JOINT EUROPEAN TORUS

JET–P(93)21

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Preprint of a paper to be submitted for publication in
Joumal of Plasma Physics

ABSTRACT

A thermodynamic analysis of tokamak plasmas is developed, starting from a single fluid model. Some general properties of the entropy production in the plasma are discussed.

An approach is employed which is more general than the one based on Onsager symmetry relationships. This choice appears to be justified by experiments. The concept of "thermodynamic stability" is introduced, and a general criterion for the stability of stationary states far from thermodynamic equilibrium is assessed.

An upper limit on the particle density is found to exist as a necessary condition in order to prevent disruptions. Analysis of the entropy balance shows that this limit is modified when neutral beam heating is applied.

Radial temperature profiles are shown to be stable against external heating under specific experimental conditions, in particular when ion and electron temperatures are comparable.

An offset scaling law for the energy confinement time $\tau_E = \tau_E(P_{inp})$ (with P_{inp} total input power) is obtained for stationary states, when stability against additional heating is imposed.

I. INTRODUCTION

Many general features in the behaviour of tokamak plasmas, like the profile consistency /1,2,3,4/ and the Murakami limit /5/ are still object of investigations. A number of works have been published /6,7,8,9,10,11,12,13,14,41,48/, which try to describe the plasma behaviour starting from the properties of the plasma entropy S and of the functionals related with it. The quantity S has been defined in different manners (see e.g. /6,10/) and different points of view have been followed: either maximization of S /10/ or the so-called minimum entropy production rate principle /8,9/.

However, ambiguities can occur when dealing with entropy in non-equilibrium systems, like tokamak plasmas. (In the following, "equilibrium" is used with the meaning of "thermodynamic equilibrium"). An unambiguous definition of entropy is only possible when a complete description of the degrees of freedom of the system is given, through the usual formula

$$S = -k_B \sum_i p_i \log(p_i)$$

where k_B is Boltzmann's constant and p_i is the probability of the i-th microstate of the system, including all degrees of freedom. If such a complete description is not available, the definition itself of entropy depends on the characteristic time scale of the processes to be described: the shorter the time scale, the lower the entropy and the higher its time derivative /15,16,17,18/. Accordingly, no universal dissipation principle, apart from the second law of thermodynamics itself, can be formulated for non-equilibrium physical systems. Further, even if the prescription $\sigma = \sum X_i Y_i$ on the production σ of entropy per unit volume is given, with X_i intensive ("thermodynamic forces") and Y_i extensive quantities ("thermodynamic fluxes"), no uniqueness exists in the choice of X_i and Y_i.

The aim of this paper is to discuss some properties of entropy, which can be relevant to tokamak plasmas. Starting from these properties, stability criteria for stationary states will be proposed in the form of inequalities likely to occur in most thermodynamic relationships. (The "state" of a system described by m variables $q_1(\bar{r},t),\ldots, q_m(\bar{r},t)$ at the time t_0 is the collection of $q_1(\bar{r},t_o),\ldots, q_m(\bar{r},t_o)$ at each $\bar{r}$. If there is no dependence on t, the state is "stationary").

Instabilities can occur if these inequalities are violated thus destroying the initial stationary state, or, at least, radically modifying it. The time and space pattern of an instability will depend on its nature, on the boundary conditions and on the local values of the plasma parameters; in the following, attention will be focussed on the assessment of the stability criteria themselves.

The paper will follow the approach of non-equilibrium thermodynamics of /19,21/ and is organized as follows. In Sec. II the criteria for defining a time scale are discussed. In Sec. III the entropy balance for an "open" physical system is defined and in Sec. IV and V the properties of the contributions of the plasma bulk and boundary to the entropy balance are discussed. The concept of "thermodynamic stability" is introduced. Then, we discuss some stability conditions: against perturbations of the impurity content (Sec. VI), of the temperature profile (Sec. VII), and of the energy balance (Sec. VIII). A simple, one-fluid model (ion temperature T_i = electron temperature $T_e \equiv T$) is adopted. We simplify the analysis by using a cylindrical coordinate system to describe a circular plasma with minor radius a_p, in order to ensure the clarity of the conclusions of physical interest. Conclusions are given in Sec. IX.

II. CHOICE OF THE TIME SCALE

Different characteristic time scales are relevant to the experimental knowledge of the variables n, T, namely the macroscopic moments of the distribution function. In order to study stationary, non-equilibrium plasma states, we consider a time scale τ_s such that $\tau_D < \tau_s < \tau_{E\ (p)}$, where τ_D is the time resolution of the diagnostics system and $\tau_{E\ (p)}$ is the energy (particle) confinement time. The inequalities above compete with $\tau_D < \tau_s < \tau_F$, where τ_F is the fuelling time. Generally speaking, $\tau_F > \tau_{E\ (p)}$. In the special case of pellet fuelling, when $\tau_F \leq \tau_D$, a thermodynamic description of the plasma on the time scale τ_s is not feasible during the ablation period and, as it corresponds to a large (entropy source) change in the particle flow which cannot be translated in an entropy flow, we have to consider the ablation period as discontinuity point in thermodynamic terms. As we expect the plasma to be far from equilibrium already before the pellet injection, no continuity of states is to be expected across the ablation period.

In the following we shall assume a formulation of entropy based on the relationship /22/

$$dG = -SdT + Vdp + \sum_k \mu_k dN_k$$

where G is the Gibbs potential, p is the (total) pressure, N_k and μ_k particle number and chemical potential respectively of the k-th chemical species: the sum reduces to μdN for a one-fluid system (unessential multiplication constants are omitted). This relationship is equivalent to the Gibbs-Duhem relationship /21/.

We stress the point that the contributions of electromagnetic polarization to G are neglected; that is, we neglect plasma diamagnetism ($\beta_t \equiv 2\mu_o n T B^{-2} << 1$ with $\mu_o = 4\pi 10^{-7} T m A^{-1}$) and both wave excitation and absorption (since the contribution of plasma polarization is neglected): this latter condition imposes a new lower boundary on τ_s ; if τ_{RF} is the transient time scale in wave-plasma interaction, then $\tau_{RF} < \tau_s$ is a condition for describing stationary states.

We shall further assume that the hypothesis of local thermodynamical equilibrium (LTE) /19/ is valid: the entropy S in a point $\bar{x}$ of the system at time t (as well as other thermodynamic quantities) is a function of $n(\bar{x},t)$, $T(\bar{x},t)$,..., $S(\bar{x},t) = S(n(\bar{x},t), T(\bar{x},t),...)$ just in the same manner as in thermodynamic equilibrium $S = S(n,T,...)$; but n, T, ... are uniform at equilibrium, while in a plasma they depend on $\bar{x}$ and t. Dependence of S on t is implicit only, through n, T, ... An explicit expression for S in toroidal plasmas is given in /47/. The LTE hypothesis implies that the microscopic mean free path is much shorter than the linear dimensions of the system (problems arising in the banana regime, where this inequality can be violated, are discussed in Sec. V).

Finally, a suitable relaxation mechanism of the actual distribution function to a locally near Maxwelliam distribution function must be provided. Collisions usually perform this task; in strongly collisionless plasmas, the role of electric and magnetic field fluctuations must be considered. If fluctuations are present, the interaction between particles and fluctuating fields can provide a mechanism for approaching LTE on a time scale $\tau_{LTE} \approx \left(\bar{k} \bullet \bar{v}_{th}\right)^{-1}$, where $\bar{k}$ is the fluctuating field wave-vector and $\bar{v}_{th}$is the particle thermal velocity. For $\left|\bar{k}\right|^{-1} \approx r_{LI}$ion Larmor radius (corresponding in JET to broadband fluctuations up to 100 kHz at T_i = 5 keV) we have $\tau_{LTE} < 10^{-6}$ seconds for a stationary magnetic field of 3 Tesla. Then the condition $\tau_{LTE} < \tau_s$ is satisfied and the LTE approximation holds.

III. ENTROPY BALANCE

The general form of the entropy balance is

$$dS = d_e S + d_i S \tag{3.1}$$

where $d_i S$ is the increase of entropy due to irreversible processes within the system, and $d_e S$ is due to energy and matter exchange through the boundary.

In the following, we neglect short-range interactions such as viscosity, chemical and nuclear reactions. The fluid-like nature of the system is maintained by longrange interactions which dominate the plasma behaviour. We assume LTE and $T_i = T_e \equiv T$ at all times.

We show in Appendix A that Eq. 3.1) can be written in the form

$$\frac{d\,(n_m s)}{dt} = -\nabla \cdot \bar{J}_s + \sigma \qquad (3.2)$$

where n_m is the mass density, s the entropy per unit mass, $\bar{J}_s = \bar{J}_q T^{-1} - \bar{J}_p \mu T^{-1}$ the entropy flux, $\bar{J}_q$ the heat flux, $\bar{J}_p$ the particle flux, and

$$\sigma = \sigma_0 + \sigma_\Omega$$

is the entropy production density, where

$$\sigma_o = \bar{J}_q \cdot \nabla T^{-1} - \bar{J}_p \cdot \nabla \mu T^{-1}$$

$$\sigma_\Omega = \frac{\bar{E} \cdot \bar{J}}{T}$$

and $\bar{E}$, $\bar{J}$ are the electric field and current density respectively.

It is shown in Appendix A that $\bar{J}_q$ represents heat sources and losses with the exception of convective and of electromagnetic contributions on a time scale τ_s. Then, both radiofrequency heating and radiation losses due to atomic processes (recombination, ionization) and collisions (Bremsstrahlung) appear in the entropy balance through $\bar{J}_q$. Including short-range interactions leads to new additive terms in σ. In a multi-species theory a contribution to σ appears due to collisional heat exchange between ions and electrons, as T_i differs from T_e. Neutral beam heating supplies heat (contribution to $\bar{J}_q$) and matter; since entropy is an additive quantity, new terms will have to be considered, i.e. contributions to $\bar{J}_p$ and to collisional terms in σ_0).

In the following, we shall investigate some of the properties of the quantities which appear in Eqs. 3.1), 3.2). These properties will be employed to analyse overall plasma stability conditions.

IV. PROPERTIES OF $d_e S/dt$

Let us consider a region Ω of plasma surrounded by two magnetic surfaces, an inner one A1 and an outer one A2. The boundary of Ω is $\cap\,\Omega$. The surface vector outcoming from Ω is parallel to $\bar{e}_r$ on A2, antiparallel to $\bar{e}_r$ on A1, (fig. 1). It is supposed that the safety factor q is > 1 on A1, so that no sawtooth instability occurs within Ω. If $\bar{v} = 0$ then (in the following we denote with w the radial component of a vector $\overline{w}$, $w = \overline{w} \bullet \bar{e}_r$)

$$\frac{d_e S}{dt} = -\int_{\cap\Omega} \bar{J} \cdot d\bar{A} = -\left(J_q A T^{-1}\right)_2 + \left(J_q A T^{-1}\right)_1 + \left(J_p \mu A T^{-1}\right)_2 - \left(J_p \mu A T^{-1}\right)_1 \tag{4.1}$$

Conservation of energy in stationary conditions implies

$$(J_q A)_1 = (J_q A)_2$$

Since $T_1 > T_2$ we have

$$-(J_q A T^{-1})_2 + (J_q A T^{-1})_1 = (J_q A)_1 (T_1^{-1} - T_2^{-1}) \tag{4.2}$$

We define

$$L_T^{-1} \equiv -T^{-1} \frac{dT}{dr}$$

$$L_n^{-1} \equiv -n^{-1} \frac{dn}{dr}$$

$$z \equiv \mu T^{-1}$$

Then $\nabla T^{-1} = L_T^{-1} T^{-1}$ and

$$\sigma_0 = J_q T^{-1} L_T^{-1} - J_p \frac{dz}{dr} \tag{4.3}$$

$$J_p = \left(J_q T^{-1} L_T^{-1} - \sigma_0\right)\left(\frac{dz}{dr}\right)^{-1} \tag{4.4}$$

and

$$\frac{d_e S}{dt} = (J_a A)_1 \left(T_1^{-1} - T_2^{-1}\right) + \left(J_q T^{-1} L_T^{-1} - \sigma_0\right) z A \left(\frac{dz}{dr}\right)^{-1} \Big|_1^2 \tag{4.5}$$

We write $(J_q A)_1 \equiv P_{cd}$, which is proportional to the power lost from the system through non-convective mechanisms (conduction + radiation), and $(J_p T A)_1 \equiv$

P_{cv}, which is proportional to the power lost from the system through convection. It is useful to define

$$J_{eq} = J_q T^{-1} - \frac{3}{2} J_p$$

For a perfect gas /24/ $z = \ln(n_m) - \frac{3}{2}\ln(T) + k$, k = const. Then /25/

$$\sigma_0 = J_q T^{-1} L_T^{-1} - J_p \nabla z = J_{eq} L_T^{-1} + J_p L_n^{-1} = P_{cd} A^{-1} T^{-1} L_T^{-1} + P_{cv} A^{-1} T^{-1}\left(L_n^{-1} - \frac{3}{2} L_T^{-1}\right) \quad (4.6)$$

(for $\nabla(n_m)/n_m = \nabla(n)/n$), and

$$\frac{d_e S}{dt} = P_{cd}\left(T_1^{-1} - T_2^{-1}\right) + \left(\frac{3}{2} L_T^{-1} - L_n^{-1}\right) P_{cv} z \left(T \frac{dz}{dr}\right)^{-1} \Big|_1^2 \quad (4.7)$$

Since $\frac{dz}{dr} = \frac{3}{2} L_T^{-1} - L_n^{-1}$, we write

$$\frac{d_e S}{dt} = P_{cd}\left(T_1^{-1} - T_2^{-1}\right) + P_{cv} z T^{-1} \Big|_1^2 \quad (4.8)$$

Let the radial profiles of particle density and temperature be "pseudo-parabolic" /26,27/

$$n(r) = n_o \left(1 - \frac{r^2}{a_p^2}\right)^{\alpha} + n_{edge} \quad (4.9)$$

$$T(r) = T_o \left(1 - \frac{r^2}{a_p^2}\right)^{\beta} + T_{edge} \quad (4.10)$$

with $n_{edge} \ll n_0$, $T_{edge} \ll T_0$. In stationary conditions both P_{cd} and P_{cv} do not depend on the surface label, i.e. on the surface they are computed on. Then, we can take the surface A1 as reference, with no loss of generality. Let us consider the behaviour of the two terms on the R.H.S. of the expression for $d_e S/dt$ in the limit $r_2 \rightarrow a_p$, where r_2 is the minor radius of the surface A2. Divergent terms are $-P_{cd} T_2^{-1}$ and $P_{cv}(zT^{-1})_2 = P_{cv}(\ln(n_2 T_2^{-\frac{3}{2}}) + k) T_2^{-1}$, where $n_2 \equiv n(r_2)$, $T_2 \equiv T(r_2)$.

The ratio W between these two terms is $W = -P_{cv} P_{cd}^{-1}(k + \ln(n_2 T_2^{-3/2}))$; it behaves like
$-P_{cv} P_{cd}^{-1}(k + \lim \ln(x^{\alpha - \frac{3}{2}\beta}))$ as $x \equiv 1 - \frac{r^2}{a_p^2} \rightarrow 0$ (k includes $\ln(n_0 T_0^{-\frac{3}{2}})$.

Experimentally, particle density profiles are found to be flatter than temperature profiles (Fig. 2). Taking $\alpha < 3\beta/2$, the limit is $\gg 1$ (divergence is removed for non-zero T_{edge}, n_{edge}). Then W «-1, and the absolute value of the P_{cv} term is much larger than the absolute value of the first term; since the first term is < 0, it is clear that d_eS/dt is dominated by a positive, convective term in the external region. Loss of particle through the external boundary makes the plasma entropy to increase. This result is confirmed by the observation that z has a maximum for $r/a_p \approx 1$ in many JET discharges under different experimental conditions (Fig. 3).

Even when dealing with viscous, multi-species plasmas the expression of d_eS/dt remains unchanged. The same holds for chemical and nuclear reactions (their effects enter d_eS/dt through J_q only). Usually, in the plasma bulk $P_{cd} > P_{cv}$, but near the wall $dz/dr \rightarrow 0$. It is clear from these considerations that the entropy balance of tokamak plasmas is actually an "open system" balance.

Further consequences of the expression for $\frac{d_eS}{dt}$ will be investigated below.

V. PROPERTIES OF d_i S/dt

We want to point out some useful properties of $P = \frac{d_iS}{dt}$. If the "internal" entropy production density is written in the form (see Eq. 4.6)

$$\sigma = J_{eq}L_T^{-1} + J_pL_n^{-1} + \bar{E}\cdot\bar{J}T^{-1} = \sum_i X_iY_i \tag{5.1}$$

then a possible definition of thermodynamic forces X_i is (i = 1,2,3)

$$X_1 = L_T^{-1};\ X_2 = L_n^{-1};\ X_3 = \bar{E} \tag{5.2}$$

with corresponding thermodynamic fluxes

$$Y_1 = J_{eq};\ Y_2 = J_p;\ Y_3 = \bar{J}T^{-1} \tag{5.3}$$

This choice is not unique; another choice for X_1, X_2 are $\frac{\nabla T}{T}$, $\frac{\nabla n}{n}$ or $\frac{\nabla p}{p}$, $\frac{\nabla T}{T}$ /14,23,28/ and $X_3 = \bar{E}T^{-1}$, $Y_3 = \bar{J}$ /25/. Relationships between X's and Y's are often supposed to be linear

$$Y_i = \sum_j L_{ij}X_j \tag{5.4}$$

and the invariance under time reversal of the equations of motion for the individual particles in the system can be expressed by Onsager symmetry relationships /29/

$$L_{ij} = \pm L_{ji} \tag{5.5}$$

where the sign depends on the parity of the quantities involved with respect to the magnetic field and the angular velocity. Furthermore, it is often assumed that the coefficients L_{ij} are almost constant and uniform.
Starting from these hypotheses and taking the sign + in Onsager relationships (these assumptions are often referred with the collective name of "linear non-equilibrium thermodynamics", LNET), it has been shown /30,44/ that the value of P is the minimum value compatible with given boundary conditions in stationary non-equilibrium states. If the non-equilibrium state is non stationary (e.g. if some perturbation occurs at a given time), then

$$\frac{dP}{dt} < 0; \tag{5.6}$$

and, in stationary states only,

$$\frac{dP}{dt} = 0. \tag{5.7}$$

This is the so called "minimum entropy production rate principle", quoted above. This principle was first applied to problems of chemical reactions with particle diffusion and later extended /31/ to include heat diffusion.

The minimum entropy production rate principle can provide a stability criterion against perturbations. A small variation dP of P due to a perturbation of a stationary state will relax to zero, provided that dP is positive at all times. If dP < 0 at any time, then its absolute value will increase with time beyond control.

We stress the point that neither the condition S = const. -which has been utilised to describe stationary states /47/ - nor the condition S = maximum /12/ are general conditions of stability against perturbations, since plasma stationary states are far from equilibrium.

The minimum entropy production rate principle has been employed in plasma physics /6,7,8,9 and Refs. therein, 14,48/. However, its validity is limited not only within the domain of LNET, but also by the choice of the time scale. If effect, an interesting counter-example has been found /8/. The minimum entropy production rate principle is violated when acoustic and Alfvén wave effects are included in a linearized resistive MHD treatment of an one-fluid plasma (however, it is then possible to define a suitable "dissipative potential" which goes monotonically to zero in non-stationary states and reaches zero when a stationary state is reached).

Irreversible processes considered in Ref. /8/ are ohmic heating and viscosity; heat and particle transport is neglected. Nevertheless, it is worthwhile to assess the applicability of the minimum entropy production rate principle to real tokamak plasmas. In LNET $\sigma = \sum_i X_i Y_i$ even in the banana regime. (A flux surface average of σ in toroidal, axisymmetric geometry is computed in Ref. /23/, starting from the neoclassical transport theory. It is shown that, even in the banana regime, it is always possible to write the flux-surface-average entropy production density in the form $\sigma = \sum_i X_i Y_i$, provided that L_{ij} is a suitable function of the neoclassical transport coefficients.).

For the symmetric form $L_{ij} X_i Y_j$ to be non-negative (see Appendix A) a necessary condition is that all principal cofactors of $L^S_{ij} \equiv (L_{ij} + L_{ji})/2$ be non-negative. This implies $L_{11} > 0$, $L_{22} > 0$, $(L_{12} + L_{21})^2 - 4L_{11}L_{22} < 0$. If we write /25/

$$J_{eq} = \alpha_T L_T^{-1} + \alpha_n L_n^{-1} \tag{5.8}$$

$$J_p = \beta_T L_T^{-1} + \beta_n L_n^{-1} \tag{5.9}$$

the above conditions give

$$\alpha_T > 0 \tag{5.10}$$

$$\beta_n > 0 \tag{5.11}$$

$$4\alpha_T \beta_n > (\alpha_n + \beta_T)^2 \tag{5.12}$$

and LNET implies

$$\alpha_n = \beta_T \tag{5.13}$$

Introducing $c \equiv \beta_n / \alpha_T$:

$$4c\alpha_T^2 > 4\alpha_n^2;\ c > \left(\frac{\alpha_n}{\alpha_T}\right)^2 \equiv c_{\min} \tag{5.14}$$

Multiplication of J_p by L_n and division by α_n gives

$$L_n \frac{J_p}{\alpha_n} - \frac{L_n}{L_T} = c\frac{\alpha_T}{\alpha_n} = c c_{\min}^{-\frac{1}{2}} > c_{\min}^{\frac{1}{2}} \tag{5.15}$$

and

$$L_n \frac{J_p}{\alpha_n} > \frac{L_n}{L_T} + c_{\min}^{1/2} \tag{5.16}$$

Definition of J_{eq} gives $J_q = TJ_{eq} + (3/2)J_pT$. Neglecting the convective term in the plasma bulk we obtain:

$$J_q \approx TJ_{eq}$$

Be K_{th} defined by $J_q = nTK_{th}L_T^{-1}$. Then, the expressions for J_{eq}, J_q and K_{th} give

$$\alpha_n^{-1} = \frac{1 + c_{\min}^{\frac{1}{2}} L_T L_n^{-1}}{nK_{th} c_{\min}^{\frac{1}{2}}} \tag{5.17}$$

and, together with the inequality for $L_n J_p / \alpha_n$, we obtain the following necessary condition for the validity of LNET

$$\frac{\left(1 + c_{\min}^{\frac{1}{2}} L_T L_n^{-1}\right) L_n J_p}{nK_{th} c_{\min}^{\frac{1}{2}}} > \frac{L_n}{L_T} + c_{\min}^{\frac{1}{2}} \tag{5.18}$$

This inequality may be checked using values of the relevant parameters either measured directly (n, T, L_n, L_r), or through power balance analysis, (K_{th}, $c_{\min}$). The result of such a comparison is shown in Fig. 4, for data obtained on JET for various plasma conditions, to rule out the validity of LNET in many cases /43/.

However, a "general evolution criterion" has been shown to hold in many cases beyond the domain of validity of LNET. If particle transport only is present, if

LTE is valid, and if boundary conditions are stationary, then the following "general evolution criterion" holds /32/

$$\frac{d_x P}{dt} \leq 0 \tag{5.19}$$

with

$$\frac{d_x P}{dt} \equiv \int_v \sum_i J_i \frac{dX_i}{dt} dv$$

when V is the volume of the system and

$$\frac{d_x P}{dt} = 0 \tag{5.20}$$

in stationary states only. The general evolution criterion is a generalization of the minimum entropy production rate principle.

Starting from the general evolution criterion, the following statement has been shown to hold /19/. Let a scalar parameter y exist, such that y corresponds to one and only one stationary state of the physical system. If thermodynamic equilibrium corresponds to y_{eq} let a small $\partial X_i(\bar{r})$ perturb at $t = t_0$ the thermodynamic force X_i of the system in a stationary state y. (Nothing changes if a set of parameters $y_1, \ldots, y_m$ is considered. The following arguments will be separately repeated for each y_i, $i = 1, \ldots, m$). The corresponding perturbation of P will be $\partial P(t_0)$. Then, an interval $- y_1 < y - y_{eq} < + y_1$, $y_1 > 0$ ("thermodynamic branch", TB) exists, such that if y belongs to TB,

1. $\partial P(t) > 0$ for $t > t_0$

2. $\partial P(t)$ is maximum at $t = t_0$, and decreases monotonically $\rightarrow 0$ as $t \rightarrow \infty$, and, consequently:

3. the stationary state has "thermodynamic stability", i.e. the system will relax again to the value of P of the initial stationary state after a suitable relaxation time.

Further, for a given $t_c \geq t_0$, the system is unstable for $t > t_c$ if $\partial P(t_c) < 0$. In this case the system will be driven away from its state at $t = t_c$, and a finite, negative variation of P develops in a finite time interval from an initial infinitesimal perturbation.

Then, a necessary condition for stability for $t > t_0$ is $\partial P(t_0) \geq 0$, and a sufficient condition for stability for $t > t_0$ is $\partial P(t) > 0$ for all $t > t_0$.

An interpretation of the stability condition within the framework of statistical mechanics is given in Ref. /39/.

Since ∂P is built by varying the X_i's only, it is not an exact differential. The non-existence of a general variational principle based on exact differentials and valid for stationary states in LTE, beyond the narrow domain of validity of LNET, has been explicitely shown in /40/. The validity itself of the general evolution criterion depends on the choice of a proper time scale /15/. Then, this validity has to be explicitely assessed for each term in σ.

As far as the heat conduction term in σ is concerned, this proof is given in /21/. The general evolution criterion is therefore valid for $\sigma = \sigma_0$. We shall check its validity for $\sigma = \sigma_0 + \sigma_\Omega$

First of all, we show that the ohmic contribution σ_Ω is not negligible in real tokamak plasmas. We compare $\sigma_\Omega = \bar{E} \bullet \bar{J} / T$ to the conduction term $J_q T^{-1} L_T^{-1}$ in σ_0. If $\bar{E} = \eta \bar{J}, \eta = \eta_o Z_{eff} T^{-\frac{3}{2}}$, where Z_{eff} is the effective charge, then $\sigma_\Omega = \eta_0^{-1} Z_{eff}^{\ 1} |\bar{E}|^2 T^{\frac{1}{2}}$. Conservation of energy in stationary ohmic discharges gives

$$\int_A J_q dA \approx 4\pi^2 r_A R_p J_q = \int_V \bar{E} \bullet \bar{J} dV = \eta_0^{-1} |\bar{E}|^2 \int_V T^{\frac{3}{2}} dV$$

for a volume V of plasma surrounded by a surface A with minor radius r_A and major radius R_p (Spitzer resistivity is assumed). Then

$$J_q \approx \eta_0^{-1} |\bar{E}|^2 r_A^{-1} \int_V^{r_A} r' T^{\frac{3}{2}}(r') dr'$$

and the ratio between the conduction and the ohmic term is

$$F(p) = a_p L_T^{-1} (3\beta + 2)^{-1} p^{-3\beta/2} (1 - p)^{-1/2} (1 - p^{1 + 3\beta/2})$$

where $p \equiv 1 - r_A^2 a_p^{-2}$ and $T(r) = T_0(1 - r^2 a_p^{-2})^\beta$. For realistic values of $\beta (\beta = 1.5)$ at $r_A = .7a_p$, $F = .9a_p L_T^{-1}$. The conclusion from this estimate is that conduction and ohmic contribution play comparable roles in entropy production (in ohmic discharges at least). Introducing neoclassical resistivity does not affect this conclusion significantly.

In order to apply the general evolution criterion, it must be shown that in a tokamak plasma

$$\frac{d_x}{dt}\int_V \frac{\bar{E}\bullet\bar{J}}{T}dV < 0$$

This task is performed in Appendix B.

In the next Sections we shall apply the stability conditions which have been derived above from the general evolution principle. We shall make use of small perturbations of the thermodynamic forces. These perturbations will be suitable, arbitrarily choosen functions of the radial coordinate, in order to obtain useful necessary criteria for thermodynamic stability.

A different approach to the plasma entropy is discussed in Appendix C.

Analogously, the small "displacement vectors" of MHD stability theory are suitable trial functions of the radial coordinate, in order to derive MHD stability criteria, e.g. Suydam's criterion /27/.

VI. STABILITY AGAINST PERTURBATIONS OF THE EFFECTIVE CHARGE

VI.1 Ohmic Case.

We suppose that a small perturbation ∂Z_{eff} of the effective charge Z_{eff} occurs in a stationary ohmic plasma. In a multi-species plasma model this perturbation corresponds to small deformations of density radial profiles of various impurities. For simplicity, we choose a perturbation which does not change both the z profile and the $\bar{J}/T$ profile. For given $\bar{J}(\bar{r})$, $T(\bar{r})$, a perturbation $\partial\bar{E}$ in the thermodynamical force $\bar{E}$ arises from the Z_{eff} dependence of the plasma resistivity. Moreover, there is a perturbation $\partial(dS/dt) = \partial P + \partial(d_eS/dt)$, ∂P being caused by $\partial\bar{E}$ (we keep $\partial L_n^{-1} = 0$, $\partial L_T^{-1} = 0$). Stability requires $\partial P(t) \to 0$ as $t \to \infty$. Then, after a suitable relaxation time, the non-vanishing perturbation of the entropy total derivative is just the boundary contribution

$$\partial \frac{dS}{dt} = \partial \frac{d_eS}{dt} \tag{6.1}$$

We have shown that the inclusion of the most external region of the plasma makes d_eS/dt to be dominated by the convective term: since z is unchanged, we write

$$\partial \frac{d_eS}{dt} \propto \partial \frac{P_{cv}}{T} \propto R_p a_p \partial J_p \tag{6.2}$$

If we write the mass conservation law in an integral form

$$4\pi^2 R_p a_p J_p = 2\pi^2 a_p{}^2 R_p n/\tau_p \tag{6.3}$$

we get

$$J_p = n a_p/(2\tau_p)$$

In the case of constant density $\partial J_p = \frac{1}{2} n a_p \partial(1/\tau_p)$ and

$$\partial \frac{d_eS}{dt} \propto R_p a_p{}^2 n \partial(1/\tau_p) \tag{6.4}$$

Now an expression for $\partial(dS/dt)$ can be written. The heat produced through ohmic dissipation by a current I_p in a plasma with temperature T in a small time interval dt is

$$dQ \propto Z_{eff} T^{-3/2} R_p a_p{}^{-2} I_P^2 \, dt$$

since R_p and a_p^2 are proportional to the length and the cross section of the plasma column respectively. Accordingly, the total entropy change is

$$\frac{dS}{dt} \propto \frac{dQ}{Tdt} \propto Z_{eff} T^{-5/2} R_p a_p{}^{-2} I_p^2.$$

If a perturbation ∂Z_{eff} occurs, the resulting variation in total entropy change is

$$\partial \frac{dS}{dt} \propto T^{-5/2} R_p a_p{}^{-2} I_p^2 \partial Z_{eff} \tag{6.5}$$

(Here the radiative contribution is considered to be much smaller than the ohmic contribution; consequently, we can suppose that for small ∂Z_{eff} the ohmic power is much higher than the power lost by radiation both in the initial state and in the perturbed state. Radiation will be taken into account below). Since $\partial(dS/dt) = \partial(d_eS/dt)$, we can write

$$\frac{\partial \tau_p^{-1}}{\partial Z_{eff}} \propto T^{-5/2} n^{-1} a_p^{-4} I_p^2 \qquad (6.6)$$

Using the Bennett relationship /33/ $I_p^2 \propto n a_p^2 T$, which is qualitatively equivalent to the Grad-Shafranov MHD equilibrium equation for low β_t, cylindrical plasmas, we can write $T^{-5/2} \propto I_p^{-5} n^{5/2} a_p^5$ and (6.7)

$$\frac{\partial \tau_p^{-1}}{\partial Z_{eff}} \propto I_p^{-3} n^{3/2} a_p. \qquad (6.8)$$

For $q_{edge} \equiv a_p B_{tor} R_p^{-1} B_{pol}^{-1}$, $B_{pol} \propto I_p a_p^{-1}$, and

$$a_p^2 \frac{\partial \tau_p}{\partial Z_{eff}} \propto \left(\frac{n R_p q_{edge}}{B_{tor} I_p}\right)^{3/2} \qquad (6.9)$$

Let us suppose that the perturbation ∂Z_{eff} does not lead to a severe disruption. Then, a finite upper limit on the L.H.S. of this proportionality exists. In fact, if a finite upper limit does not exist, τ_p^{-1} goes to ∞ for finite ∂Z_{eff}, the particle diffusion coefficient $D_p \equiv a_p^2 \tau_p^{-1}$ diverges and particles are no more confined. This is a necessary condition for stability of ohmic stationary states against perturbation of Z_{eff}. It can be written as

$$c_1 B_{tor} R_p^{-1} q_{edge}^{-1} > n I_p^{-1} \qquad (6.10)$$

(c_i's are finite positive constant). Given a class of perturbations $\partial Z_{eff}(\bar{r})$, a corresponding necessary criterion for stability may be derived from similar considerations. The existence of such a condition is a matter of thermodynamics; its detailed structure depends on the particular features of the physics of the discharge. A simple example can be considered.

Ohmic heating must supply enough power to sustain radiation losses due to impurities. If we just consider Bremsstrahlung losses due to collisions against ions of atomic number Z_i and density n_i, we can write

$$c_2 Z_{eff} T^{-3/2} I_p^2 R_p a_p^{-2} > c_3 R_p a_p^2 n n_i Z_i^2 T^{1/2}$$

Assuming $nn_i \propto n^2$ and only one $Z_i > 1$ ion impurity species present, then $nn_i = (Z_{eff} - Z_i)\,(Z_i^2 - Z_i)^{-1} n^2$. Bennett relationship gives $f(Z_i, Z_{eff}) > I_p$, where f is a function of Z_i, Z_{eff} only. Substitution in the necessary condition for stability gives

$$n_\Omega \equiv c_4 B_{tor} R_p^{-1} q_{edge}^{-1} > n \qquad (6.11)$$

which is the form of the well-known Murakami limit on particle density for ohmic plasmas /5/.

Even in this extremely simplified model, the Murakami limit is likely to be a necessary condition for stability of stationary states of ohmic plasmas against perturbation of impurity contents. More precisely, if the necessary criterion for stability is fulfilled, convective losses on the boundary (the J_p term in d_eS/dt) can "remain under control" as Z_{eff} varies, under the restriction of having no unbalanced radiative losses. In the opposite case, the confinement of particles is lost, and the initial state of the plasma is destroyed. In this latter case, the entropy balance behaves as follows. P goes from its initial positive value, ($\sigma > 0$, see Appendix A) to zero, since:

1. Radial profiles of particle density and temperature flatten and L_n^{-1}, $L_T^{-1} \rightarrow 0$.

2. Convective loss of plasma energy increases sharply, since $\tau_p \rightarrow 0$.
   Then, $T \rightarrow 0$, $\sigma_\Omega \propto T^{1/2} \rightarrow 0$.

Correspondingly, a finite, negative variation of P occurs, starting from an infinitesimal perturbation (see Sec. VI). The value of the boundary contribution d_eS/dt to the time derivative of the entropy becomes exceedingly high, since $J_p \rightarrow \infty$ as $\tau_p \rightarrow 0$. Then, a sudden, net increase of the entropy time derivative occurs. There is a sharp transition from the ordered state of a confined plasma to an highly disordered final state. Our stability condition is equivalent to requiring that the evolution of the entropy increment exhibits no explosive behaviour.

However, it can be noted that the occurrence of thermodynamic instability in plasmas is not likely to coincide necessarily with hard disruptions. For instance, the minimum absolute value of the quantity P is zero, since $\sigma \geq 0$; and it is reached when $L_n^{-1} \rightarrow 0$, $L_T^{-1} \rightarrow 0$, $\sigma_\Omega \rightarrow 0$. But there are sudden transitions to flat radial profiles of temperature and density in the plasma bulk with no vanishing σ_Ω, as the L-mode to H-mode transitions. Thus, the disruptive nature - if any - of a thermodynamic instability has to be separately verified in each case.

The analysis has been limited to perturbations of dS/dt induced by ∂Z_{eff}, and to the necessary condition for stability against these perturbations. An impurity influx model of fast disruptions of practical interest for tokamak plasmas is given in Ref./45/. Analytical theory and stability conditions of the energy balance of a plasma with marfes and near the density limit - including radiative losses - is given in Ref./46/. According to the models developed in these Refs., a suitable increase of impurity ions leads to disruption.

VI.2 Neutral beam heating case.

We have shown in Sec. III that additional heating enters the entropy balance through J_q. In the particular case of neutral beam injection (NBI), extra mass is injected into the plasma. Consequently, the entropy balance is somehow modified, since entropy is an additive quantity. We did not show explicitely that the general evolution criterion holds for NBI-heated plasmas. However, it is worthwhile to analyse the entropy balance. Let us suppose that an ohmic stationary state with current I_p and total ohmic resistance R_Ω is brought by NBI to a new stationary state with current $I'_p = I_{oh} + I_{cd}$ (I_{oh} is the ohmic current of the final state, I_{cd} is the non-inductive term of the final state, due to NBI). The ohmic resistance of the final state is R'_{oh}. We can keep the applied voltage V_o fixed: $V_o = R_\Omega I_p = R'_\Omega I_{oh}$. Since the temperature T is increased by NBI and the ohmic resistivity is a decreasing function of T, $R_\Omega > R'_\Omega$, and $I_p < I_{oh}$.

The energy E_b of the beam is described by $dE_b/dt = E_b/\tau_{eq}$, with energy equipartition time $\tau_{eq} \propto n^{-1}T^{3/2}$, as beam particle velocity is smaller than the electron thermal velocity. Then, for a given neutral beam source, the neutral beam power absorbed by the plasma is $P_b \propto \tau_{eq}^{-1} \propto nT^{-3/2}$.

The total entropy time derivative becomes

$$\frac{dS}{dt} = \frac{R'_\Omega I^2_{oh} + P_b}{T} \tag{6.12}$$

Let us perturb Z_{eff} in the ohmic and in the NBI case. We want to compute the relative variation $\partial(dS/dt)/(dS/dt)$ of the entropy total time derivative dS/dt due to a variation of Z_{eff} in both ohmic and NBI case. In order to perform this task, we choose a particular ∂Z_{eff}, such that the absolute variation $\partial(dS/dt)$ of dS/dt is the same in both cases, i.e.: $\partial(dS/dt)_{oh} = \partial(dS/dt)_{NBI} = \partial R_{\Omega} I_p^2 T^{-1}$. ($\partial(dS/dt)_{oh}$ and $\partial(dS/dt)_{NBI}$ are two different functionals of the same function $\partial Z_{eff}(\bar{r})$). Then, the relative variation is

$$\frac{\partial \frac{dS}{dt}}{\left.\frac{dS}{dt}\right._{oh}} = \frac{\partial R_{\Omega}}{R_{\Omega}}$$

in the ohmic case, and

$$\frac{\partial \frac{dS}{dt}}{\left.\frac{dS}{dt}\right._{NBI}} = \frac{I_p^2 \partial R_{\Omega}}{R'_{\Omega} I^2_{oh} + P_b}$$

in the NBI case. For $P_b > 0$ the relative variation is lower than in the ohmic case ($R'_{\Omega} I^2_{oh} = V_0 I_{oh} > V_0 I_p = R_{\Omega} I_p^2$). (We could have kept the current fixed; nothing changes, provided that P_b is high enough). If $P_b \gg R'_{\Omega} I^2_{oh}$ the relative variation in the NBI case is obtained from the relative variation in the ohmic case through multiplication of the latter by a factor $w \equiv R_{\Omega} I_p^2 P_b^{-1}$. All quantities but P_b refer to the initial ohmic state. From the Bennett relationship and the expression for P_b we get:

$$w \propto R_{\Omega} a_p^2 T^{5/2} \propto R_{\Omega} a_p^{-3} I_p^5 n^{-5/2}$$

In the initial ohmic state the density is of course lower than the Murakami limit: $n < n_{\Omega} \propto I_p$. Substitution in the expression above gives (apart from a factor $R_{\Omega} a_p^{-3}$)

$$w > c_5 I_p^{5/2} \qquad (6.13)$$

We can conclude that the ohmic value of the relative variation of the total derivative of the entropy for a particular class of perturbations ∂Z_{eff} is multiplied by a factor $w > c_5 I_p^{5/2}$ if NBI is applied.

In our previous discussion of the ohmic case, an upper limit was imposed on $\partial(dS/dt)$ in order to preserve particle confinement. Here, $\partial(dS/dt)$ is the same in the ohmic and the NBI case. Since, for our ∂Z_{eff}, just the relative variation is different in the two cases, we come back to the ohmic problem if we multiply dS/dt by 1/w. But dS/dt is linear in the particle density n; then, our NBI problem is simply equivalent to the ohmic one, provided that n is scaled by a factor 1/w. Intuitively, the smaller the relative variation of dS/dt, the smaller $\partial(dS/dt)$ in comparison to the initial dS/dt, and the weaker the restriction on n, since this restriction is just obtained by imposing an upper limit on dS/dt. Then, the new, NBI-modified Murakami limit will be $n_\Omega w^{-1}$ instead on n_Ω, and

$$n < n_\Omega w^{-1} < c_6 n_\Omega I_p^{-5/2} \tag{6.14}$$

We remark that the above argument does not depend on the actual validity of the general evolution criterion for NBI-heated plasmas, because we investigated the stability condition against a particular perturbation; and, for this perturbation, the stability problem reduces to the ohmic problem.

If NBI increases density at a given rate, the maximum final density will be reached for maximum initial density, that is $n = n_\Omega$. Then $P_b \propto nT^{-3/2} = n_\Omega T^{-3/2}$, and $n_\Omega \propto P_b T^{3/2}$.

Finally, in order to avoid runaway, the streaming parameter $\Gamma \equiv I_p/(nev_{th-e})$, e electronic charge, v_{th-e} electron thermal speed, must be lower than some maximum value Γ_{max}; since $\Gamma \propto I_p/(nT^{1/2})$, the inequality $I_p^{-1} > c_7 n^{-1} T^{-1/2}$ holds. This inequality and Bennett relationship give $I_p > c_8 T^{1/2}$ and consequently $T^{3/2} < c_9 I_p^3$. Then, the above relationship for n_Ω may be written in the form $n_\Omega < c_{13} P_b I_p^3$. Substitution in the expression of the NBI-modified density limit gives

$$n < c_{10} P_b I_p^{1/2} \tag{6.15}$$

This condition is compatible with experiments /34,35/.

VII. STABILITY AGAINST PERTURBATIONS OF THE RADIAL TEMPERATURE PROFILE

Let us consider a region of plasma where the convective and the radiative losses are negligible in comparison with the conduction losses. A good example of such a region in L-mode discharges of tokamaks is the "confinement zone", i.e. the intermediate zone lying between the central sawtooth zone and the external region where atomic processes are relevant.

Entropy production density is $\sigma = \sigma_0 + \sigma_\Omega$, where

$\sigma_0 = P_{cd}A^{-1}T^{-1}L_T^{-1} + P_{cv}A^{-1}T^{-1}(L_n^{-1} - \frac{3}{2}L_T^{-1}) \approx P_{cd}A^{-1}T^{-1}L_T^{-1} = K'_{th}L_T^{-2}$ where
$K'_{th} \equiv nK_{th} = K'_{th}(n,T,B,\nabla T,\nabla n, ...)$ and
$\sigma_\Omega = \eta_0^{-1} \mid \bar{E} \mid^2 T^{1/2}$.

Then

$$P = \int_V \sigma \, dV = \int_V K'_{th}L_T^{-2} \, dV + \int_V \eta_0^{-1} \mid \bar{E} \mid^2 T^{1/2} dV \qquad (7.1)$$

Since $dV = 4\pi^2 Rrdr$ and $d(\eta_0^{-1} \mid \bar{E} \mid^2)/dr \approx 0$ as $a_p/R_p \ll 1$ it is useful to define $P' \equiv P/(4\pi^2 R_p)$; then

$$P' = \int K'_{th}L_T^{-2} rdr + \eta_0^{-1} \mid \bar{E} \mid^2 \int T^{1/2} rdr \equiv F + G \qquad (7.2)$$

where F and G are the first and second term respectively.

As usual, we take pseudo-parabolic profiles for n(r), T(r) for stationary states in L-mode discharges. For this profile $L_T^{-1} = 2\beta r(1 - r^2 a_p^{-2})^{-1} a_p^{-2}$.

Let us perturb the temperature profile of an initial stationary state:

$\beta \to \beta + \partial\beta$.

Here we neglect perturbations of the particle density profile, and assume T(r), n(r) to be pseudo-parabolic at all times. Then, a perturbation of the thermodynamic force

$L_T^{-1} \to L_T^{-1} + \partial L_T^{-1}$

occurs, where

$\partial L_T^{-1} = 2r(1 - r^2 a_p^{-2})^{-1} a_p^{-2} \partial\beta$

for small $d(\partial\beta)/dr$. We write the corresponding perturbation of P' (i.e., of P)

$\partial P' = \partial F + \partial G$

If we write

$r = a_p^2(1 - r^2 a_p^{-2}) L_T^{-1}(2\beta)^{-1}$

it is straightforward to show that $G \rightarrow G + \partial G$, $\partial G = (\partial\beta/\beta)G$. Developing to first order

$$K'_{th} \rightarrow K'_{th} + \frac{\partial K'_{th}}{\partial L_T^{-1}} \partial L_T^{-1}$$

and, in the variable L_T^{-1}, with $r \propto L_T^{-1}$

$$r L_T^{-2} \rightarrow r L_T^{-2} + \frac{3}{2} a_p^2 \beta^{-1}(1 - r^2 a_p^{-2}) L_T^{-2}\, \partial L_T^{-1}$$

we obtain, as $\partial L_T^{-1} / L_T^{-1} = \partial\beta/\beta$,

$$\partial F = \left(\frac{\partial\beta}{\beta}\right)\int K'_{th} L_T^{-2}\left(3 + L_T^{-1} \frac{\partial\, 1n(K'_{th})}{\partial L_T^{-1}}\right) r\, dr$$

(Integrals have been computed for $0 < r < r_p$ for the sake of simplicity).

The second addendum on the R.H.S. can often be neglected, at least when $K'_{th} > 0$. Then

$$\partial P = 4\pi^2 R_p (3F + G)\beta^{-1}\partial\beta \qquad (7.3)$$

Stability of the initial state against the perturbation ∂L_T^{-1} depends on the sign of ∂P. Additional heating modifies both the peak temperature and the radial temperature profile, i.e. both β and the total energy U

$$U \equiv \int \frac{3}{2} n(r) T(r) dV = c_{11} T_0 (\alpha + \beta + 1)^{-1}$$

Let us investigate the sign of $\partial\beta$ in the case where we apply additional heating to an initial stationary state, at constant density and applied voltage V_0. We consider a simultaneous perturbation

$\beta \rightarrow \beta + \partial\beta$

$U \rightarrow U + \partial U$

with $\partial U > 0$ at all time during the application of additional heating. The corresponding perturbation in central temperature T_0 due to additional heating is

$$\partial T_0 = c_{11}^{-1}(U\partial\beta + (\alpha + \beta + 1)\partial U)$$

The ohmic power is

$$P_\Omega \equiv \int \bar{E} \bullet \bar{J}\, dV = c_{12} T_0^{3/2}(3\beta + 2)^{-1}$$

and another expression may be derived for ∂T_0

$$\partial T_0 = c_{12}^{-2/3}(2P_\Omega^{2/3}(3\beta + 2)^{-1/3}\partial\beta + \frac{2}{3} P_\Omega^{-1/3}(3\beta + 2)^{2/3}\partial P_\Omega)$$

By equating these two expressions for ∂T_0 and dividing by T_0 we get

$$\partial\beta((\alpha + \beta + 1)^{-1} - 2(3\beta + 2)^{-1}) = \frac{2}{3}(\frac{\partial P_\Omega}{P_\Omega}) - \frac{\partial U}{U} \qquad (7.4)$$

Since $V_0 = const.$, $P_\Omega = V_0 I_p$ and $\partial P_\Omega / P_\Omega = \partial I_p / I_p$. The R.H.S. of Eq. 7.4) equals $\partial(\ln(I_p^{2/3} U^{-1}))$. For $\beta_p = 1$, $U \propto I_p^2$ and $\partial(\ln(I_p^{2/3} U^{-1})) = \partial(\ln(U^{-2/3})) < 0$ for $\partial U > 0$. Then $\partial\beta > 0$ for $\beta < 2\alpha$. If the additional heating raises the poloidal beta, I_p increases more weakly with U, and our argument is even more valid. Since $\partial U > 0$, $\partial\beta > 0$ also. As ∂P is linear in $\partial\beta$ and has the same sign, then $\partial P(t) > 0$ during additional heating.

Additional heating produces, through $\partial\beta$, a perturbation of the thermodynamical force L_T^{-1}, which makes a positive perturbation ∂P of P. Then, as a consequence of the general evolution criterion (Sec. V), the perturbation ∂P monotonically decreases from its initial (maximum) value to zero after a suitable relaxation time. As usual, no information can be obtained about this relaxation time from our purely thermodynamic treatment; it depends on the details of the heat transport mechanism.

We have shown that $\partial\beta$ is proportional to ∂P through a positive constant at all times. Then, since $\partial P \rightarrow 0$, $\partial\beta \rightarrow 0$ and additional heating leaves temperature profiles unchanged, provided that the applied voltage is kept fixed. This result is relevant to the so-called "profile consistency" /1,2,3,4/. Our demonstration is subject to a number of restrictive hypotheses.

Pseudo-parabolic profiles for n(r), T(r) are assumed.

All integrals are computed from 0 to a_p; that is,

the radiation-dominated zone be small

$r_{q=1} \ll a_p$, $r_{q=1}$ the q=1 radius.

However, if an uniform dT/dr is assumed in the confinement zone /36/, the relationship $\partial P \propto \partial |\nabla T|$ is straightforward, and the validity of the above result in this zone follows immediately (ohmic dissipation increases with $|\nabla T|$ for fixed voltage).

We made use of a single fluid model, that is, we supposed $T_i = T_e$. This is unrealistic when describing additional heating, unless high density plasmas are considered. A complete description of the entropy evolution in a plasma with additional heating requires a multi-species model, which separates entropy production of ions and electrons and takes into account the entropy term linked with collisional heat exchange between particles of different species.

However, if the particle density is low the collisional coupling is weak; if ohmic-heated electrons are initially hotter than ions, our argument can be repeated, provided that ion and electron contributions to P are written separately. The ion temperature gradient is smaller than the electron one in the initial state. Then, ions contribute to the total P of the plasma much less than electrons do. In this case, we expect that the ion temperature profile will change with additional power, and that the electron temperature will not. This is true, unless the ion contribution to J_q is exceedingly larger than the electron one.

If a population of fast ions is present, a corresponding new term appears in P, and our argument cannot be applied straightforwardly.

For fixed $\partial U/U$, $\partial\beta$ is an increasing function of β. Assuming pseudo-parabolic profiles and Spitzer resistivity, β is an increasing function of q_{edge} for fixed safety factor on the axis $q(r = 0)$ (see also /3/ and, for Gaussian profiles, /4/). Then, for fixed $\partial U/U$ at a given time the perturbation of the temperature profile increases with q_{edge}. Then, we expect that profile consistency is relaxed at high q_{edge} (see /37/).

We remark that we have dealt with profiles of temperature T, not of current density j. However, ohmic and conduction terms in σ are comparable (Sec. V), and we could just repeat our arguments using j instead of T, as the phaenomenological relationship j = j(T) is monotonic in the confinement zone with dj/dT > 0, and T_e replacing T in the multifluid case.

Another interesting consequence results from Eq. 7.4). If any physical process subtracts energy from the plasma ($\partial U < 0$) and the $\partial\beta$ due to this process is non-zero, then $\partial P < 0$ and thermodynamic instability occurs. The end of the discharge is an example of this evolution. In fact, a particular case is the external voltage pre-programming at the end of the discharge (for $\beta = \beta(q_{edge})$) (the switching-off of additional heating is not: if heat transport leaves β unchanged during additional heating, the same occurs when additional heating is switched off). Within the limits of this model, we can say that a singularity in $\partial(1/\tau_p)/\partial Z_{eff}$ does not occur, if the stability condition on density of Sec. VI is satisfied at all times.

VIII STABILITY AGAINST PERTURBATION OF THE ENERGY BALANCE

The simplest way to write the energy balance of a plasma is

$$\frac{dU}{dt} = - P_{loss} + P_{inp} \tag{8.1}$$

where U is the total plasma energy, and P_{inp}, P_{loss} represent power input and power losses respectively. We suppose that conduction is the dominant loss process $P_{loss} = P_{cd}$. The energy confinement time is therefore $\tau_E \equiv U/P_{loss} = U/P_{cd}$. In stationary conditions $P_{loss} = P_{inp}$. Let the particle density profile n(r) and the temperature profile $T(r)/T_0$ be fixed. Then $U \propto T$. We apply a small perturbation to the energy balance, i.e. $P_{inp} \rightarrow P_{inp} + \partial P_{inp}$. As ∂P_{inp} perturbs the thermodynamic force L_T^{-1}, stability requires that the associated perturbation of the entropy production $\partial P = 4\pi^2 R_p \partial(F + G)$ goes to zero after a suitable time. We can write $\partial F \propto \partial(P_{loss}/T) \propto \partial(P_{loss}/U) \propto \partial(1/\tau_E)$. In the unperturbed state $P_{inp} = P_\Omega$

+ P_{aux}; leaving the ohmic power P_{Ω} unchanged and perturbing P_{aux} only (i.e., the perturbation in P_{inp} at the onset is due to additional heating only), we can write $G \propto P_{\Omega}/T$ and $\partial G \propto \partial(P_{\Omega}/T) \propto \partial(1/T) \propto \partial(1/U)$. The condition $\partial P \rightarrow 0$ gives

$$-\partial(1/\tau_E) = a\partial(1/U) \quad (8.2)$$

with a dimensional, positive constant. Integration gives

$$g - \tau_E^{-1} = aU^{-1}$$

(g integration constant), which can be rewritten

$$\tau_E = q(1 - hU^{-1})^{-1}$$

(here $q \equiv 1/g, h \equiv aq$). Since $U = P_{inp}\tau_E$ in stationary conditions, we get

$$\tau_E = q + hP_{inp}^{-1} \quad (8.3)$$

as $\tau_E > 0, g > 0$. The quantity h depends on various plasma parameters through a and is arbitrary. Eq. 8.3) describes therefore a whole family of possible functions $\tau_E(P_{inp})$.

The thermodynamic stability discussed in Sec. V against perturbations of the energy balance due to additional heating requires that $\tau_E(P_{inp})$ belong to this family, at least for fixed n(r) and T(r) with conduction as the dominant loss process. Rebut-Lallia scaling law for τ_E /38/ satisfies this condition.

Straightforward generalization to non-stationary states within the time scale τ_s requires the substitution $P_{loss} = P_{inp} \rightarrow P_{loss} = P_{inp} - dU/dt$, and $\tau_E = q + h/(P_{inp} - dU/dt)$.

Since we cannot apply this theory during a pellet ablation period (Sec. II), we would conclude that the Rebut-Lallia confinement law does not fit well pellet-fuelled discharges. This statement is not in contrast with experiments /49/.

IX CONCLUSIONS

We have discussed the features of the production of entropy in a low beta, large aspect ratio tokamak plasma.

A single fluid model has been utilized, with $T_i = T_e$. Ambiguities in definition of entropy have been overcome through suitable choice of a macroscopic time scale. Pseudo-parabolic radial profiles for particle density n, temperature T and current density j /26/ have been assumed.

The validity of the approach of linear non-equilibrium thermodynamics, based on Onsager symmetry relationships, is shown to be questionable in real plasmas.

A general evolution criterion /21/ has been assessed for the time evolution of the entropy production is low beta tokamak plasmas.

The behaviour of stationary ($\partial/\partial t = 0$) states of the plasma is investigated, when a small perturbation of any thermodynamical force occurs. Either relaxation to the level of entropy production in the plasma bulk of the initial state or uncontrolled evolution of this production occurs. If the former case occurs, the initial stationary state is said to have "thermodynamic stability".

A set of stable stationary states actually exists ("thermodynamic branch"). Plasmas near to thermodynamic equilibrium belong to the thermodynamic branch; nevertheless, no linearization of phaenomenological relationships is used in the model.

The actual behaviour of the entropy production in the plasma bulk depends on the macroscopic parameters (n, T, magnetic field B), on their gradients, and on the particular thermodynamic force which is actually perturbed. Various necessary conditions for thermodynamic stability are found.

Stability against small changes of the effective charge Z_{eff} in ohmic plasmas implies that an upper density limit exists:

$$n < c_4 B_{tor} R_p{}^{-1} q_{edge}^{-1}$$

with B_{tor} toroidal magnetic field, R_p major plasma radius, q_{edge} boundary safety factor, c_4 positive constant. This result is similar to the well-known Murakami limit /5/.

Here and in the following formulas, the actual value of multiplicative dimensional constants (c_4, c_{10}, ...) cannot be computed starting from a purely thermodynamic approach; the existence of stability conditions is a matter of thermodynamics, their detailed structure depends on the particular features of the physics of the discharge.

Entropy balance is strongly affected by neutral beam heating: when the latter is applied, the upper limit on density becomes:

$$n < c_{10} P_b I_p^{1/2}$$

with P_b neutral beam heating power and I_p plasma current. The constraint on density is weakened.

It is shown that, if these conditions are violated, the particle confinement time goes to zero and confinement is lost.

Thermodynamic stability against perturbation of radial temperature profile requires that additional heating does not change the slope of T(r) in the "confinement zone", i.e. the plasma region where conduction is the dominant loss process. This conclusion is valid only if a number of experimental conditions are verified; for practical purposes, the most stringent ones are $T_i = T_e$ and the absence of a fast particle population. Invariance of T(r) is relaxed at high q_{edge}. Qualitative discussion of the multi-species problem seems to show that the invariance property is more likely to be a property of electron rather than of ion temperature profiles. Generally speaking, these arguments remain valid if the current density is used instead of T.

Thermodynamic stability against perturbations of the energy balance requires that the energy confinement time τ_E is written in the form $\tau_E = a + bP_{inp}^{-1}$ with a, b positive constant, U total internal energy, P_{inp} total input power, leading to $P_{inp} \rightarrow P_{inp} - dU/dt$ for non-stationary discharges. Conduction is considered as the dominant loss process. The Rebut-Lallia scaling law for τ_E /38/ satisfies this condition.

We have tried to employ concepts and methods of thermodynamics in the description of tokamak plasmas, through an oversimplified model. Further developments are required for a multi-species treatment. Possible occurrence of non-disruptive thermodynamic instabilities (like L-H mode transition) may be investigated.

A number of necessary conditions for stability of stationary states has been obtained. As usual with thermodynamics, the existence itself of such a condition imposes a constraint on any self-consistent microscopic theory. Thus the usefulness of a rigorous - even if rather abstract - investigation of plasma entropy properties is emphasized.

ACKNOWLEDGEMENTS

Dr. L.C. Woods' invaluable suggestions and comments helped us to clarify the discussion of general properties of entropy. One of us (M.B.) acknowledges useful discussions with Prof. J.D. Callen. One of us (A.D.V.) would like to thank Dr. F. Rosatelli of Ansaldo Ricerche for his useful comments.

APPENDIX A. ALTERNATIVE FORMS OF THE ENTROPY BALANCE

We consider in this Appendix alternative forms of the entropy balance to Eq. 3.1) under the same physical hypotheses as listed in Sec. III of the text.

The second law of thermodynamics implies $d_i S \geq 0$. If the irreversible processes within a small volume dV increase the entropy by $\sigma dVdt$ in a time interval dt, then integration over the volume V gives

$$P \equiv \frac{d_i S}{dt} = \int_V \sigma \, dV \qquad (A.I.1)$$

and, by the second law of thermodynamics,

$$\sigma > 0. \qquad (A.I.2)$$

Let $\bar{J}_{Stot}$ be the "entropy flux density" through the boundary $\cap V$ of the volume V with infinitesimal surface $d\bar{A}$, i.e. $\bar{J}_{Stot} \bullet d\bar{A}dt$ is the entropy crossing the infinitesimal surface in a time interval dt. The entropy flux density is positive when pointing outwards and is associated to both energy and particle transport through the boundary. Then

$$\frac{d_e S}{dt} = -\int_{\cap V} \bar{J}_{Stot} \bullet d\bar{A} \qquad (A.I.3)$$

Let

$$S = \int_V n_m s \, dV$$

with n_m mass density and s entropy per unit mass. Then, the entropy balance (eq. 3.1)) in a fixed small volume dV inside the system is

$$\frac{\partial n_m s}{\partial t} = -\nabla \bullet \bar{J}_{Stot} + \sigma \qquad (A.I.4)$$

where $\frac{\partial \bullet}{\partial t}$ is the usual Eulerian time derivative. Eq. A.I.4) has the structure of a continuity equation and represents the local form of the entropy balance 3.1). Then, it is valid for both compressible and non-compressible fluids. It is useful to write it in a somewhat different form.

The lagrangian time derivative is: $\frac{d(\bullet)}{dt} = \frac{\partial(\bullet)}{\partial t} + \bar{v} \bullet \nabla (\bullet)$ where $\bar{v}$ is the macroscopic velocity. Mass conservation is given by:

$$\frac{\partial n_m}{\partial t} = - \nabla \bullet (n_m \bar{v}) \qquad (A.I.5)$$

Then, a third form of the entropy balance is:

$$n_m \frac{ds}{dt} = - \nabla \bullet \bar{J}_s + \sigma \qquad (A.I.6)$$

with

$$\bar{J}_s = \bar{J}_{Stot} - n_m s \bar{v}. \qquad (A.I.7)$$

Another version of the entropy balance can be considered. Neglecting electromagnetic fields $\bar{E}$ and $\bar{B}$ on the time scale τ_s, energy conservation implies /21/

$$\frac{\partial n_m e}{\partial t} = - \nabla \bullet \bar{J}_e \qquad (A.I.8)$$

where $e = \frac{v^2}{2} + u$ is the total energy per unit mass, the total energy flux is $\bar{J}_e = n_m e \bar{v} + \bar{J}_q$ and $\bar{J}_q$ is the heat flux, defined as $n_m \frac{dq}{dt} = - \nabla \bullet \bar{J}_q$. We define the vector $\bar{J}_p$ such that $\frac{dn}{dt} = - \nabla \bullet \bar{J}_p$.

Thermodynamics supplies an expression for $\frac{ds}{dt}$. Differentiation of Gibbs potential /21,22/ $G = U - TS + pV$ (U internal energy) together with the relationship $dG = -SdT + Vdp + \mu dN$ gives: $TdS = dU + pdV - \mu dN$. We apply these equations to the small volume $V_u \equiv n_m^{-1}$, i.e. the unit mass volume. Then $dN = V_u dn$, n particle density. In the following s, u, e, dq are entropy, internal energy, total energy (kinetic + internal) and non-exact heat differential respectively in the volume V_u, i.e. per mass unit. In this volume $ds = \frac{du}{T} + \frac{p}{T} d\,(n_m^{-1}) - \frac{\mu}{T} \frac{dn}{n_m}$ and the first principle of thermodynamics $dU = dQ - pdV$ (dQ non-exact heat differential) gives $du = dq - pd\,(n_m^{-1})$ so that $ds = \frac{dq}{T} - \frac{\mu dn}{n_m T}$. Differentiation upon time and multiplication by n_m gives $n_m \frac{ds}{dt} = \frac{n_m}{T} \frac{dq}{dt} - \frac{\mu}{T} \frac{dn}{dt}$.

Then we can write

$$n_m \frac{ds}{dt} = -\frac{1}{T}\nabla \bullet \bar{J}_q + \frac{\mu}{T}\nabla \bullet \bar{J}_p \qquad (A.I.9)$$

Ambiguities arising from defining $\bar{J}_q$ and $\bar{J}_p$ through their divergences are removed assuming suitable boundary conditions and a privileged radial direction ($\nabla(\bullet) \rightarrow \bar{e}_r \bullet \frac{\partial(\bullet)}{\partial r}$ with $\bar{v} \perp \bar{e}_r$). This is a natural choice in toroidal axisymmetric plasmas, where /23/ only surface-averaged fluxes of particles and heat are measurable along the poloidal flux gradient.

The expression for $n_m \frac{ds}{dt}$ may be easily rewritten as follows

$$n_m \frac{ds}{dt} = -\nabla \bullet \bar{J}_s + \sigma \qquad (A.I.10)$$

where:

$$\sigma \equiv \sigma_0 = \bar{J}_q \bullet \nabla T^{-1} - \bar{J}_p \bullet \nabla \mu T^{-1}$$

$$\bar{J}_s = \bar{J}_q T^{-1} - \bar{J}_p \mu T^{-1}$$

The mass conservation law gives $n_m \frac{dV_u}{dt} = \nabla \bullet \bar{v}$. Then, for $\bar{v} \perp \bar{e}_r$, $\nabla \bullet \bar{v} = 0$ and $\frac{dn_m}{dt} = 0$ and we obtain

$$n_m \frac{ds}{dt} = \frac{d(n_m s)}{dt} \qquad (A.I.11)$$

and $-\nabla \bullet \bar{J}_q = n_m \frac{dq}{dt} = \frac{d(n_m q)}{dt}$ so that the definitions of $\bar{J}_p$ and $\bar{J}_q$ have a similar structure.

Electromagnetic fields $\bar{E}$, $\bar{B}$ acting on the time scale τ_s modify $\bar{J}_q$ and σ as follows /21/:

$$\frac{\partial\left(n_m e + \frac{1}{2}\varepsilon_0 \left|\bar{E}\right|^2 + \frac{1}{2}\mu_0 \left|\bar{B}\right|^2\right)}{\partial t} = -\nabla \bullet \bar{J}_e \qquad (A.I.12)$$

where

$$\bar{J}_q = \bar{J}_e - n_m e \bar{v} - \frac{\left(\bar{E} \wedge \bar{B}\right)}{\mu_0}$$

$$\sigma = \sigma_0 + \sigma_\Omega \qquad (A.I.13)$$

$$\sigma_\Omega = \frac{\bar{E} \bullet \bar{J}}{T} \qquad (A.I.14)$$

$\varepsilon_0 = 8.85x10^{-12}$ in MKSA units and $\bar{J}$ is the current density ($\bar{J} \perp \bar{e}_r$). Both σ_0 and $\bar{J}_s$ are unchanged.

Eqs. A.I.10), A.I.11), A.I.13) give Eq. 3.2).

The heat production density deduced from the expression for σ corresponds to the quantity H in ref /14/.

APPENDIX B. A MATHEMATICAL PROOF

We have to show that

$$\frac{d_x}{dt}\int_v \frac{\bar{E}\bullet\bar{J}}{T}dV < 0 \qquad (A.II.1)$$

We assume a low beta, high aspect ratio, axysimmetric MHD equilibrium of the unperturbed initial state with minor plasma radius $a_p \ll$ major radius R_p. The thermodynamic flux is $\bar{J}/T$, and its corresponding thermodynamic force is $\bar{E}$. The volume of the plasma is V and its external boundary surface is S.

Through Maxwell's equations we may write

$$\frac{d_x}{dt}\int_v \frac{\bar{E}\bullet\bar{J}}{T}dV = \int_v \frac{\partial\bar{E}}{\partial T}\bullet\frac{\bar{J}}{T}dV = c^2\int_v \nabla\wedge\bar{B}\bullet\frac{\bar{J}}{T}dV - \int_v \frac{|\bar{J}|^2}{\varepsilon_0 T}dV \qquad (A.II.2)$$

where ε_0 is the vacuum permittivity and c the speed of light in vacuum. The term in $|\bar{J}|^2$ is always negative. The term in $\nabla\wedge\bar{B}$ may be developed as follows

$$\int_v \nabla\wedge\bar{B}\bullet\frac{\bar{J}}{T}dV = \int_v \nabla\bullet\left(\frac{\bar{B}\wedge\bar{J}}{T}\right)dV + \int_v \bar{B}\bullet\nabla\wedge\left(\frac{\bar{J}}{T}\right) \qquad (A.II.3)$$

The first integral may be reduced to the surface integral of ∇p on the external surface S (p plasma pressure), since the MHD equilibrium equation $\bar{J}\wedge\bar{B} = \nabla p$ holds. For $\bar{J} = 0$ and/or $\nabla p = 0$ on the external boundary, this term is zero. The argument of the second integral may be developed as follows

$$\bar{B}\bullet\nabla\wedge\left(\frac{\bar{J}}{T}\right) = \bar{B}\bullet\nabla\frac{1}{T}\wedge\bar{J} + \frac{\bar{B}}{T}\bullet\nabla\wedge\bar{J} \qquad (A.II.4)$$

Then

$$\bar{B}\bullet\nabla\frac{1}{T}\wedge\bar{J} = -\frac{1}{T^2}\bar{B}\bullet\nabla T\wedge\bar{J} = -\frac{1}{T^2}\nabla T\bullet\nabla p \qquad (A.II.5)$$

Since $T > 0$, $\nabla T \bullet \nabla p > 0$, the contribution of the first addendum on the R.H.S. of Eq. A.II.4) to the integral in Eq. A.II.3) is negative.

The only contribution which can be positive is Y:

$$Y \equiv \int_v \frac{\bar{B}}{T}\bullet\nabla\wedge\bar{J}\,dV \qquad (A.II.6)$$

From the MHD equilibrium equation

$$\bar{J} = h(\bar{r})\bar{B} + \left(\bar{B} \wedge \nabla p\right)\left|\bar{B}\right|^{-2} \tag{A.II.7}$$

with h unknown function of $\bar{r}$.

From Ampere's law

$$\nabla \wedge \bar{J} = \mu_0 h\bar{J} + \nabla h \wedge \bar{B} + \nabla \wedge \left(\left(\bar{B} \wedge \nabla p\right)\left|\bar{B}\right|^{-2}\right) \tag{A.II.8}$$

($\mu_0 \equiv c^{-2}\varepsilon_0$), with

$$\nabla \wedge \left(\left(\bar{B} \wedge \nabla p\right)\left|\bar{B}\right|^{-2}\right) = \bar{B}\left(\nabla \bullet \left(\left|\bar{B}\right|^{-2} \nabla p\right)\right) + \left(\left|\bar{B}\right|^{-2} \nabla p \bullet \nabla\right)\bar{B} - \left(\bar{B} \bullet \nabla\right)\left(\left|\bar{B}\right|^{-2} \nabla p\right) \tag{A.II.9}$$

since $\nabla \bullet \bar{B} = 0$ and, in Ampere's law, $|\nabla \wedge \bar{B}|$, $|\mu_0 \bar{J}| \gg |\partial \bar{E}/\partial t|$ for small time-dependent perturbations. (The role of the $\partial \bar{E}/\partial$t contribution to Eq. A.II.8) will be discussed below). Then, we can write

$$\bar{B} \bullet \nabla \wedge \bar{J} = \mu_0 \left|\bar{J}_{\parallel}\right|^2 + \left|\bar{B}\right|^2 \nabla \bullet \left(\left|\bar{B}\right|^{-2} \nabla p\right) + \left|\bar{B}\right|^{-2} \bar{B} \bullet (\nabla p \bullet \nabla)\bar{B} - \bar{B} \bullet \left(\bar{B} \bullet \nabla\right)\left(\left|\bar{B}\right|^{-2} \nabla p\right) \tag{A.II.10}$$

where $\bar{J}_{//} \equiv h\bar{B}$.

The integral in $|\bar{J}_{//}|^2$ and the integral in $|\bar{J}|^2$ add in Eq. A.II.2): a negative contribution in $|\bar{J} - \bar{J}_{//}|^2$ remains.

The second term on R.H.S. of Eq. A.II.10) gives

$$|\bar{B}|^2 \nabla \bullet (|\bar{B}|^{-2} \nabla p) = \nabla^2 p + |\bar{B}|^2 \nabla p \bullet \nabla |\bar{B}|^{-2} \tag{A.II.11}$$

with

$$\nabla |\bar{B}|^{-2} = -|\bar{B}|^{-4} \nabla(\bar{B} \bullet \bar{B}) = -2|\bar{B}|^{-4}(\bar{B} \wedge (\nabla \wedge \bar{B}) + (\bar{B} \bullet \nabla)\bar{B}) = \tag{A.II.12}$$

$$= 2|\bar{B}|^{-4}(\mu_0 \nabla p - (\bar{B} \bullet \nabla)\bar{B}) - c^{-2}\, \bar{B} \wedge \partial \bar{E}/\partial \mathrm{t})$$

Then,

$$|\overline{B}|^2(\nabla p \bullet \nabla)|\overline{B}|^{-2} = 2\mu_0 |\nabla p|^2 |\overline{B}|^{-2} - 2|\overline{B}|^{-2}\nabla p \bullet (\overline{B} \bullet \nabla)\overline{B} \qquad (A.II.13)$$

$$-2c^{-2}|\overline{B}|^{-2}\nabla p \bullet \overline{B} \wedge \partial \overline{E}/\partial t \quad = 2\mu_0 |\overline{J} - \overline{J}_{//}|^2 - 2|\overline{B}|^{-2}\nabla p \bullet (\overline{B} \bullet \nabla)\overline{B}$$

$$-2c^{-2}|\overline{B}|^{-2}\nabla p \bullet \overline{B} \wedge \partial \overline{E}/\partial t$$

The term in $\nabla^2 p$ on the R.H.S. of Eq. A.II.11) gives to the L.H.S. of Eq. A.II.1) a contribution

$$c^2 \int_V T^{-1}\nabla^2 p \, dV$$

whose sign depends on the pressure profile. This term is negative for pseudo-parabolic profiles (see text) of temperature and particle density and for Gaussian profiles, if the density profile is not too peaked.

The term in μ_0 on the R.H.S. of Eq. A.II.13) simplifies with the negative contribution in $|\overline{J} - \overline{J}_{//}|^2$ on the R.H.S. of Eq. A.II.10) and a positive contribution in

$$|\overline{J} - \overline{J}_{\|}|^2 \propto \frac{|\nabla p|^2}{|\overline{B}|^2} \propto \beta_{tor}$$

remains where $\beta_{tor} \equiv 2\mu_0 nT|\overline{B}|^{-2}$, which is assumed to be a small quantity. The c^{-2} term is $\sim \beta_{tor}$.

We note that ∇p is parallel to the poloidal flux gradient. Moreover the radial component of the magnetic field is zero and axisymmetry is assumed, i.e. the toroidal component of the gradient is zero.

The fourth term on the R.H.S. of Eq. A.II.10) may be written as follows

$$-\overline{B} \bullet (\overline{B} \bullet \nabla)(|\overline{B}|^{-2}\nabla p) = -\overline{B} \bullet [|\overline{B}|^{-2}(\overline{B} \bullet \nabla)\nabla p + \nabla p(\overline{B} \bullet \nabla |\overline{B}|^{-2})] \qquad (A.II14)$$

The contribution of the last addendum on the R.H.S. of Eq. A.II.14) is zero since $\overline{B} \bullet \nabla p = 0$. In the following we shall make use of the formula D.13 (and following ones) in Appendix D of Ref. /42/, which can be applied for $a_p \ll R_p$.

The first term of the R.H.S. Eq. A.II.14) gives to the L.H.S. of A.II.1) a contribution which is proportional to $|B_{pol}/B_{tor}|^2 \ll 1$.

Finally, the third addendum in the R.H.S. of Eq. A.II.10) is proportional to

$$\frac{\frac{d\left(|\overline{B}|^2\right)}{dr}}{|\overline{B}|^2}$$

The second term in the R.H.S. of Eq. A.II.13) gives to the LHS of A.*II*.1 a contribution which is proportional to

$$|\overline{B}|^{-2}|\overline{B}_{pol}|^2$$

Both these terms vanish as the toroidal field is much larger than the poloidal field.

We neglected a term $hc^{-2}\partial \overline{E}/\partial t$ in Eq. A.II.8). Its contribution to the L. H.S. of Eq. A.II.1) is

$$\int_v \frac{h\overline{B}}{T} \bullet \frac{\partial \overline{E}}{\partial t} dV$$

It is possible to show that the only non-negative contribution to this integral is vanishingly small, as ap << Rp. (The same procedure which brings to Eq. A.II.6) is followed. Integrals are computed in the coordinate system of Ref. /42/, App.D - $B_{pol} << B_{tor}$ is assumed).

Then, the only positive contributions in the L.H.S. of Eq. A.II.1) vanish in the limit of small toroidal beta and small $a_p R_p^{-1}$. Negative contributions do not vanish in the same limit.

APPENDIX C. A DIFFERENT APPROACH TO THE PLASMA ENTROPY

The ohmic entropy production density σ_Ω does not contribute to d_eS/dt. This is true even for a multi-species, viscous plasma with chemical and nuclear reactions. Then, we could justify the following approach to plasma entropy:

1. the plasma is considered as an insulated system which satisfies suitable boundary conditions (e.g. given values of total electromagnetic energy and/or helicity, conductive shell, etc.);

2. the entropy is a functional of current density, charge density and electromagnetic fields only, through given phaenomenological relationships $\bar{J} = \bar{J}\left(\bar{E}, T, \ldots\right)$.

This point of view is developed in Refs. /10,11,12,13,20,41/. We stress that the plasma can be described as an isolated system if electromagnetic degrees of freedom only are taken into account. A sound basis for the point of view of Refs. /10,11,12,13,20,41/ is the fact that the heat transport equations and the magnetic field diffusion equation are effectively decoupled for times $t < \sqrt{(\tau_E \tau_R)}$, τ_E energy confinement time and τ_R resistive diffusion time /12/. Then, it is possible to apply the usual statistical mechanics of equilibrium systems to the plasma, through a suitable definition of a generalized temperature, which describes thermal interaction with the external world /41/.

However, for $t > \sqrt{(\tau_E \tau_R)}$ we have to take into account the non-ohmic irreversible processes (like heat transport), which appear in d_eS/dt and in P. Generally speaking, physical processes with loss of energy confinement ($\tau_E \rightarrow 0$) cannot be described by the theory developed in Refs. /10,11,12,13,14/. The same argument holds for particle confinement, since this approach does not take into account explicitely the entropy increase due to particle diffusion. A physical process with loss of particle confinement is studied in Section VI.

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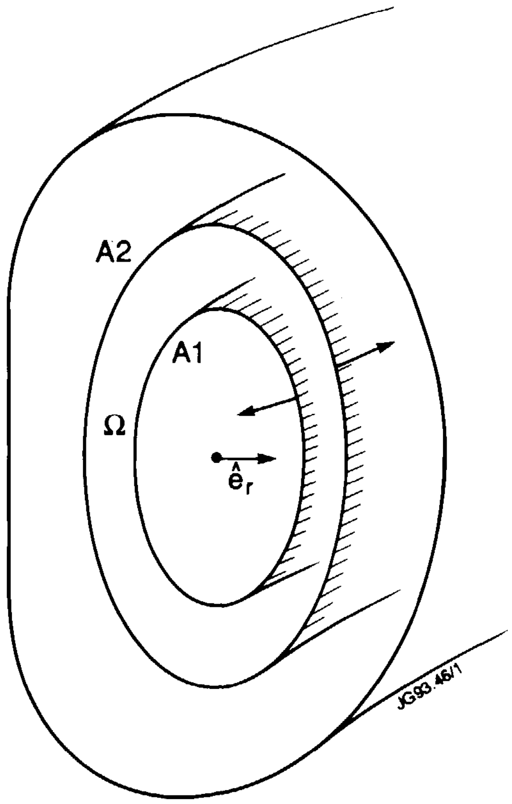


Fig. 1. The plasma geometry considered in the entropy balance calculation.

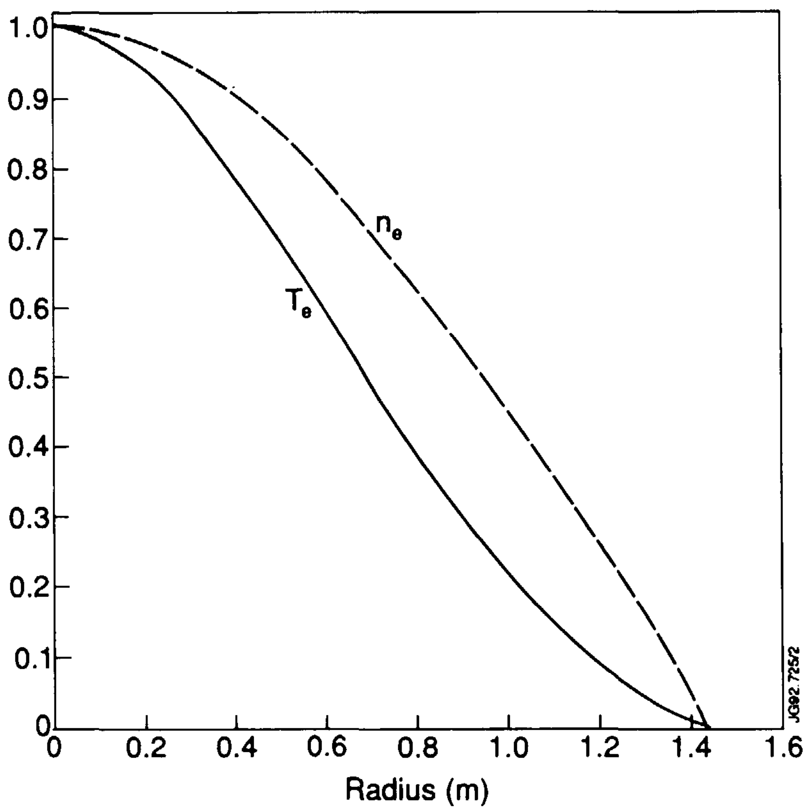
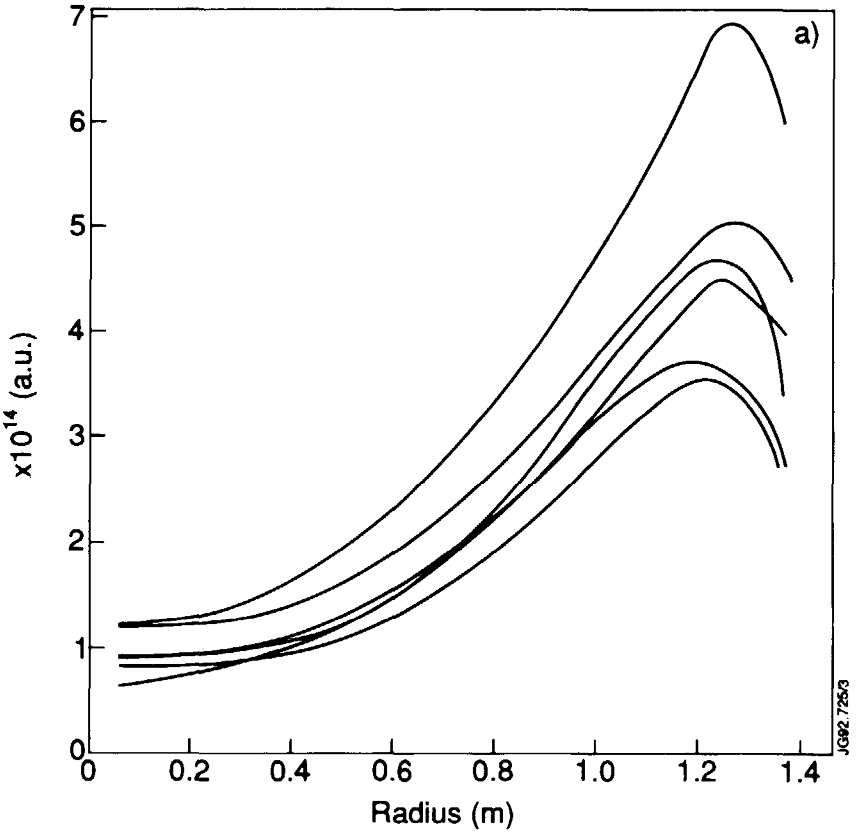
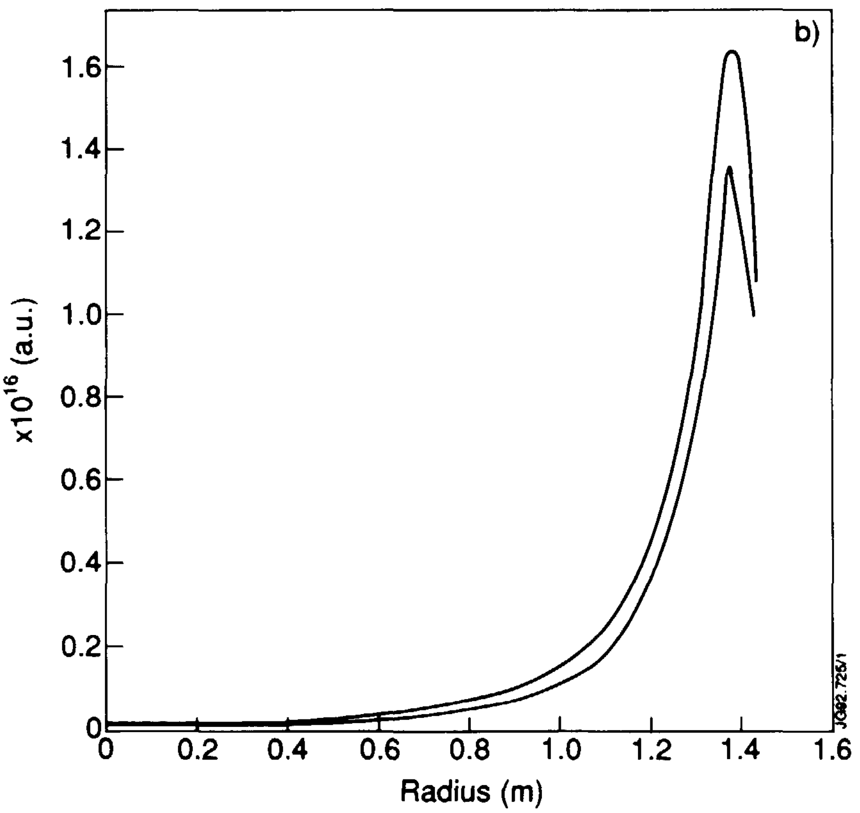


Fig. 2. Normalised electron temperature and density profiles in JET (typical) vs average minor radius. The lower temperature gradient in the outer layers enhances the boundary contribution to the entropy flow.


a)
x10¹⁴ (a.u.)
Radius (m)
0
1
2
3
4
5
6
7
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
JG92.725/3


b)
x10¹⁶ (a.u.)
Radius (m)
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
JG92.725/1

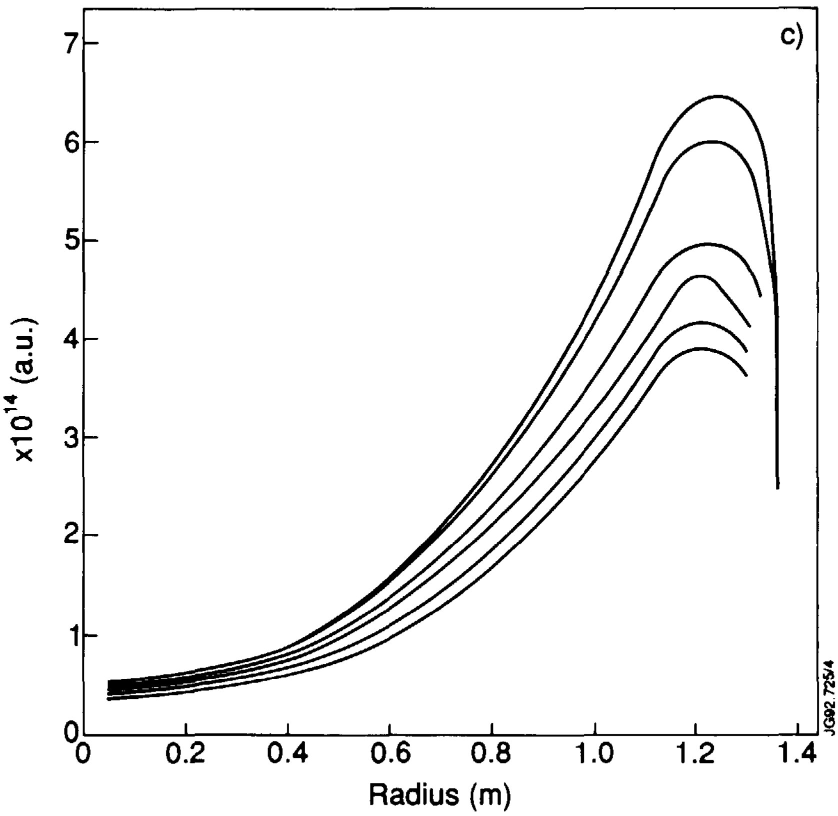


Fig. 3. Plot of $nT^{-3/2}$ vs average minor radius in JET for different discharge conditions; a) H-mode phase; b) off-axis ICRH heating; c) on-axis ICRH heating. The behaviour is such that the total entropy production in the plasma centre and in the outer plasma layers is dominated by the entropy flow.

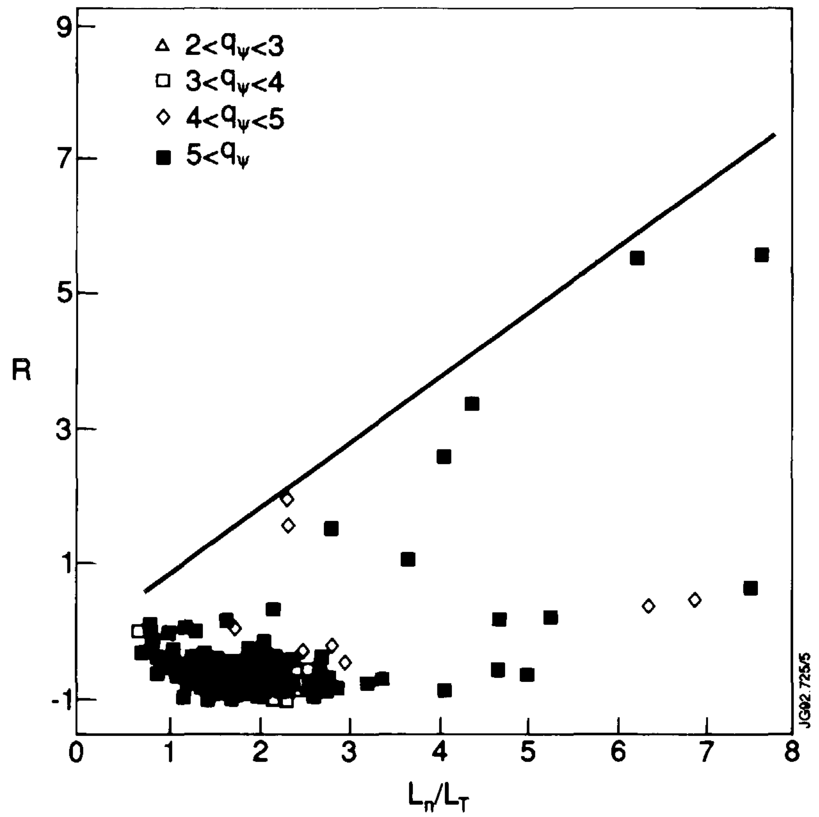


Fig. 4. Diagram showing the plasma thermodynamic state compared with the range of validity of linear non-equilibrium thermodynamics, in a series of ohmic and additionally heated plasmas at different current and magnetic fields.

The experimental data are evaluated at 3/4 radius and the different symbols refer to different flux q values. The solid line is the lower boundary of validity of the linear domain. Typical error bars are shown.