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ABSTRACT

Expressions, suitable for numerical evaluation, are given for the parameters characterizing the locations of the R–cutoff (upper X–mode cutoff), the L–cutoff (lower X–mode cutoff) and the O–mode cutoff for electromagnetic waves in relativistic plasmas. The ion motion is ignored, and it is assumed that the plasma is collisionless and in thermodynamic equilibrium. A new derivation of the relativistic cutoff expressions, starting from first principles, is given, together with an heuristic explanation for the relativistic shift of the O–mode cutoff.

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1 INTRODUCTION

Relativistic modifications to the dielectric properties, besides absorption, of plasmas at the temperatures found in present fusion experiments can have practical consequences for a number of microwave diagnostics used in experiments on magnetically confined fusion plasmas. In particular it has been found that in millimetre wave Thomson scattering [BINDSLEV, 1991a] and in reflectometry [BINDSLEV, 1991b and 1992; MAZZUCATO, 1992] it may be essential to take relativistic effects into account when analyzing results, while diagnostics relying on electron cyclotron emission can be affected by the relativistic increase in the cutoff density [COSTLEY and BARTLETT, 1993].

The need to take relativistic dielectric effects into account in a range of applications including those mentioned above adds to the desirability of having accurate relativistic expressions which may be evaluated with a minimum of computing time, and which are readily accessible to workers who are unfamiliar with relativistic calculations.

Fully relativistic expressions for the locations of cutoffs in plasmas have been derived by a number of authors, all starting from expressions for the relativistic dielectric tensor. For an isotropic Maxwellian momentum distribution expressions were given in BATCHELOR, GOLDFINGER and WEITZNER (1984). This work was extended for the X–mode R–cutoff (upper cutoff) to a loss cone distribution by BORNATICI and RUFFINA (1988), who also presented approximate relations for the R–cutoff. A weakly relativistic expression was given by ROBINSON (1986a).

In Section 2 of this paper we give an alternative derivation of the fully relativistic expressions for the cutoffs with an heuristic explanation for the relativistic shift of the O–mode cutoff. Approximate expressions suitable for numerical evaluation are presented in Section 3, and their accuracies are compared numerically in Section 4.

In the expressions presented here the ion motion is ignored, and it is assumed that the plasma is collisionless (i.e. the dynamics of the electrons in the plasma can be described by the Vlasov equation). The unperturbed velocity distribution of the electrons is taken to be the isotropic relativistic Maxwellian.

2 ALTERNATIVE DERIVATION OF THE RELATIVISTIC EXPRESSIONS FOR THE LOCATIONS OF CUTOFFS

In this section the relativistic expressions for the locations of cutoffs are derived *ab initio*, without reference to expressions for the dielectric tensor. By leaving out the complications involved in the derivation of the dielectric tensor this approach facilitates a better appreciation of the underlying physics in the relativistic shifts of the cutoffs.

Mathematically a wave is cut off when its wavelength is infinite, which implies that the refractive index and spatial derivatives of quantities relating to the wave are zero. From Maxwell's equations we then find that cutoffs occur when the second derivative of the dielectric displacement, $\mathbf{D}$, is identically zero,

$$\frac{\partial^2 \mathbf{D}}{\partial t^2} = 0 \quad . \tag{1}$$

The dielectric displacement consists of the vacuum dielectric displacement, $\varepsilon_0 \mathbf{E}$, and the plasma dielectric displacement, $\mathbf{P}$ ($\partial \mathbf{P}/\partial t$ is the plasma current), so (1) takes the form

$$\varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{\partial^2 \mathbf{P}}{\partial t^2} = 0 \quad . \tag{2}$$

The time derivative of the plasma current, $\partial^2 \mathbf{P}/\partial t^2$, is given by

$$\frac{\partial^2 \mathbf{P}}{\partial t^2} = q_e \int \mathbf{v} \frac{\partial f^{(1)}}{\partial t} \, d\mathbf{p} \quad , \tag{3}$$

where $\mathbf{v} = \mathbf{p}/\gamma m_e$ is the velocity, $\mathbf{p}$ is the momentum, $\gamma = (1 + p^2)^{1/2}$, $p = |\mathbf{p}|/m_e c$, m_e is the electron rest mass, c is the vacuum velocity of light and $f^{(1)}(\mathbf{p}, t)$ is the first order perturbation to the electron momentum distribution. $f^{(1)}$ satisfies the linearized Vlasov equation with $\partial/\partial \mathbf{r} = 0$:

$$\frac{\partial f^{(1)}}{\partial t} + q_e (\mathbf{v} \times \mathbf{B}^{(0)}) \cdot \frac{\partial f^{(1)}}{\partial \mathbf{p}} = -q_e \mathbf{E} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{p}} \quad . \tag{4}$$

$\mathbf{B}^{(0)}$ is the static magnetic field and $f^{(0)}(\mathbf{p})$ is the unperturbed momentum distribution, which in thermodynamic equilibrium is given by

$$f^{(0)} = \frac{n^{(0)}\mu\exp\{-\mu\gamma\}}{4\pi(m_e c)^3 K_2(\mu)} \quad , \tag{5}$$

where $n^{(0)}$ is the equilibrium electron density, $\mu = m_e c^2/T_e$, T_e is the electron temperature and K_2 is the modified Bessel function of the second kind and order 2.

In the absence of a wave vector (it is identically zero) and with an isotropic (or gyrotropic) momentum distribution there is only one preferred direction; that of the static magnetic field, $\mathbf{B}^{(0)}$, and the system is thus invariant under rotation about this axis. Unless two or more solutions for the electric field are degenerate (exist at the same frequency) the solutions for the electric field must also satisfy this symmetry. This implies that $\mathbf{E}$ is either linearly polarized in the direction of $\mathbf{B}^{(0)}$, corresponding to the O–mode cutoff, or circularly polarized in the plane perpendicular to $\mathbf{B}^{(0)}$, in which case the rotation is either right handed (clockwise rotation viewed in the direction of $\mathbf{B}^{(0)}$) corresponding to the R–cutoff, or left handed corresponding to the L–cutoff. The solutions are only degenerate when $\mathbf{B}^{(0)} = 0$.

Since $\mathbf{E}$ is parallel to $\mathbf{B}^{(0)}$ at the O–mode cutoff it follows that this cutoff does not depend on $\mathbf{B}^{(0)}$. From the linearized Vlasov equation (4) we thus have

$$\frac{\partial f^{(1)}}{\partial t} = -q_e \mathbf{E} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{p}} \tag{6}$$

which upon insertion in the expression (3) for the derivative of the plasma current gives

$$\frac{\partial^2 \mathbf{P}}{\partial t^2} = -q_e^2 \int \mathbf{v} \left(\mathbf{E} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{p}} \right) d\mathbf{p} \quad . \tag{7}$$

Inserting (5) for $f^{(0)}$ and integrating we find

$$\frac{\partial^2 \mathbf{P}}{\partial t^2} = \varepsilon_0 \mathbf{E} \frac{\omega_p^2 \mu^2}{3K_2(\mu)} \int_0^\infty \frac{p^4 \exp\{-\mu\gamma\}}{\gamma^2} \, dp \quad . \tag{8}$$

Noting that $\partial^2 \mathbf{E}/\partial t^2 = -\omega_0^2 \mathbf{E}$, where ω_0 is the O–mode cutoff frequency, we recover the fully relativistic expression for the O–mode cutoff from equation (2):

$$\omega_0^2 = \omega_p^2 \frac{\mu^2}{3K_2(\mu)} \int_0^\infty \frac{p^4 \exp\{-\mu\gamma\}}{\gamma^2} \, dp \quad . \tag{9}$$

Here $\omega_p = \left(n^{(0)} q_e^2/(m_e \varepsilon_0)\right)^{1/2}$ is the cold electron plasma frequency.

For a simple heuristic explanation for the relativistic shift in the O-mode cutoff we return to expression (7) for the derivative of the plasma current, $\partial^2\mathbf{P}/\partial t^2$. By partial integration we find

$$\frac{\partial^2\mathbf{P}}{\partial t^2} = q_e^2 E_i \int \frac{d\mathbf{v}}{dp_i} f^{(0)}\, d\mathbf{p} \quad , \tag{10}$$

where p_i is the i'th component of $\mathbf{p}$ and similarly for other vectors. Noting that $d\mathbf{p}/dt = q_e(\mathbf{E} + \mathbf{v} \times \mathbf{B}^{(0)})$ we can write (10) as

$$\frac{\partial^2\mathbf{P}(t)}{\partial t^2} = q_e \int \mathbf{a}(\mathbf{p}, t) f^{(0)}\, d\mathbf{p} \quad , \tag{11}$$

where $\mathbf{a}(\mathbf{p}, t)$ is the acceleration at time t that an electron with momentum $\mathbf{p}$ experiences due to the electric field, $\mathbf{E}(t)$. In the present case the acceleration due to the magnetic field does not contribute to $\partial^2\mathbf{P}/\partial t^2$ because $f^{(0)}$ is isotropic (this holds also if $f^{(0)}$ is just gyrotropic).

In a non-relativistic treatment the acceleration of an electron experiencing a force $\mathbf{F}$ is simply $\mathbf{a} = \mathbf{F}/m_e$. This dynamic relation, with $\mathbf{F} = q_e\mathbf{E}$, leads to the cold expression for the cutoff. By contrast the relativistic equation for the acceleration of an electron experiencing a force $\mathbf{F}$ is:

$$\begin{aligned} a_i &= \frac{dv_i}{dp_j} F_j \\ &= \left(\frac{\delta_{ij}}{m_e\gamma} - \frac{p_i p_j}{m_e\gamma^3 (m_e c)^2} \right) F_j \quad . \end{aligned} \tag{12}$$

It is noteworthy that relativity affects the acceleration in two ways. Firstly, there is a relativistic mass increase which manifests itself in the first term in (12). Secondly, when $\mathbf{F} \cdot \mathbf{p} \neq 0$ some of the work that the force does on the particle goes into increasing the mass rather than the velocity of the particle. This effect is represented by the second term in (12).

At the X-mode cutoffs, where, as noted above, the electric field is circularly polarized in the plane perpendicular to the magnetic field, we must retain the static magnetic field in the linearized Vlasov equation (4). Integrating (4) along characteristics we find

$$f^{(1)}(\mathbf{p}, t) = \frac{q_e \mu f^{(0)}}{\gamma (m_e c)^2} \int_{-\infty}^{0} \mathbf{E}(\tau + t) \cdot \mathbf{p}'(\tau)\, d\tau \tag{13}$$

where $\mathbf{p}'(\tau)$ is the unperturbed orbit in momentum space of the electron which at $\tau = 0$ has the momentum $\mathbf{p}'(0) = \mathbf{p}$ ($\mathbf{p}$ is one of the arguments of $f^{(1)}$). The electrons gyrate with an angular frequency of ω_c/γ, where $\omega_c = B^{(0)}|q_e|/m_e$ is the non–relativistic electron cyclotron frequency. The electric field rotates with the angular cutoff frequency, ω_s, where $s = 1$ if the field rotates in the same direction as the electrons gyrate and $s = -1$ if it rotates in the opposite direction. With these definitions we can write the integrand in (13) as

$$\mathbf{E}(\tau + t) \cdot \mathbf{p}'(\tau) = |\mathbf{E}(0)||\mathbf{p}_\perp| \cos\left((\omega_s - \frac{s\omega_c}{\gamma})\tau + \omega_s t - \phi\right) \quad ,$$

where ϕ is the azimuthal angle of $\mathbf{p}$, and $\mathbf{p}_\perp$ is the component of $\mathbf{p}$ perpendicular to $\mathbf{B}^{(0)}$.

To carry out the integration in (13) we assume that the field dies out slowly as $\tau \to -\infty$. With this assumption we find after integration with respect to τ:

$$f^{(1)}(\mathbf{p}, t) = \frac{q_e \mu f^{(0)}}{\gamma (mc)^2} |\mathbf{E}(0)||\mathbf{p}_\perp| \frac{\sin(\omega_s t - \phi)}{(\omega_s - s\omega_c/\gamma)}$$

and after differentiation with respect to t:

$$\frac{\partial f^{(1)}(\mathbf{p}, t)}{\partial t} = \frac{q_e \mu f^{(0)}}{(mc)^2 (\gamma - s\omega_c/\omega_s)} \mathbf{E}(t) \cdot \mathbf{p} \quad . \tag{14}$$

Inserting expression (14) in (3) and integrating we find

$$\frac{\partial^2 \mathbf{P}}{\partial t^2} = \varepsilon_0 \mathbf{E} \frac{\omega_p^2 \mu^2}{3K_2(\mu)} \int_0^\infty \frac{p^4 \exp\{-\mu\gamma\}}{\gamma(\gamma - s\omega_c/\omega_s)}\, dp \quad . \tag{15}$$

Noting that $\partial^2 \mathbf{E}/\partial t^2 = -\omega_s^2 \mathbf{E}$, we recover the fully relativistic expression for the X–mode cutoffs from equation (2):

$$\omega_s^2 = \omega_p^2 \frac{\mu^2}{3K_2(\mu)} \int_0^\infty \frac{p^4 \exp\{-\mu\gamma\}}{\gamma(\gamma - s\omega_c/\omega_s)}\, dp \quad , \tag{16}$$

where $s = 1$ corresponds to the R–cutoff and $s = -1$ corresponds to the L–cutoff.

3 FULLY RELATIVISTIC EXPRESSION

Collecting the relativistic expressions for the O–mode cutoff (9), and the X–mode R– and L– cutoffs (16) in one expression, we have

$$\Pi = \frac{3K_2(\mu)}{\mu^2 \int_0^\infty \frac{p^4 \exp\{-\mu\gamma\}}{\gamma(\gamma - s\Omega)}\, dp} \quad , \tag{17}$$

$$\gamma = \sqrt{1+p^2} \quad , \qquad s = \begin{cases} 1: & \text{X–mode R–cutoff}\,, \quad 0 \le \Omega \le 1\,, \\ 0: & \text{O–mode cutoff}\,, \\ -1: & \text{X–mode L–cutoff}\,, \quad 0 \le \Omega \quad , \end{cases}$$

where $\Pi = \omega_p^2/\omega_s^2$, $\Omega = \omega_c/\omega_s$ and ω_s is the relativistic cutoff frequency. Π may also be interpreted as the ratio of the relativistic cutoff density, n_s, to the cold O–mode cutoff density, n_p, for waves at a given frequency: $\Pi = n_s/n_p$. We will therefore refer to Π as the normalized cutoff density.

Expression (17) for Π, is valid for the kind of plasmas described in the Introduction with no further limitations. This expression, with $s = 0$, is identical to Expression (43) in BATCHELOR, GOLDFINGER and WEITZNER (1984) while our expression with $s = \pm 1$ follows from their expressions (39–41), with the "−" in (39) replaced by "+" (misprint). (17) with $s = 1$ is also identical to (29) in BORNATICI and RUFFINA (1988) with $j = 0$.

For a given frequency the relativistic effects increase the density at which the cutoffs occur. To illustrate this we introduce the cold normalized cutoff density, ψ,

$$\psi = 1 - s\Omega \quad , \tag{18}$$

and plot in Figure 1 the relativistic increase in the normalized cutoff density $\Delta\Pi = \Pi - \psi$, as a function of ψ. At the R–cutoff we have $0 < \psi < 1$, while $\psi = 1$ at the O–mode cutoff and $\psi > 1$ at the L–cutoff. The plots in Figure 1 thus cover the R–cutoff, the O–mode cutoff and part of the L–cutoff.

As noted by PRITCHETT (1984), the relativistic theory, unlike the cold, predicts that there is a finite density limit, Π_{lim}, below which the R–cutoff no longer exists. From expression (17) we have

$$\Pi_{\text{lim}} = \frac{3K_2(\mu)}{\mu^2 \int_0^\infty \frac{p^4 \exp\{-\mu\gamma\}}{\gamma(\gamma - 1)}\, dp} \tag{19}$$

which is identical to (31) in BORNATICI and RUFFINA (1988) with $j = 0$.

At the temperatures relevant for fusion plasmas we generally have that $\mu \gg 1$. In this case the following asymptotic expansions of (17) are useful:

$$\Pi = K/G \tag{20}$$

$$\begin{aligned} K &= \left(\frac{2\mu}{\pi}\right)^{1/2} \exp\{\mu\} K_2(\mu) && (21a) \\ &= \sum_{n=0}^{N} \frac{\Gamma(5/2+n)}{\Gamma(5/2-n)n!2^n}\,\mu^{-n} + O\left(\mu^{-(N+1)}\right) && (21b) \\ &\approx 1 + \frac{15}{8}\frac{1}{\mu} + \frac{105}{128}\frac{1}{\mu^2} - \frac{315}{1024}\frac{1}{\mu^3} + \ldots \quad , && (21c) \end{aligned}$$

$$\begin{aligned} G &= \frac{\mu^{5/2}}{3(\pi/2)^{1/2}} \int_0^\infty \frac{p^4 \exp\{-\mu(\gamma-1)\}}{\gamma(\gamma-1+\psi)}\,dp && (22a) \\ &= \sum_{n=0}^{N} \frac{\Gamma(5/2+n)\mu^{1-n}}{\Gamma(5/2-n)n!2^n} F_{n+5/2}(\zeta) \;+\; O\left(\mu^{-N}\right) , && (22b) \\ &\approx \mu F_{5/2}(\zeta) + \frac{15}{8} F_{7/2}(\zeta) + \frac{105}{128\mu} F_{9/2}(\zeta) - \frac{315}{1024\mu^2} F_{11/2}(\zeta) + \ldots . && (22c) \end{aligned}$$

Here $\zeta = \mu\psi$ and

$$F_q(z) = z^{q-1} e^z \int_z^\infty \exp\{-x\} x^q \, dx \, , \quad q = 5/2,\, 7/2,\, 9/2,\, \ldots$$

are the Dnestrovskii functions of half integer index [DNESTROVSKII *et al.*, 1964; ROBINSON, 1986b]. These functions are readily evaluated from either their series expansions or their continued fraction representations [ROBINSON, 1986b, Expressions (29), (30)[1] and (34)], and when ζ is not too large ($\zeta \lesssim 1000$, dependent on machine precision)

[1] There is an error in expression (30) in ROBINSON (1986 b). It should read

$$F_q(z) = z^{q-1} e^z \Gamma(1-q) - e^z \sum_{j=0}^{\infty} \frac{(-z)^j}{(j+1-q)j!} \quad .$$

the recursive relation that exists between the Dnestrovskii functions [ROBINSON, 1986b, Expression (14)] may be used. The real and imaginary parts of $F_{5/3}(\zeta)$, $F_{7/3}(\zeta)$ and $F_{9/3}(\zeta)$ are plotted in Figures 1 and 2 in DNESTROVSKII *et al.* (1964). Note that in the present applications ζ takes only positive real values, for which the required Dnestrovskii functions are purely real. Expression (22b) is, apart from a factor μ, identical to expression (19c) in BORNATICI and RUFFINA (1988).

Including only the first term in (21b) and the first term in (22b) we recover ROBINSON's (1986a) weakly relativistic result for Π generalized to include the O–mode cutoff and the L–cutoff.

For small to moderately large values of ζ ($\zeta \lesssim 1000$, dependent on machine precision) expression (22b) can be further simplfied to read:

$$G = \sum_{n=0}^{N} a_n f_n(\zeta) \mu^{1-n} \; + \; O\left(\mu^{-N}\right) \tag{23}$$

where the coefficients a_n and $f_n(\zeta)$ are given by the following recursion relations:

$$a_0 \;=\; 1 \quad , \qquad a_n = \frac{2n-5}{4n}\, a_{n-1} \quad ; \tag{24a}$$

$$f_0 \;=\; F_{5/2}(\zeta) \quad , \qquad f_n = \zeta f_{n-1} - b_{n+1} \quad ; \tag{24b}$$

$$b_0 \;=\; \frac{4}{3} \quad , \qquad b_n = (1/2 - n)\, b_{n-1} \quad . \tag{24c}$$

Including only the first two terms in (21b) and (23) we find the following simple expression for the normalized cutoff density:

$$\Pi \approx \frac{1}{\mu F_{5/2}(\zeta)} \left[1 + \frac{3}{4\mu} \left(\frac{5}{2} - \frac{1}{F_{5/2}(\zeta)} + \zeta \right) \right] \quad . \tag{25}$$

Π can be determined from expression (25) for the R–cutoff, the O–mode cutoff and large parts of the L–cutoff with adequate precision for most applications involving fusion plasmas (see discussion in Section 4).

For large values of ζ ($\zeta \gtrsim 1000$, dependent on machine precision) expressions (23) and (25), and the recursive relation between the Dnestrovskii functions, lead to numerical cancellation errors (small differences between large numbers need to be evaluated). An alternative simplified expression for G may be derived which is valid for medium to large values of ζ ($\zeta \gtrsim 100$) thus covering the range where the simplified expressions

(23) and (25) failed and giving a comfortable overlap in regions of validity with these expressions. Starting from (22b) and making use of the asymptotic expressions for the Dnestrovskii functions, we find

$$G = \psi^{-1} \sum_{n=0}^{N} c_n g_n(\psi) \mu^{-n} \; + \; O\left(\mu^{-N}\right) \tag{26}$$

where the coefficients c_n and $g_n(\psi)$ are given by the recursive relations

$$c_0 = 1 \quad , \qquad c_n = -(3/2 + n)\, c_{n-1} \quad ; \tag{27a}$$

$$g_0 = 1 \quad , \qquad g_n = \psi^{-1} g_{n-1} + a_n \quad ; \tag{27b}$$

and a_n is obtained from the recursion relation (24a).

For convenience we give here the first four terms in the expansion (26):

$$\begin{aligned} G \;=\; & \psi^{-1} \left[1 - \frac{5}{2} \left(\psi^{-1} - \frac{3}{4} \right) \mu^{-1} + \frac{35}{4} \left(\psi^{-2} - \frac{3}{4}\psi^{-1} + \frac{3}{32} \right) \mu^{-2} \right. \\ & \left. - \frac{315}{8} \left(\psi^{-3} - \frac{3}{4}\psi^{-2} + \frac{3}{32}\psi^{-1} + \frac{1}{128} \right) \mu^{-3} \right] + \dots \quad . \end{aligned} \tag{28}$$

If we include only the first two terms in (21b) and (26) we find the following expression for the normalized cutoff density:

$$\Pi \approx \psi + \frac{5}{2} \mu^{-1} \quad . \tag{29}$$

Expression (29) for Π is attractively simple. Unfortunately its range of validity has significant limitations which particularly affect its use for the R–cutoff (see discussion in Section 4). A similar approximation was proposed by MAZZUCATO (1992):

$$\Pi \approx \psi + \left(1 + 5\mu^{-1}\right)^{1/2} - 1 \quad . \tag{30}$$

4 SUMMARY AND DISCUSSION OF NUMERICAL PERFORMANCE

In the previous section a number of expressions have been given for the normalized cutoff density, Π. These include a fully relativistic expression and approximations of varying complexity and validity.

Expression (17) is fully relativistic and is thus valid for any temperature. The main drawback of this expression is that it requires the evaluation of an infinite integral. This integral is, however, well behaved and sufficient accuracy can generally be obtained by straightforward summation over less that 100 terms.

If the temperature is limited to values well below the rest mass energy of an electron ($m_e c^2 = 511$ keV) asymptotic expansions in the inverse temperature parameter, μ, are useful. In this regime Π is conveniently split into two factors, K and G. K depends only on μ and presents no difficulties. G on the other hand depends on both μ and ζ or alternatively on μ and ψ.

The most general expansion presented here is given by expression (22b) for G, together with (21b) for K. This expansion is a good approximation for $T_e \lesssim 200$ keV. For the range of parameters covered in Figure 1 and including three terms in the expansion (i.e. $N = 2$) this expansion predicts Π correctly to more than five significant figures. Its principal drawback is that it requires the evaluation of several Dnestrovskii functions. At moderate values of ζ ($\zeta \lesssim 1000$, dependent on machine precision) these may however be determined recursively, leaving only $F_{5/2}(\zeta)$ to be determined by its series expansion or by its continued fraction representation.

At moderate to small values of ζ the expansion (23) may be used together with (21b). Within its limitations ($\zeta \lesssim 1000$), which covers the range of parameters explored in Figure 1, this expansion yields the same accuracy as the general expansion (22b) for identical N. Its main drawback is that it requires the evaluation of $F_{5/2}(\zeta)$ and that its validity is limited to moderate values of ζ, which principally affects its usefulness for the L–cutoff.

Expression (25) is the simplest expression given here which is valid for moderate to small values of ζ. In the parameter range covered in Figure 1 it computes Π correctly to three or more significant figures. Its relative simplicity combined with reasonable accuracy makes it ideal for use in computational data analysis involving the R–cutoff (e.g. X–mode reflectometry). Its main drawback is that it also requires $F_{5/2}(\zeta)$ and has limited validity for the L–cutoff.

At moderate to large values of ζ ($\zeta \gtrsim 100$) the expansion (26) or the explicit form (28) may be used together with (21b). This expansion is useful for the L–cutoff and, at $T_e \lesssim 20$ keV, also for the O–mode cutoff. As illustrated in Figure 2 (a), its usefulness for the R–cutoff is limited. The attraction of this expansion is that it involves no special functions. Since this may encourage wide use it is important to note its limitations.

The greatest range of validity is achieved with $N = 8$. Larger values of N results in divergence of the approximation from the true value at larger values of ψ. Note that the term $O(\mu^{-N})$ in expression (26) implies that the expansion is convergent for a given value of N and $\mu \to \infty$, but it does not imply convergence for a given value of μ and $N \to \infty$.

The simplest expression for Π given here is (29). This expression is useful for the L–cutoff and the O–mode cutoff at $T_e \lesssim 20$ keV. Its accuracy is illustrated in Figure 2 (*b*). While this expression is attractive for its simplicity it is important to note its limitations which significantly affect its usefulness for the R–cutoff. The expression (30) yields results very similar to (29), the two estimates of $\Delta\Pi$ differing by 0.0025 at $T_e = 15$keV.

The weakly relativistic expression for the normalized cutoff density, which is identical to the first term in the fully relativistic expansion (20), (21b) and (22b) with $N = 0$, is of particular interest because of the wide use of the weakly relativistic approximation to the dielectric tensor. The weakly relativistic predictions for the normalized cutoff density are plotted in Figure 2 (*c*) together with the fully relativistic predictions. This plot confirms that the weakly relativistic approximation is acceptable for most applications for $T_e \lesssim 20$ keV.

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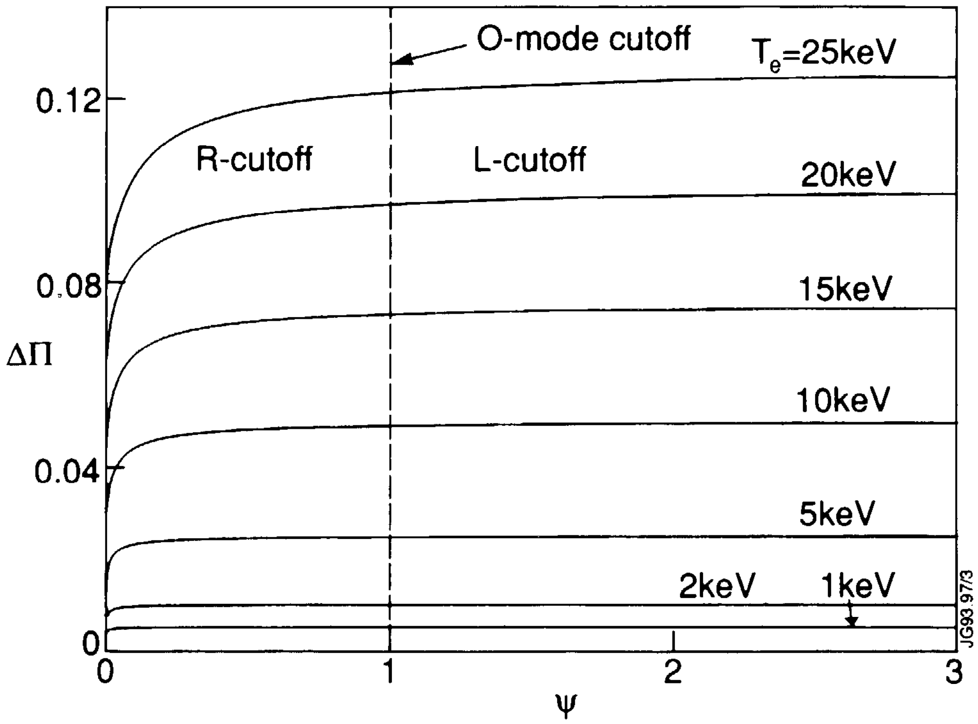

Figure 1: Relativistic increase in cutoff density normalized by cold O–mode cutoff density, $\Delta\Pi = \Pi - \psi$, as a function of $\psi = 1 - s\Omega$. R–cutoff: $0 < \psi < 1$. O–mode cutoff: $\psi = 1$. L–cutoff: $1 < \psi$.

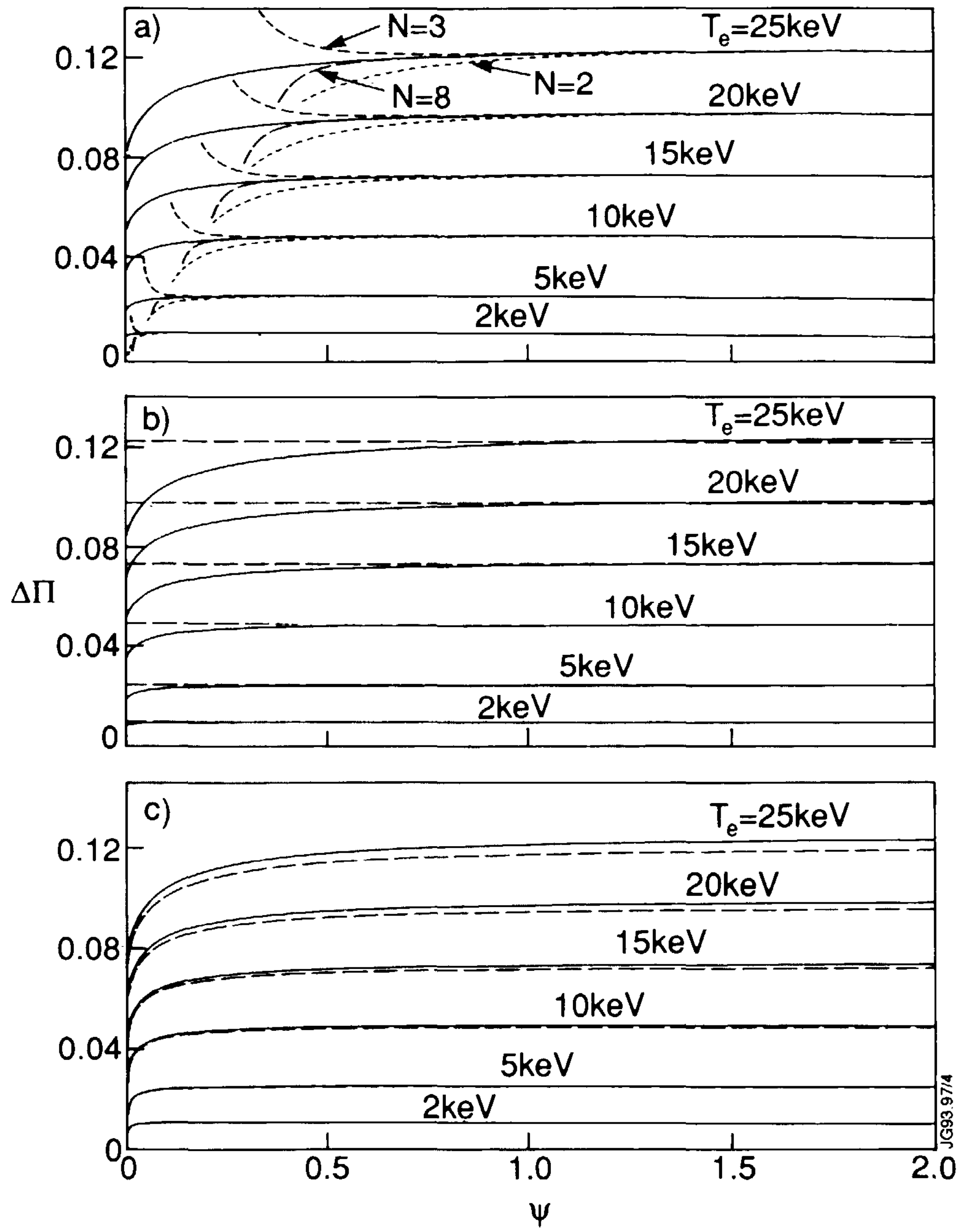

Figure 2: Relativistic increase in cutoff density normalized by cold O–mode cutoff density, $\Delta\Pi = \Pi - \psi$, as a function of ψ. The full curves are calculated with the fully relativistic expression (17) while the broken curves are calculated using a range of approximations. (*a*) Broken curves calculated with the asymptotic expansion (21b) for K and (26) for G, with $N = 2$, 3 and 8. This approximation has its greatest range of validity for $N = 8$ (see text). (*b*) Broken curves calculated with the simple approximation, (29). (*c*) Broken curves calculated with the weakly relativistic expression, which is equivalent to expressions (21b) and (22b) with $N = 0$.