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A NOTE ON THE CALCULATION OF NBI FAST ION DISTRIBUTION FUNCTIONS

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ABSTRACT

An asymptotic evaluation of the NBI fast ion distribution function is given. The result is applicable to problems in which the form of the distribution is required such as microinstability studies and charge exchange recombination spectroscopy analysis.

Calculation of the Distribution Function

In the Neutral Beam heating of a tokamak plasma the fast ions slow down to thermal energy through Coulomb collisions with the background Maxwellian ions and electrons. This relaxation to thermal energy is described by the Fokker-Planck equation in the 2-D velocity space variables v, ζ,

$$\frac{D}{v^2}\frac{\partial^2}{\partial v^2}\left\{\left(v^2+\frac{2\sigma}{v}\right)f\right\}+\frac{1}{v^2}\frac{\partial}{\partial v}\left\{\left(v^3+v_c^3\right)f\right\}+\beta\frac{v_c^3}{v^3}\frac{\partial}{\partial\zeta}\left(1-\zeta^2\right)\frac{\partial f}{\partial\zeta}$$

$$=-\frac{S\tau_s}{4\pi v^2}\,\delta(v-v_o)\,\delta(\zeta-\zeta_o), \quad (1)$$

where in standard notation, D, σ, v_c, and β are functions of the plasma parameters that describe the Coulomb scattering processes, S is the source strength, τ_s the Spitzer time, and v_o, the initial fast ion velocity and ζ_o (= cos θ_o) is the initial pitch. The solution to Eq. (1) is well known and takes the form [1]

$$f(v,\zeta)=\frac{S\tau_s}{v^3+v_c^3}\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)u^{n(n+1)}P_n(\zeta)\,P_n(\zeta_o)\exp\{-g(v)H(v-v_o)\}, \quad (2)$$

where

$$u=\left\{\frac{v_o^3+v_c^3}{v^3+v_c^3}\cdot\frac{v^3}{v_o^3}\right\}^{\pm\beta/3},$$

v_o is the injection velocity, P_n is the Legendre polynomial of order n, g(v) describes energy diffusion, the $+\beta$ exponent is to be taken when $v < v_o$ and H(x) is the Heaviside function.

While the fast ion distribution function as developed above Eq. (2) is convenient for the calculation of averaged quantities it is not particularly useful when calculating its form on or near the ion injection velocity. Close to the injection velocity $v \approx v_o$ $(u \approx 1)$ the above series converges very slowly, and many terms are required to produce acceptable convergence and in some cases n = 100 have been needed. In order to avoid this convergence problem we proceed as follows.

Writing

$$F = \sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right) u^{n(n+1)} P_n(\zeta)\, P_n(\zeta_o)$$

$$= \exp\left\{\frac{1}{4}\ell n(u)\right\} \sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right) \exp\left\{-\left(n+\frac{1}{2}\right)^2 \ell n\left(\frac{1}{u}\right)\right\} P_n(\zeta)\; P_n(\zeta_0)$$

$$= \exp\{\alpha/4\} \sum_{n=-\infty}^{\infty} \int_0^{\infty} x dx e^{-\alpha x^2} \delta\left\{x-n-\frac{1}{2}\right\} P_{x-\frac{1}{2}}(\zeta) P_{x-\frac{1}{2}}(\zeta_o)$$

where $\alpha = \ell n\left(\frac{1}{u}\right)$.

Then introducing a Fourier series representation for the delta function $\delta\left(x-n-\frac{1}{2}\right)$ gives

$$F = \exp\{\alpha/4\} \sum_{n=-\infty}^{\infty} (-1)^n \int_0^{\infty} x dx e^{-\alpha x^2 + i2\pi n x}\; P_{x-\frac{1}{2}}(\zeta) P_{x-\frac{1}{2}}(\zeta_o),$$

and under a change of integration variable $\tau = x\sqrt{\alpha}$ we have,

$$F = \frac{\exp\{\alpha/4\}}{\alpha} \sum_{n=-\infty}^{\infty} (-1)^n \int_0^{\infty} \tau d\tau\, e^{-\left(\tau - i\pi n/\sqrt{\alpha}\right)^2 - \frac{\pi^2 n^2}{\alpha}}\; P_{\frac{\tau}{\sqrt{\alpha}}-\frac{1}{2}}(\zeta)\; P_{\frac{\tau}{\sqrt{\alpha}}-\frac{1}{2}}(\zeta_o)$$

For $n \neq 0$, $\alpha \to 0$ it is easily shown using the Riemann-Labesque lemma [2] that

$$\int_0^\infty \tau d\tau\, e^{-\tau^2 + i\frac{2\pi n}{\sqrt{\alpha}}\tau}\; P_{\frac{\tau}{\sqrt{\alpha}}-\frac{1}{2}}(\zeta)\; P_{\frac{\tau}{\sqrt{\alpha}}-\frac{1}{2}}(\zeta_o) \quad \text{is of} \quad 0\left(\frac{\sqrt{\alpha}}{2\pi n}\right)$$

consequently we only need to consider the n = 0 term in the series.

Hence

$$F = \frac{\exp\{\alpha/4\}}{\alpha} \int_0^\infty \tau d\tau\; e^{-\tau^2}\; P_{\frac{\tau}{\sqrt{\alpha}}-\frac{1}{2}}(\zeta)\; P_{\frac{\tau}{\sqrt{\alpha}}-\frac{1}{2}}(\zeta_o),$$

and since the principle contribution to the integral is when $\tau \neq 0$, and since $\alpha<<1$, and $\tau/\sqrt{\alpha}>>0$, we can use the following asymtotic form for the Legendre functions of large order [3]

$$P_{\upsilon-\frac{1}{2}}(\cos\phi) = \sqrt{\frac{2}{\upsilon\pi\, \sin\phi}}\cos\left(\upsilon\phi - \frac{\pi}{4}\right),$$

where $\varepsilon < \phi < \pi - \varepsilon$. The product of the Legendre functions appearing in the above integral can now be expressed in the following form

$$P_{\upsilon-\frac{1}{2}}(\cos\theta)\; P_{\upsilon-\frac{1}{2}}(\cos\theta_0) = \frac{1}{\pi}(\sin\theta\; \sin\theta_0)^{-\frac{1}{2}}\; \frac{1}{\upsilon}\{\cos\upsilon(\theta-\theta_0) + \sin\upsilon(\theta+\theta_0)\}.$$

Hence

$$F = \frac{\exp\{\alpha/4\}}{\alpha} \int_0^\infty \tau d\tau\; e^{-\tau^2}\; \frac{1}{\pi}\,(\sin\theta\; \sin\theta_0)^{-\frac{1}{2}}\; \frac{\sqrt{\alpha}}{\tau}\left\{\cos\tau\left(\frac{\theta-\theta_0}{\sqrt{\alpha}}\right) + \sin\tau\left(\frac{\theta+\theta_0}{\sqrt{\alpha}}\right)\right\}$$

$$= \frac{\exp\{\alpha/4\}}{\sqrt{\alpha}}\frac{1}{\pi}\,(\sin\theta\; \sin\theta_0)^{-\frac{1}{2}} \int_0^\infty d\tau\; e^{-\tau^2}\left\{\cos\tau\left(\frac{\theta-\theta_0}{\sqrt{\alpha}}\right) + \sin\tau\left(\frac{\theta+\theta_0}{\sqrt{\alpha}}\right)\right\}$$

The integral appearing in this expression is of a standard type and can be readily evaluated. Integrating along the imaginary axis from $i\frac{\theta \pm \theta_o}{\sqrt{\alpha}}$ to 0 and then along the real axis to ∞ we obtain

$$\int_o^\infty d\tau e^{-\tau^2}\{......\} \quad = \frac{\sqrt{\pi}}{2} e^{-\frac{(\theta-\theta_o)^2}{4\alpha}} + e^{-\frac{(\theta+\theta_o)^2}{4\alpha}} \int_o^{\frac{\theta+\theta_o}{2\sqrt{\alpha}}} e^{x^2} dx$$

$$\simeq \frac{\sqrt{\pi}}{2} e^{-\frac{(\theta-\theta_o)^2}{4\alpha}} + O\left(\frac{\sqrt{\alpha}}{\theta+\theta_o}\right).$$

Introducing $\zeta = \cos\theta$, $\zeta_o = \cos\theta_o$,

$$\theta - \theta_o = \cos^{-1}\zeta - \cos^{-1}\zeta_o = \frac{1}{\sqrt{1-\zeta_o^2}}(\zeta - \zeta_o),$$

where $|\zeta - \zeta_o| << 1$, we finally obtain for particles injected initially into the regions of velocity space, where $\zeta_o^2 < 1$.

$$F = \frac{e^{-\frac{(\zeta-\zeta_o)^2}{4\alpha}}}{2\sqrt{\pi\alpha}} \quad ,$$

where α is now defined as

$$\alpha = \frac{\pm\beta\left(1-\zeta_o^2\right)}{3} \ell n \left\{ \frac{1+(v_c/v)^3}{1+(v_c/v_o)^3} \right\} \quad .$$

The fast ion distribution function then takes the form

$$f(v, \zeta) = F(v, \zeta) \exp\{-g(v)\, H(v-v_o)\} \, ,$$

where $$F(v, \zeta) = \frac{S\tau_S}{v^3 + v_c^3} \cdot \frac{e^{-\frac{(\zeta-\zeta_o)^2}{4\alpha}}}{2\sqrt{\pi\alpha}}$$

Finally, it is worth mentioning that for velocities $v \leq v_L$, where

$$v_L \sim v_c \left[\left\{ 1 + \left(\frac{v_c}{v_o} \right)^3 \right\} e^{\frac{3}{4\beta} \frac{(1-|\zeta_o|)}{(1+|\zeta_o|)}} - 1 \right]^{-1/3} ,$$

the fast ions have undergone consideration pitch angle scattering and the expression for the distribution function, Eq. (1) should be used. For this velocity range Eq. (1) converges rapidly and only a few terms have to be summed.

[1] CORDEY, J.G. and CORE, W.G.F., Phys. Fluids 17 (1974).

[2] WHITTAKER, E.T., and WATSON, G.N., A course of Modern Analysis, Cambridge University Press (63)

[3] ABRAMOWITZ, M. and STEGUN, I.A., Handbook of Mathematical Functions, Dover Publications, Inc., New York (70).