\mathrm{a}')}\frac{\mathbf{j}^{(1\mathrm{b})}\cdot\mathbf{k}^{\mathrm{a}}}{\omega^{\mathrm{a}}} + \mathbf{E}^{(1\mathrm{b})}\frac{\mathbf{j}^{(1\mathrm{a}')}\cdot\mathbf{k}^{\mathrm{b}}}{\omega^{\mathrm{b}}}\right) \\
& + \frac{iq_e}{m_e\omega^\sigma}\left(\mathbf{j}^{(1\mathrm{b})}\times\mathbf{B}^{(1\mathrm{a}')} + \mathbf{j}^{(1\mathrm{a}')}\times\mathbf{B}^{(1\mathrm{b})}\right) .
\end{aligned} \tag{72}$$

Here use has been made of the linearized momentum equations and the linearized current equations for waves [1a] and [1b].

Comparison of expression (72) with the kinetic expression (33) reveals a number of differences. In particular the first term, which is usually the most significant in Thomson scattering, differs in the two expressions by $\omega^\sigma/\omega^{\mathrm{a}}$.

3.2 Magnetized plasma

The traditional fluid derivation of the source current in a magnetized plasma follows the same lines as that of the unmagnetized case. Here we therefore only give those steps in the derivation for a magnetized plasma which differ from the derivation for an unmagnetized plasma.

Expression (64) for $\mathbf{j}^{(2\sigma)}$ is again the starting point. In the magnetized case the momentum equation (65) is modified to read

$$\begin{aligned}
& \frac{\partial\mathbf{u}^{(2\sigma)}}{\partial t} + u_i^{(1\mathrm{a})}\frac{\partial\mathbf{u}^{(1\mathrm{b})}}{\partial r_i} + u_i^{(1\mathrm{b})}\frac{\partial\mathbf{u}^{(1\mathrm{a})}}{\partial r_i} \\
& \quad = \frac{q_e}{m_e}\left(\mathbf{E}^{(2\sigma)} + \mathbf{u}^{(2\sigma)}\times\mathbf{B}^{(0)} + \mathbf{u}^{(1\mathrm{a})}\times\mathbf{B}^{(1\mathrm{b})} + \mathbf{u}^{(1\mathrm{b})}\times\mathbf{B}^{(1\mathrm{a})}\right) .
\end{aligned} \tag{73}$$

Fourier–Laplace transforming equation (73), assuming [1a] is monochromatic and ignoring terms involving the sum frequency, we find

$$\begin{aligned}\mathbf{\Pi}\cdot\mathbf{u}^{(2\sigma^-)} &= \frac{q_e}{m_e}\left(\mathbf{E}^{(2\sigma)}+\mathbf{u}^{(1\mathrm{a}')}\times\mathbf{B}^{(1\mathrm{b})}+\mathbf{u}^{(1\mathrm{b})}\times\mathbf{B}^{(1\mathrm{a}')}\right) \\ &\quad - i\mathbf{u}^{(1\mathrm{a}')}\mathbf{k}^\sigma\cdot\mathbf{u}^{(1\mathrm{b})} - i\mathbf{u}^{(1\mathrm{b})}\mathbf{k}^\sigma\cdot\mathbf{u}^{(1\mathrm{a}')}\,,\end{aligned} \tag{74}$$

where the tensor $\mathbf{\Pi}$ has the form

$$\Pi_{ij} = -i\omega^\sigma\delta_{ij}+\omega_c\epsilon_{ijk}\hat{B}_k^{(0)}\,, \qquad \hat{\mathbf{B}}^{(0)}=\mathbf{B}^{(0)}/|\mathbf{B}^{(0)}|\,. \tag{75}$$

The inverse of $\mathbf{\Pi}$ is related to the susceptibility associated with the scattered wave:

$$\frac{i\omega_p^2}{\omega^\sigma}\mathbf{\Pi}^{-1} = \boldsymbol{\chi}^\sigma\,. \tag{76}$$

From expressions (64) and (74) one then finds that the source current for the scattered field in the magnetized case is given by

$$\begin{aligned}\mathbf{j}^{\sigma^-} &= -\frac{i\varepsilon_0}{n^{(0)}}\left(\omega^{\mathrm{a}}\boldsymbol{\chi}^{\mathrm{a}}\cdot\mathbf{E}^{(1\mathrm{a}')}n^{(1\mathrm{b})}+\omega^{\mathrm{b}}\boldsymbol{\chi}^{\mathrm{b}}\cdot\mathbf{E}^{(1\mathrm{b})}n^{(1\mathrm{a}')}\right) \\ &\quad - i\varepsilon_0\,\omega^\sigma\boldsymbol{\chi}^\sigma\cdot\left(\mathbf{u}^{(1\mathrm{a}')}\times\mathbf{B}^{(1\mathrm{b})}+\mathbf{u}^{(1\mathrm{b})}\times\mathbf{B}^{(1\mathrm{a}')}\right) \\ &\quad - \frac{\varepsilon_0 m_e}{q_e}\omega^\sigma\boldsymbol{\chi}^\sigma\cdot\left[\mathbf{u}^{(1\mathrm{a}')}\left(\mathbf{k}^\sigma\cdot\mathbf{u}^{(1\mathrm{b})}\right)+\mathbf{u}^{(1\mathrm{b})}\left(\mathbf{k}^\sigma\cdot\mathbf{u}^{(1\mathrm{a}')}\right)\right].\end{aligned} \tag{77}$$

By using the linearized continuity equation, the linearized momentum equation, $u_i^{(1\mathrm{a}')} = -i\omega^{\mathrm{a}}\varepsilon_0\chi_{ij}^{\mathrm{a}}E_j^{(1\mathrm{a}')}/q_e n^{(0)}$, and Faraday's law, all for [1a], expression (77) can be written as

$$\begin{aligned}j_i^{\sigma^-} &= -\frac{i\varepsilon_0}{n^{(0)}}\omega^{\mathrm{a}}\chi_{ij}^{\mathrm{a}}E_j^{(1\mathrm{a}')}n^{(1\mathrm{b})} - \frac{q_e\varepsilon_0\omega^\sigma\omega^{\mathrm{a}}}{m_e\omega_p^2}\chi_{ij}^\sigma\epsilon_{jkm}\chi_{kl}^{\mathrm{a}}E_l^{(1\mathrm{a}')}B_m^{(1\mathrm{b})} \\ &\quad + \left\{\frac{-i\varepsilon_0\omega^\sigma}{\omega^{\mathrm{a}}}\chi_{ij}^\sigma\left[k_j^{\mathrm{a}}\delta_{lm}-\delta_{jl}k_m^{\mathrm{a}}-\frac{(\omega^{\mathrm{a}})^2}{\omega_p^2}\chi_{kl}^{\mathrm{a}}\left(\delta_{jk}k_m^{\mathrm{b}}+\delta_{jm}k_k^{\mathrm{b}}\right)\right]\right. \\ &\quad \left. - i\varepsilon_0\chi_{kl}^{\mathrm{a}}k_k^{\mathrm{a}}\delta_{im}\right\}E_l^{(1\mathrm{a}')}u_m^{(1\mathrm{b})}\,.\end{aligned} \tag{78}$$

This expression for the source current of the scattered field in a magnetized plasma is identical to expression (20.3) in AKHIEZER *et al.* (1967) and expression (11.5) in SITENKO (1967), except that we have used SI units.

The first term and the last term in (78) are symmetric with regard to the interchange of waves [1a] and [1b] which can be seen by noting that they stem from the two first terms in (77), or made explicit by rewriting the last term as

$$- i\varepsilon_0 \chi^{\mathrm{a}}_{kl} k^{\mathrm{a}}_k \delta_{im} E^{(1\mathrm{a}')}_l u^{(1\mathrm{b})}_m = -\frac{i\omega^{\mathrm{b}}\varepsilon_0}{n^{(0)}} \chi^{\mathrm{b}}_{ij} E^{(1\mathrm{b})}_j n^{(1\mathrm{a}')} \, . \tag{79}$$

This term can therefore be interpreted as the interaction between the electric field fluctuations [1b] and the density perturbations associated with the incident wave [1a]. The assertion by AAMODT and RUSSELL (1992) p. 747 that the second term in their expression (10) is new is thus not correct. Like AAMODT and RUSSELL's result and our kinetic result, AKHIEZER *et al.* and SITENKO's fluid result for a magnetized plasma is symmetric with respect to interchange of waves [1a] and [1b]. This is not explicit in their final result (78), but it is evident in expression (77).

The differences between the traditional fluid expression and the new kinetic expression are best appreciated by comparing expressions (77) and (57). We find that in the first two terms, representing scattering due to the interaction of electric fields with density perturbations ($\mathbf{E}^{(1\mathrm{a}')}n^{(1\mathrm{b})}$ and $\mathbf{E}^{(1\mathrm{b})}n^{(1\mathrm{a}')}$), the factor $\omega^\sigma \boldsymbol{\chi}^\sigma$ in the kinetic expression (57) is replaced in the first and second terms of the traditional fluid expression (77) by $\omega^{\mathrm{a}}\boldsymbol{\chi}^{\mathrm{a}}$ and $\omega^{\mathrm{b}}\boldsymbol{\chi}^{\mathrm{b}}$ respectively. In the third and fourth terms, representing interaction between particle fluxes and magnetic fields ($\overline{\mathbf{v}}^{(1\mathrm{b})} \times \mathbf{B}^{(1\mathrm{a}')}$ and $\overline{\mathbf{v}}^{(1\mathrm{a}')} \times \mathbf{B}^{(1\mathrm{b})}$), we find complete agreement between the two results (the difference between $\mathbf{v}/n^{(0)}$ and $\mathbf{u}$ is of higher order and not significant at this point). In the fifth and sixth terms we find considerable differences between the two expressions.

While the differences are of minor practical importance to most laser scattering experiments, they are not negligible for the millimetre wave scattering experiments planned at JET [COSTLEY *et al.*, 1988] and TFTR [WOSKOV *et al.*, 1988], and from a theoretical point of view these discrepancies are clearly not satisfactory.

4 CORRECTION OF THE FLUID APPROACH

Since the fluid description has been used so extensively in the modelling of Thomson scattering it is of interest to see where the traditional approach goes wrong, and how the correct expression for the source current may be derived by the fluid approach.

There are two main problems with the traditional fluid approach. To expose the first problem we write down the exact definitions of the fluid variables associated with each wave in terms of the momentum distribution of the plasma:

$$n = \sum_a n^{(a)} , \quad (80a)$$

$$n^{(a)} = \int f^{(a)} \, d\mathbf{p} , \quad (80b)$$

$$\mathbf{u} = \sum_a \mathbf{u}^{(a)} , \quad (81a)$$

$$\mathbf{u}^{(a)} = \frac{1}{n} \int \mathbf{v} f^{(a)} \, d\mathbf{p} . \quad (81b)$$

We note that while $n^{(a)}$ depends only on $f^{(a)}$ this is not the case for $\mathbf{u}^{(a)}$: through the division by n, $\mathbf{u}^{(a)}$ depends on the density perturbations associated with all the other waves that are present in the plasma.

The differences between the fluid velocities associated with a given wave but with different sets of additional waves present is of second or higher order and it is therefore acceptable to ignore the differences in linear problems. Since the problem of scattering or three wave mixing is manifestly nonlinear, second order terms in the definition of the fluid velocity may not be ignored as is done in the traditional fluid approach.

The second problem with the traditional fluid approach is that the pressure term, $\overline{\mathbf{ww}}$,

$$\overline{\mathbf{ww}} = \frac{1}{n} \int (\mathbf{v} - \mathbf{u})(\mathbf{v} - \mathbf{u}) f \, d\mathbf{p} , \quad (82)$$

in the momentum equation was ignored. While it may be acceptable at low temperatures to assume that the first order pressure term, $\overline{\mathbf{ww}}^{(1)}$, is small compared with the other terms in the linearized momentum equation, this is not the case for $\overline{\mathbf{ww}}^{(2)}$ in relation to the other terms in the second order momentum equation, as will be shown in subsections 4.1 and 4.2. Consequently, it may be acceptable to ignore the pressure term in linear problems but not in scattering and three wave mixing.

To overcome the first problem we use the electron flux, $\overline{\mathbf{v}}$,

$$\overline{\mathbf{v}} = \sum_a \overline{\mathbf{v}}^{(a)} , \quad (83a)$$

$$\overline{\mathbf{v}}^{(a)} = \int \mathbf{v} f^{(a)} \, d\mathbf{p} , \quad (83b)$$

as the second fluid variable instead of the fluid velocity, **u**. The second problem we overcome by including the pressure term in the momentum equation. This, of course, implies that the energy equation is required. Our basic set of equations are thus, in addition to Maxwell's equations; the current equation,

$$\mathbf{j} = q_e \overline{\mathbf{v}} \ , \tag{84}$$

and the zeroth, first and second order moments of the Vlasov equation,

$$\frac{\partial}{\partial t} n + \frac{\partial}{\partial r_i} \overline{v_i} = 0 \ , \tag{85}$$

$$\frac{\partial}{\partial t} \overline{v_i} + \frac{\partial}{\partial r_j} \overline{v_i v_j} - \frac{q_e}{m_e} \left(n E_i + \epsilon_{ijk} \overline{v_j} B_k \right) = 0 \ , \tag{86}$$

$$\frac{\partial}{\partial t} \overline{v_i v_j} + \frac{\partial}{\partial r_k} \overline{v_i v_j v_k} - \frac{q_e}{m_e} \left(E_i v_j + v_i E_j + (\delta_{kl} \epsilon_{ilm} + \delta_{ik} \epsilon_{jlm}) \overline{v_k v_l} B_m \right) = 0 \ . \tag{87}$$

Here we have introduced the variables $\overline{v_i v_j}$ and $\overline{v_i v_j v_k}$ defined analogously to $\overline{v_i}$:

$$\overline{v_i v_j} = \sum_a \overline{v_i v_j}^{(a)} \ , \tag{88a}$$

$$\overline{v_i v_j}^{(a)} = \int v_i v_j f^{(a)} \, d\mathbf{p} \ , \tag{88b}$$

$$\overline{v_i v_j v_k} = \sum_a \overline{v_i v_j v_k}^{(a)} \tag{89a}$$

$$\overline{v_i v_j v_k}^{(a)} = \int v_i v_j v_k f^{(a)} \, d\mathbf{p} \ . \tag{89b}$$

To first perturbation order the moments (85), (86) and (87) of the Vlasov equation give the familiar linearized fluid equations. To find the source current we need the equations for the second order terms. Making use of (83a) we find for the current associated with wave $[2\sigma]$

$$\mathbf{j}^{(2\sigma)} = q_e \overline{\mathbf{v}}^{(2\sigma)} \ . \tag{90}$$

Comparing expression (90) with the traditional fluid expression (64) we find that already at this stage in the derivation there are significant differences.

4.1 Unmagnetized plasma

We first consider an unmagnetized plasma where $\mathbf{B}^{(0)} = 0$. Making use of (83a) and (88a), and including, in addition to $[2\sigma]$ terms, only cross terms in first order quantities we find from (86) that

$$\begin{aligned} \frac{\partial \{v_i^{(2\sigma)}\}}{\partial t} &= -\frac{\partial\{\overline{v_i v_j}^{(2\sigma)}\}}{\partial r_j} + \frac{q_e}{m_e} n E_i^{(2\sigma)} \\ &+ \frac{q_e}{m_e}\left(n^{(1\mathrm{b})}E_i^{(1\mathrm{a})} + n^{(1\mathrm{a})}E_i^{(1\mathrm{b})}\right) \\ &+ \frac{1}{m_e}\epsilon_{ijk}\left(j_j^{(1\mathrm{a})}B_k^{\mathrm{b}} + j_j^{(1\mathrm{b})}B_k^{\mathrm{a}}\right) . \end{aligned} \tag{91}$$

We assume that $\overline{\mathbf{vv}}^{(0)}$, $\overline{\mathbf{vv}}^{(1)}$ and $\overline{\mathbf{vvv}}^{(2\sigma)}$ are small compared with the other terms of zeroth, first and second order respectively. This essentially amounts to limiting the plasmas to low temperatures. With this set of assumptions we can close the set of equations for $\mathbf{j}^{(2\sigma)}$ with the energy equation for $\overline{\mathbf{vv}}^{(2\sigma)}$. Retaining only second order terms it takes the form:

$$\frac{\partial}{\partial t}\overline{v_i v_j}^{(2\sigma)} = \frac{1}{m_e}\left(E_i^{(1\mathrm{a})}j_j^{(1\mathrm{b})} + E_j^{(1\mathrm{a})}j_i^{(1\mathrm{b})} + E_i^{(1\mathrm{b})}j_j^{(1\mathrm{a})} + E_j^{(1\mathrm{b})}j_i^{(1\mathrm{a})}\right) . \tag{92}$$

Here we see that $\overline{\mathbf{vv}}^{(2\sigma)}$ can indeed be similar in magnitude to the other second order terms even if $\overline{\mathbf{vv}}^{(1\mathrm{a})}$ and $\overline{\mathbf{vv}}^{(1\mathrm{b})}$ are small compared with the other first order terms. $\overline{\mathbf{vv}}^{(2\sigma)}$ must therefore be retained in the momentum equation. Fourier–Laplace transforming the current equation (90), the momentum equation (91) and the energy equation (92), eliminating $\overline{\mathbf{v}}^{(2\sigma)}$ and $\overline{\mathbf{vv}}^{(2\sigma)}$, and subtracting the linear plasma response to wave $[2\sigma]$ we find the new fluid expression for the source current where, as before, we ignore the terms involving the sum frequency:

$$\begin{aligned} \mathbf{j}^{\sigma^-} &= \frac{iq_e}{m_e\omega^\sigma}\Bigg\{ q_e\left(\mathbf{E}^{(1\mathrm{a}')}n^{(1\mathrm{b})} + \mathbf{E}^{(1\mathrm{b})}n^{(1\mathrm{a}')}\right) \\ &+ \mathbf{E}^{(1\mathrm{a}')}\frac{\mathbf{j}^{(1\mathrm{b})}\cdot\mathbf{k}^\sigma}{\omega^\sigma} + \mathbf{j}^{(1\mathrm{b})}\frac{\mathbf{E}^{(1\mathrm{a}')}\cdot\mathbf{k}^\sigma}{\omega^\sigma} + \mathbf{E}^{(1\mathrm{b})}\frac{\mathbf{j}^{(1\mathrm{a}')}\cdot\mathbf{k}^\sigma}{\omega^\sigma} + \mathbf{j}^{(1\mathrm{a}')}\frac{\mathbf{E}^{(1\mathrm{b})}\cdot\mathbf{k}^\sigma}{\omega^\sigma} \\ &+ \mathbf{j}^{(1\mathrm{b})}\times\mathbf{B}^{(1\mathrm{a}')} + \mathbf{j}^{(1\mathrm{a}')}\times\mathbf{B}^{(1\mathrm{b})}\Bigg\} . \end{aligned} \tag{93}$$

Comparing expression (93) with (33) we find complete agreement, demonstrating that the expression for the source current can be derived correctly by the fluid approach. It is interesting to note that the third to sixth term on the right hand side of (93) (the second line) all stem from the term $\partial\{\overline{v_i v_j}^{(2\sigma)}\}/\partial r_i$ in (91) which was determined by the energy equation (92).

The most important consideration used in guiding this derivation was that fluid variables relating to a particular wave should not include higher order terms relating to other waves. This helped to ensure that no second order terms were lost in the manipulations. The failure of the previous fluid approach was primarily due to the inappropriate use of the fluid velocity as one of the fluid variables. As pointed out earlier, this variable cannot be related to a single wave only, but contains second order terms from all waves present. Its use led among other errors to an incorrect expression for the scattering due to the interaction of $\mathbf{E}^{(1a')}$ with $n^{(1b)}$ which is in many cases the dominant term. Further errors were introduced by the omission of the pressure term in the momentum equation which left out terms of the same order as those included in the traditional fluid expression for the source current.

4.2 Magnetized plasma

For a magnetized plasma we again derive the expression for the electron flux, $\overline{\mathbf{v}}^{(2\sigma)}$, associated with wave $[2\sigma]$ from the momentum and energy equations (86) and (87), but now assume that $\mathbf{B}^{(0)} \neq 0$. Including only second order terms and only cross terms in first order variables we find that the momentum equation gives

$$\begin{aligned} \frac{\partial \overline{v_i}^{(2\sigma)}}{\partial t} + \frac{\partial \overline{v_i v_j}^{(2\sigma)}}{\partial r_j} - \frac{q_e}{m_e}\left(n^{(0)}E_i^{(2\sigma)} + n^{(1a)}E_i^{(1b)} + n^{(1b)}E_i^{(1a)}\right) & \\ -\frac{q_e}{m_e}\epsilon_{ijk}\left(\overline{v_j}^{(2\sigma)}B_k^{(0)} + \overline{v_j}^{(1a)}B_k^{(1b)} + \overline{v_j}^{(1b)}B_k^{(1a)}\right) &= 0 \,. \end{aligned} \tag{94}$$

From the energy equation we find

$$\begin{aligned} \frac{\partial \overline{v_i v_j}^{(2\sigma)}}{\partial t} - \frac{q_e}{m_e}\epsilon_{klm}(\delta_{ik}\delta_{jn} + \delta_{in}\delta_{jk})B_m^{(0)}\overline{v_l v_n}^{(2\sigma)} & \\ -\frac{q_e}{m_e}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\left(\overline{v_l}^{(1a)}E_k^{(1b)} + \overline{v_l}^{(1b)}E_k^{(1a)}\right) &= 0 \,. \end{aligned} \tag{95}$$

Fourier–Laplace transforming equation (94) gives

$$\begin{aligned}
\Pi_{ij}\overline{v_j}^{(2\sigma^-)} &= \frac{q_e}{m_e}\left(n^{(0)}E_i^{(2\sigma)} + n^{(1a')}E_i^{(1b)} + n^{(1b)}E_i^{(1a')}\right) \\
&+ \frac{q_e}{m_e}\epsilon_{ijk}\left(\overline{v_j}^{(1a')}B_k^{(1b)} + \overline{v_j}^{(1b)}B_k^{(1a')}\right) - ik_j^{(2\sigma)}\overline{v_i v_j}^{(2\sigma^-)} \; .
\end{aligned} \tag{96}$$

The tensor $\mathbf{\Pi}$ is defined in expression (75). Fourier–Laplace transforming equation (95) gives

$$-\,i\omega^\sigma \Gamma_{abij}\,\overline{v_i v_j}^{(2\sigma^-)} = \frac{q_e}{m_e}(\delta_{ak}\delta_{bl} + \delta_{al}\delta_{bk})\left(\overline{v_l}^{(1a')}E_k^{(1b)} + \overline{v_l}^{(1b)}E_k^{(1a')}\right) \; . \tag{97}$$

Here the rank four tensor Γ_{abij} has the form

$$\Gamma_{abij} = \delta_{ai}\delta_{bj} + \frac{i\omega_c}{\omega^\sigma}(\epsilon_{kim}\hat{B}_m^{(0)}\delta_{ak}\delta_{bj} + \epsilon_{kjm}\hat{B}_m^{(0)}\delta_{ai}\delta_{bk}) \; . \tag{98}$$

We have made use here of the symmetry of $\overline{v_i v_j}$ with respect to interchange of i and j in order to bring Γ_{abij} into a form which facilitates the finding of the inverse of $\mathbf{\Gamma}$ (see below).

Orienting the coordinate system such that $\hat{\mathbf{z}} = \hat{\mathbf{B}}^{(0)}$, then Γ_{abij} can be written

$$\Gamma_{abij} = \delta_{ai}\delta_{bj} + i\Omega(\xi_{ai}\delta_{bj} + \delta_{ai}\xi_{bj}) \; , \tag{99}$$

where $\Omega = \omega_c/\omega^\sigma$. ξ_{ij} is defined with η_{ij} and ζ_{ij} in expressions (44). We define the inverse of Γ_{abij} as the rank four tensor, Γ^{-1}_{ijab}, which satisfies the relation

$$\Gamma^{-1}_{ijab}\Gamma_{ablm} = \delta_{il}\delta_{jm} \; . \tag{100}$$

This tensor has the form

$$\begin{aligned}
\Gamma^{-1}_{ijab} &= \frac{1}{1-4\Omega^2}\left[-i\Omega\left(\eta_{ia}\xi_{jb} + \xi_{ia}\eta_{jb}\right) - 2\Omega^2\xi_{ia}\xi_{jb} + (1-2\Omega^2)\eta_{ia}\eta_{jb}\right] \\
&+ \frac{1}{1-\Omega^2}\left[\eta_{ia}\zeta_{jb} + \zeta_{ia}\eta_{jb} - i\Omega(\zeta_{ia}\xi_{jb} + \xi_{ia}\zeta_{jb})\right] \; + \; \zeta_{ia}\zeta_{jb} \; .
\end{aligned} \tag{101}$$

Solving for $\overline{\mathbf{vv}}^{(2\sigma^-)}$ in (97), inserting the resulting expression in (96) and solving for $\overline{\mathbf{v}}^{(2\sigma^-)}$ in this equation we find the flux and thus the current associated with wave $[2\sigma]$.

Subtracting the linear plasma response to the wave itself gives the following expression for the source current:

$$\begin{aligned} j_h^{\sigma-} &= \frac{-i\omega^\sigma \varepsilon_0}{n^{(0)}} \chi_{hi}^\sigma \Bigg\{ n^{(1a')} E_i^{(1b)} + n^{(1b)} E_i^{(1a')} \\ &\quad + \epsilon_{ijk} \left(\overline{v_j}^{(1a')} B_k^{(1b)} + \overline{v_j}^{(1b)} B_k^{(1a')} \right) \\ &\quad + \frac{k_j^\sigma}{\omega^\sigma} \Gamma_{ijab}^{-1} (\delta_{ak}\delta_{bl} + \delta_{al}\delta_{bk}) \left(\overline{v_l}^{(1a')} E_k^{(1b)} + \overline{v_l}^{(1b)} E_k^{(1a')} \right) \Bigg\} . \end{aligned} \tag{102}$$

Orienting the coordinate system such that in addition to $\hat{\mathbf{z}} = \hat{\mathbf{B}}^{(0)}$ we also have that $\mathbf{k}^\sigma = k_\perp^\sigma \hat{\mathbf{x}} + k_\parallel^\sigma \hat{\mathbf{z}}$, then

$$\frac{k_j^\sigma}{\omega^\sigma} \Gamma_{ijab}^{-1} (\delta_{ak}\delta_{bl} + \delta_{al}\delta_{bk}) = \frac{1}{c} X_{hlk}^\sigma$$

where X_{hlk}^σ was defined in (53), and we find that the new fluid expression (102) for the source current in a magnetized plasma is in complete agreement with the kinetic result (57).

5 INTERACTION OF A MICROSCOPIC AND A MACROSCOPIC WAVE

In the previous sections we have considered non–linear interaction between macroscopic waves, i.e. waves for which the ensemble averages of their microscopic fields and distributions did not vanish. These descriptions of scattering do not include the scattering of a macroscopic wave by thermal fluctuations, which is most readily seen by the fact that the ensemble averages of the microscopic fields and microscopic distributions associated with the thermal fluctuations and the resulting scattered fields are identically zero.

Here we must resort to a microscopic description of the plasma, introducing the ensemble average at a later stage in the development. Let $N(\mathbf{r}, \mathbf{p}, t)$ be KLIMONTOVICH's microscopic distribution function [KLIMONTOVICH, 1967 and 1982; SCHRAM, 1991], and $\mathbf{E}^{\mathrm{M}}(\mathbf{r}, t)$ and $\mathbf{B}^{\mathrm{M}}(\mathbf{r}, t)$ be the microscopic fields. The microscopic distribution function satisfies the dynamic equation [KLIMONTOVICH, 1982, Section 24]

$$\frac{\partial N}{\partial t} + \mathbf{v} \cdot \frac{\partial N}{\partial \mathbf{r}} + q_e \left(\mathbf{E}^{\mathrm{M}} + \mathbf{v} \times \mathbf{B}^{\mathrm{M}} \right) \cdot \frac{\partial N}{\partial \mathbf{p}} = 0 . \tag{103}$$

We now split the microscopic quantities [N, $\mathbf{E}^{\mathrm{M}}$, $\mathbf{B}^{\mathrm{M}}$] into a macroscopic part [f, $\mathbf{E}$, $\mathbf{B}$] and microscopic fluctuations [$\tilde{N}$, $\tilde{\mathbf{E}}$, $\tilde{\mathbf{B}}$], defined respectively as

$$f = \langle N \rangle \ , \quad \mathbf{E} = \left\langle \mathbf{E}^{\mathrm{M}} \right\rangle \ , \quad \mathbf{B} = \left\langle \mathbf{B}^{\mathrm{M}} \right\rangle \ , \tag{104}$$

and

$$\tilde{N} = N - \langle N \rangle \ , \quad \tilde{\mathbf{E}} = \mathbf{E}^{\mathrm{M}} - \left\langle \mathbf{E}^{\mathrm{M}} \right\rangle \ , \quad \tilde{\mathbf{B}} = \mathbf{B}^{\mathrm{M}} - \left\langle \mathbf{B}^{\mathrm{M}} \right\rangle \ , \tag{105}$$

where $\langle\rangle$ denotes ensemble average. Note that from the definitions (105) it follows that $\left\langle \tilde{N} \right\rangle = \left\langle \tilde{\mathbf{E}} \right\rangle = \left\langle \tilde{\mathbf{B}} \right\rangle = 0$, as stated in the first paragraph of this section. It is convenient to introduce also the fluctuating microscopic force $\tilde{\mathbf{F}} = q_e(\tilde{\mathbf{E}} + \mathbf{v} \times \tilde{\mathbf{B}})$.

Taking the ensemble average of the dynamic equation (103) we get the Vlasov equation plus the collision integral, $\partial \left\langle \tilde{N} \tilde{F}_i \right\rangle / \partial p_i$ [KLIMONTOVICH, 1982],

$$\frac{\partial f}{\partial t} + v_i \cdot \frac{\partial f}{\partial r_i} + F_i \frac{\partial f}{\partial p_i} + \frac{\partial}{\partial p_i} \left\langle \tilde{N} \tilde{F}_i \right\rangle = 0 \ , \tag{106}$$

which governs the macroscopic waves.

Subtracting (106) from (103) we obtain the dynamic equation for the fluctuating microscopic distribution,

$$\frac{\partial \tilde{N}}{\partial t} + v_i \cdot \frac{\partial \tilde{N}}{\partial r_i} + \frac{\partial}{\partial p_i} \left\{ F_i \tilde{N} + \tilde{F}_i f + \left(\tilde{F}_i \tilde{N} - \left\langle \tilde{F}_i \tilde{N} \right\rangle \right) \right\} = 0 \ . \tag{107}$$

We see from (107) that the fluctuating microscopic distribution is modified by the presence of macroscopic waves through the terms $\mathbf{F}\tilde{N}$ and $\tilde{\mathbf{F}}f$ (the term $\partial \left(\tilde{\mathbf{F}}\tilde{N} - \left\langle \tilde{\mathbf{F}}\tilde{N} \right\rangle \right) / \partial \mathbf{p}$ represents fluctuations in the collision integral and can be ignored for the present purpose). This modification is the scattered wave resulting from the interaction between macroscopic waves and the microscopic fluctuations (e.g. thermal fluctuations). Let [δ] represent the microscopic fluctuations in the absence of macroscopic waves and let [i] be the macroscopic waves. The dynamic equation for the first Born approximation to the scattered field, [s], is obtained from (107) by retaining, in addition to the homogeneous part of (107), those terms which involve cross products between quantities relating to [δ] and [i]. The result is

$$\mathcal{L}\tilde{N}^{\mathrm{s}} = -\mathbf{F}^{\mathrm{s}} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{p}} - \mathbf{F}^{\mathrm{i}} \cdot \frac{\partial \tilde{N}^{\delta}}{\partial \mathbf{p}} - \tilde{\mathbf{F}}^{\delta} \cdot \frac{\partial f^{\mathrm{i}}}{\partial \mathbf{p}} \ . \tag{108}$$

Noting that equation (108) has the same form as (12) we have that the scattered field is given by

$$\mathbf{\Lambda}(\mathbf{k}^\sigma,\omega^\sigma)\cdot\tilde{\mathbf{E}}^{s}(\mathbf{k}^\sigma,\omega^\sigma)=\frac{-i}{\omega^\sigma\varepsilon_0}\tilde{\mathbf{j}}^\sigma(\mathbf{k}^\sigma,\omega^\sigma)\ , \tag{109}$$

where $\tilde{\mathbf{j}}^\sigma$ is the source current,

$$\tilde{\mathbf{j}}^\sigma \;=\; q_e\int \mathbf{v}\tilde{N}^\sigma\, d\mathbf{p}\ , \tag{110a}$$

$$\mathcal{L}\tilde{N}^\sigma \;=\; -\mathbf{F}^{\mathrm{i}}\cdot\frac{\partial\tilde{N}^\delta}{\partial\mathbf{p}}-\tilde{\mathbf{F}}^\delta\cdot\frac{\partial f^{\mathrm{i}}}{\partial\mathbf{p}}\ . \tag{110b}$$

With the assumptions made in equations (34) to (36) but relating to $\tilde{N}^\sigma$ instead of f^σ, and assuming a coordinate system where $\hat{\mathbf{z}}=\hat{\mathbf{B}}^{(0)}$ and $\mathbf{k}^\sigma=k_\perp^\sigma\hat{\mathbf{x}}+k_\parallel^\sigma\hat{\mathbf{z}}$, the source current takes the form

$$\begin{aligned}\tilde{j}_i^\sigma \;=\; \frac{-i\omega^\sigma\varepsilon_0}{n^{(0)}}\int\Big\{\ &\chi_{il}^\sigma\left[\tilde{n}^\delta E_l^{\mathrm{i}}+n^{\mathrm{i}}\tilde{E}_l^\delta+\ \epsilon_{lmn}\left(\tilde{v}_m^\delta B_n^{\mathrm{i}}+\overline{v}_m^{\mathrm{i}}\tilde{B}_n^\delta\right)\right]\\ &+X_{ijl}^\sigma\frac{1}{c}\left(\tilde{v}_j^\delta E_l^{\mathrm{i}}+\overline{v}_j^{\mathrm{i}}\tilde{E}_l^\delta\right)\Big\}\,\frac{d\mathbf{k}^{\mathrm{i}}\,d\omega^{\mathrm{i}}}{(2\pi)^4}\end{aligned} \tag{111}$$

where $\mathbf{k}^\delta=\mathbf{k}^\sigma-\mathbf{k}^{\mathrm{i}}$, $\omega^\delta=\omega^\sigma-\omega^{\mathrm{i}}$ and

$$\tilde{n}^\delta \;=\; \int\tilde{N}^\delta\, d\mathbf{p}\ , \tag{112}$$

$$\tilde{\mathbf{v}}^\delta \;=\; \int\mathbf{v}\tilde{N}^\delta\, d\mathbf{p}\ . \tag{113}$$

With the use of the relations

$$B_i^{\mathrm{i}} \;=\; \frac{1}{\omega^{\mathrm{i}}}\epsilon_{ijl}k_j^{\mathrm{i}}E_l^{\mathrm{i}}\ , \tag{114}$$

$$\overline{v}_i^{\mathrm{i}} \;=\; -\frac{i\omega^{\mathrm{i}}\varepsilon_0}{q}\chi_{il}^{\mathrm{i}}E_l^{\mathrm{i}}\ , \tag{115}$$

$$n^{\mathrm{i}} \;=\; -\frac{i\varepsilon_0}{q}k_i^{\mathrm{i}}\chi_{il}^{\mathrm{i}}E_l^{\mathrm{i}}\ , \tag{116}$$

$\mathbf{B}^{\mathrm{i}}$, $\overline{\mathbf{v}}^{\mathrm{i}}$ and n^{i} can all be eliminated from the expression for the source current:

$$\tilde{j}_i^\sigma = \frac{-i\omega^\sigma\varepsilon_0}{n^{(0)}}\int\frac{d\mathbf{k}^{\rm i}\,d\omega^{\rm i}}{(2\pi)^4}\left\{\chi^\sigma_{il}\,n^\delta - \chi^\sigma_{ij}\epsilon_{jmk}\frac{i\omega^{\rm i}\varepsilon_0}{q_e}\chi^{\rm i}_{ml}\,B^\delta_k\right. \tag{117}$$

$$+\left(\chi^\sigma_{ij}\epsilon_{jkm}\epsilon_{mnl}\frac{k^{\rm i}_n}{\omega^{\rm i}} + X^\sigma_{ikl}\frac{1}{c}\right)\overline{v}^\delta_k$$

$$\left.-\left(\chi^\sigma_{ik}\frac{i\varepsilon_0}{q_e}k^{\rm i}_j\chi^{\rm i}_{jl} + X^\sigma_{ijk}\frac{i\omega^{\rm i}\varepsilon_0}{cq_e}\chi^{\rm i}_{jl}\right)E^\delta_k\right\}E^{\rm i}_l\,.$$

The expression for the source current, $\tilde{\mathbf{j}}^\sigma$, can be written in a number of other ways, for instance by the use of relations between the variables relating to the fluctuations [δ]. For specific applications various simplifying assumptions may be in order. For the purpose of generating a general routine for numerical evaluation of the scattering cross–section or equation of transfer for a scattering system the form given in equation (117) is convenient.

6 EQUATION OF TRANSFER FOR A SACTTERING SYSTEM

The equation of transfer for a scattering diagnostic has been developed and investigated under a range of conditions by a number of authors including SIMONICH and YEH (1972), BRETZ (1987), HUGHES and SMITH (1989) and BINDSLEV (1991). To ease the use of the expressions for the source current derived in the previous sections we demonstrate here, using expression (117) as an example, how these results may be intgrated with the equation of transfer derived in BINDSLEV (1991).

When a monochromatic incident beam, [i], is scattered in a plasma by fluctuations, [δ], then the scattered power per unit angular frequency, $\partial P^{\rm s}(\omega^{\rm s},t_i)/\partial\omega^{\rm s}$, received by a coherent detector in the time interval $t = t_i$ to $t = t_i + T$ is given by equation (52) in BINDSLEV (1991). In most plasma Thomson scattering experiments the anti–Hermitian part of the dielectric tensor associated with the scattered wave, $(\boldsymbol{\varepsilon}^a)^{\rm s}$, is negligible; this simplifies the equation of transfer considerably [BINDSLEV, 1991] and we have

$$\frac{\partial P^{\rm s}(\omega^{\rm s},t_i)}{\partial\omega^{\rm s}} = \frac{1}{4\pi\varepsilon_0 c\mathcal{S}^{\rm s}}\frac{\left\langle|J(\omega^{\rm s},t_i)|^2\right\rangle}{T} \tag{118}$$

where $\mathcal{S}$ is the normalized flux,

$$\mathcal{S} = \frac{1}{\varepsilon_0 c}\left|\left\{\mathbf{S}_{ij} - \frac{\omega\varepsilon_0}{2}\frac{\partial\varepsilon^h_{ij}}{\partial\mathbf{k}}\right\}e^*_i e_j\right|\,. \tag{119}$$

The Poynting tensor, $\mathbf{S}_{ij}$, is defined as

$$\{\mathbf{S}_{ij}\}_l = \frac{k_k}{2\mu_0\omega}\left(-\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il} + 2\delta_{lk}\delta_{ij}\right) , \tag{120}$$

where $\mathbf{e}$ is the unit electric field vector and $\boldsymbol{\varepsilon}^h$ is the Hermitian part of $\boldsymbol{\varepsilon}$. $J(\omega^{\mathrm{s}}, t_i)$ is defined as

$$J(\omega^{\mathrm{s}}, t_i) = (\mathbf{e}^{\mathrm{s}})^* \cdot \int_{t_i}^{t_i+T} \int_{R^3} \sqrt{\mathcal{I}^{\mathrm{s}}(\mathbf{r})}\tilde{\mathbf{j}}^{\sigma}(\mathbf{r}, t) e^{-i(\mathbf{k}^{\mathbf{s}}\cdot\mathbf{r} - \omega^{\mathbf{s}} t)}\, d\mathbf{r}\, dt , \tag{121}$$

where $\mathcal{I}$ is the normalized beam intensity [BINDSLEV, 1991]. It is assumed that the receiver has a polarization filter which ensures that only scattered light from one mode is accepted. $\mathbf{k}^{\mathrm{s}} = \mathbf{k}^{\mathrm{s}}(\omega^{\mathrm{s}})$ and $\mathbf{e}^{\mathrm{s}}$ are respectively the wave vector and the unit electric field vector associated with this mode in the scattering region.

Let the incident field be a monochromatic beam:

$$\mathbf{E}^{\mathrm{i}}(\mathbf{r}, t) = E''^{\mathrm{i}} \sqrt{\mathcal{I}^{\mathrm{i}}(\mathbf{r})}\, \frac{\mathbf{e}^{\mathrm{i}} e^{i(\mathbf{k}^{\mathrm{i}}\cdot\mathbf{r} - \omega^{\mathrm{i}} t)} + (\mathbf{e}^{\mathrm{i}})^* e^{-i(\mathbf{k}^{\mathrm{i}}\cdot\mathbf{r} - \omega^{\mathrm{i}} t)}}{2} . \tag{122}$$

The total power P^{i} in this beam is

$$P^{\mathrm{i}} = \frac{\varepsilon_0 c}{2} \mathcal{S}^{\mathrm{i}} \left(E''^{\mathrm{i}}\right)^2 \tag{123}$$

The Fourier–Laplace transform of the incident field is

$$\mathbf{E}^{\mathrm{i}}(\mathbf{k}, \omega) = \pi E''^{\mathrm{i}} \left(\mathbf{e}^{\mathrm{i}} \sqrt{\mathcal{I}^{\mathrm{i}}(\mathbf{k} - \mathbf{k}^{\mathrm{i}})}\delta(\omega - \omega^{\mathrm{i}}) + (\mathbf{e}^{\mathrm{i}})^* \sqrt{\mathcal{I}^{\mathrm{i}}(\mathbf{k} + \mathbf{k}^{\mathrm{i}})}\delta(\omega + \omega^{\mathrm{i}})\right) . \tag{124}$$

Replacing $(\mathbf{k}^{\mathrm{i}}, \omega^{\mathrm{i}})$ by $(\mathbf{k}, \omega)$ in expression (117), inserting expression (124) and integrating over ω, gives

$$
\begin{aligned}
\tilde{j}_i^{\sigma-} &= \frac{-i\omega^\sigma \varepsilon_0 E'^{\mathrm{i}}}{2n^{(0)}} \int \frac{d\mathbf{k}}{(2\pi)^3} \sqrt{\mathcal{I}^{\mathrm{i}}(\mathbf{k}-\mathbf{k}^{\mathrm{i}})} \\
&\qquad \left[\chi_{il}^\sigma n^\delta - \chi_{ij}^\sigma \epsilon_{jmk} \frac{i\omega^{\mathrm{i}}\varepsilon_0}{q_e} \chi_{ml}^{\mathrm{i}} B_k^\delta \right. \\
&\qquad + \left(\chi_{ij}^\sigma \epsilon_{jkm} \epsilon_{mnl} \frac{k_n^{\mathrm{i}}}{\omega^{\mathrm{i}}} + X_{ikl}^\sigma \frac{1}{c} \right) \overline{v}_k^\delta \\
&\qquad \left. - \left(\chi_{ik}^\sigma \frac{i\varepsilon_0}{q_e} k_j^{\mathrm{i}} \chi_{jl}^{\mathrm{i}} + X_{ijk}^\sigma \frac{i\omega^{\mathrm{i}}\varepsilon_0}{cq_e} \chi_{jl}^{\mathrm{i}} \right) E_k^\delta \right] e_l^{\mathrm{i}} \, .
\end{aligned} \tag{125}
$$

Here $\omega^\delta = \omega^\sigma - \omega^{\mathrm{i}}$, $\mathbf{k}^\delta = \mathbf{k}^\sigma - \mathbf{k}$ and we have again ignored terms involving fluctuations at the sum frequency $\omega^\sigma + \omega^{\mathrm{i}}$. Expressing the convolution over $\mathbf{k}$ as a Fourier transform of a product and assuming that the beam widths and the integration time T are large relative to the plasma dielectric response length and time respectively [BINDSLEV, 1991] we find

$$
J(\omega^{\mathrm{s}}, t_i) = \frac{-i\omega^{\mathrm{s}}\varepsilon_0}{2n^{(0)}} E'^{\mathrm{i}} \left[\mathcal{N}\widehat{n}^\delta + \mathcal{B}_k \widehat{B}_k^\delta + \mathcal{U}_k \widehat{v}_k^\delta + \mathcal{E}_k \widehat{E}_k^\delta \right] , \tag{126}
$$

where $\widehat{n}^\delta = \widehat{n}^\delta(\omega^{\mathrm{s}}, t_i)$ and

$$
\widehat{n}^\delta(\omega^{\mathrm{s}}, t_i) = \int_{t_i}^{t_i+T} \int_{R^3} \sqrt{\mathcal{I}^{\mathrm{i}}\mathcal{I}^{\mathrm{s}}}\, \tilde{n}^\delta(\mathbf{r}, t)\, e^{-i(\mathbf{k}^{\mathrm{s}}\cdot\mathbf{r} - \omega^{\mathrm{s}} t)}\, d\mathbf{r}\, dt \, , \tag{127}
$$

with similar expressions for $\widehat{\mathbf{B}}^\delta$, $\widehat{\mathbf{v}}^\delta$ and $\widehat{\mathbf{E}}^\delta$. The coefficients $\mathcal{N}$, $\mathcal{B}_k$, $\mathcal{U}_k$ and $\mathcal{E}_k$ are given by

$$
\begin{aligned}
\mathcal{N} &= (e_i^{\mathrm{s}})^* \chi_{il}^{\mathrm{s}} e_l^{\mathrm{i}} \, , & (128) \\
\mathcal{B}_k &= -(e_i^{\mathrm{s}})^* \chi_{ij}^{\mathrm{s}} \epsilon_{jmk} \frac{i\omega^{\mathrm{i}}\varepsilon_0}{q_e} \chi_{ml}^{\mathrm{i}} e_l^{\mathrm{i}} \, , & (129) \\
\mathcal{U}_k &= (e_i^{\mathrm{s}})^* \left(\chi_{ij}^{\mathrm{s}} \epsilon_{jkm} \epsilon_{mnl} \frac{k_n^{\mathrm{i}}}{\omega^{\mathrm{i}}} + X_{ikl}^{\mathrm{s}} \frac{1}{c} \right) e_l^{\mathrm{i}} \, , & (130) \\
\mathcal{E}_k &= -(e_i^{\mathrm{s}})^* \left(\chi_{ik}^{\mathrm{s}} \frac{i\varepsilon_0}{q_e} k_j^{\mathrm{i}} \chi_{jl}^{\mathrm{i}} + X_{ijk}^{\mathrm{s}} \frac{i\omega^{\mathrm{i}}\varepsilon_0}{cq_e} \chi_{jl}^{\mathrm{i}} \right) e_l^{\mathrm{i}} \, . & (131)
\end{aligned}
$$

The complete equation of transfer in a cold plasma can now be written as

$$\frac{\partial P^{\mathrm{s}}}{\partial \omega^{\mathrm{s}}} = P^{\mathrm{i}}\, O_b \left(\lambda_0^{\mathrm{i}}\right)^2 r_e^2\, n^{(0)} \frac{1}{2\pi} \sum_{\alpha\beta} G_{\alpha\beta} S_{\alpha\beta}(\mathbf{k}^\delta, \omega^\delta) \;, \tag{132}$$

where $r_e = q_e^2/4\pi\varepsilon_0 m_e c^2$ is the classical electron radius, $\mathbf{k}^\delta = \mathbf{k}^{\mathrm{s}} - \mathbf{k}^{\mathrm{i}}$, $\omega^\delta = \omega^{\mathrm{s}} - \omega^{\mathrm{i}}$ and O_b is the beam overlap [BINDSLEV, 1991],

$$O_b = \int_{R^3} \mathcal{I}^{\mathrm{i}}(\mathbf{r}) \mathcal{I}^{\mathrm{s}}(\mathbf{r})\, d\mathbf{r} \;. \tag{133}$$

In the summation α and β independently take the values n, B_k, v_k and E_k. The spectral densities $S_{\alpha\beta}(\mathbf{k}^\delta, \omega^\delta)$ are defined as

$$\begin{aligned}
S_{nn}(\mathbf{k}^\delta, \omega^\delta) &= \frac{\left\langle |\widehat{n}^\delta|^2 \right\rangle}{n^{(0)} O_b T} \;, && (134\mathrm{a}) \\
S_{B_k B_{k'}}(\mathbf{k}^\delta, \omega^\delta) &= \frac{\left\langle \widehat{B}_k^\delta \left(\widehat{B}_{k'}^\delta\right)^* \right\rangle}{n^{(0)} O_b T} \;, && (134\mathrm{b}) \\
S_{n B_{k'}}(\mathbf{k}^\delta, \omega^\delta) &= \frac{\left\langle \widehat{n}^\delta \left(\widehat{B}_{k'}^\delta\right)^* \right\rangle}{n^{(0)} O_b T} \;, && (134\mathrm{c}) \\
S_{B_k n}(\mathbf{k}^\delta, \omega^\delta) &= \frac{\left\langle \widehat{B}_{k'}^\delta \left(\widehat{n}^\delta\right)^* \right\rangle}{n^{(0)} O_b T} \;, \quad \text{etc..} && (134\mathrm{d})
\end{aligned}$$

The dielectric form factors are defined as

$$G_{\alpha\beta} = \frac{(\omega^{\mathrm{s}})^4}{\omega_p^4} \frac{\mathcal{C}_{\alpha\beta}}{\mathcal{S}^{\mathrm{i}} \mathcal{S}^{\mathrm{s}}} \tag{135}$$

where

$$\begin{aligned}
\mathcal{C}_{nn} &= \mathcal{N}\mathcal{N}^* \;, && (136\mathrm{a}) \\
\mathcal{C}_{B_k B_{k'}} &= \mathcal{B}_k \mathcal{B}_{k'}^* \;, && (136\mathrm{b}) \\
\mathcal{C}_{n B_{k'}} &= \mathcal{N}\mathcal{B}_{k'}^* \;, && (136\mathrm{c}) \\
\mathcal{C}_{B_k n} &= \mathcal{B}_k \mathcal{N}^* \;, \quad \text{etc..} && (136\mathrm{d})
\end{aligned}$$

We note that those products of dielectric form factors and spectral densities which involve cross terms such as $G_{nB_{k'}}S_{nB_{k'}}$ and $G_{B_{k}n}S_{B_{k}n}$ are generally complex, but occur in pairs which are mutual complex conjugates, ensuring that the result of the summation in (132) is real.

Although the complete equation of transfer for scattering in a cold plasma is extensive we have attempted to present it in a systematic manner which together with the tensorial formulation should facilitate the writing of codes. Previous derivations (e.g. AAMODT and RUSSELL, 1992) have introduced approximations earlier in the derivations. This is undoubtedly acceptable in many situations but the risk remains that some physical phenomenon is lost in the process. With the complete expression given here the significance of any terms that are left out of a computation can more readily be quantified. A code that includes all terms will have the advantage of being valid in the widest possible ranges of parameters. The added computation time is not likely to be a major problem.

In Appendix D of BINDSLEV (1991) we discussed the symmetry of the equation of transfer. We now believe that this discussion has a fault and that the conclusion that the equation of transfer should be symmetrical in incident and scattered field quantities does not hold. A detailed and corrected discussion of the symmetries of the equation of transfer was given in BINDSLEV (1992) Chapter 9 Section 3: the results presented in this paper satisfy these symmetries.

For completeness we give also the expression for the differential scattering cross section,

$$\frac{\partial^3 \Sigma}{\partial \hat{\mathbf{k}}^s \partial \omega^s} = r_e^2 \frac{(N^s)^2}{|\hat{\mathbf{k}}^s \cdot \hat{\mathbf{v}}_g^s|} \frac{1}{2\pi} \sum_{\alpha\beta} G_{\alpha\beta} S_{\alpha\beta}(\mathbf{k}^\delta, \omega^\delta) . \tag{137}$$

Here $\hat{\mathbf{v}}_g = \mathbf{v}_g / |\mathbf{v}_g|$ where $\mathbf{v}_g$ is the group velocity.

Finally it should be pointed out that none of the expressions derived in this section assume that the difference in frequency between incident and scattered radiation is small.

7 SUMMARY

Using a kinetic model of the plasma, expressions for the source current, $\mathbf{j}^\sigma$, have been derived in the low temperature limit for both unmagnetized and magnetized plasmas. Our result for magnetized plasmas confirms the leading terms of the result derived by AAMODT and RUSSELL (1992). Both these results and our result for unmagnetized plasmas are at variance with the traditional fluid results. This disagreement is explained as being due to errors in the traditional fluid approach, resulting from an inappropriate use of the fluid velocity in the non-linear fluid equations and the omission of the pressure term from the second order momentum equation. It has further been demonstrated that the fluid approach can be applied correctly to the problem of three wave mixing and scattering by replacing the fluid velocity by the particle flux and including the pressure term in the second order momentum equation. With these corrections the fluid approach yields the same expressions for the source current as the kinetic approach in the low temperature limit.

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