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J. O'Rourke
and JET Team

An Analytic Solution of the Inhomogeneous Coupled Diffusion Problem

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J. O'Rourke and JET Team*

JET-Joint Undertaking, Culham Science Centre, OX14 3DB, Abingdon, UK

* *See Annex*

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1. INTRODUCTION

The determination of transport coefficients from the evolution of transient perturbations is an area of active research in Tokamak physics /1/. Recently, considerable effort has been devoted to the simultaneous measurement of electron density and temperature perturbations /2, 3, 4, 5/ and to the analysis of these perturbations in terms of coupled transport equations /6, 7, 8, 9, 10, 11/. Analytic solutions of these equations are of interest because they can provide physical insights which are not afforded by numerical simulations and because they can reduce the computer time required to analyse data.

A common feature of the analytic /7, 8, 9, 11/ and semi-analytic /10/ analysis techniques which have been published is that the solution is restricted to spatial (or temporal) regions in which the perturbed sources vanish. That is to say, the equations solved are of the form:

$$\frac{\partial}{\partial t}\begin{pmatrix}\tilde{n}/n_0\\ \tilde{T}/T_0\end{pmatrix} = \begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\bullet\nabla^2\begin{pmatrix}\tilde{n}/n_0\\ \tilde{T}/T_0\end{pmatrix} \qquad [1]$$

In this note the evolution of perturbations in the presence of non- vanishing harmonic perturbations of the particle and heat sources is considered. The solution is elaborated in section 2. The validity of far-field approximations for the analysis of modulation experiments is discussed in section 3.

2. SOLUTION OF THE INHOMOGENEOUS COUPLED DIFFUSION EQUATIONS

First we consider coupled equations with harmonic point sources of the form:

$$\frac{\partial}{\partial t}\begin{pmatrix}\tilde{n}/n_0\\ \tilde{T}/T_0\end{pmatrix}=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\bullet\nabla^2\begin{pmatrix}\tilde{n}/n_0\\ \tilde{T}/T_0\end{pmatrix}+\begin{pmatrix}S_n\\ S_T\end{pmatrix}\bullet\frac{\delta(r-r_0)}{4\pi^2 rR}e^{i\omega t} \qquad [2]$$

in a cylinder of length $2\pi R$ and radius a, with constant transport coefficients A_{ij} and boundary conditions: $\tilde{n}(a,t)=\tilde{T}(a,t)=(\partial\tilde{n}/\partial r)_{r=0}=(\partial\tilde{T}/\partial r)_{r=0}=0$. The solution of the corresponding scalar equation:

$$\frac{\partial y}{\partial t}=D\nabla^2 y+S_0\frac{\delta(r-r_0)}{4\pi^2 rR}e^{i\omega t} \qquad [3]$$

is given by:

$$y(r,t,r_0,D)$$

$$=\frac{S_0 e^{i\omega t}}{4\pi^2 RD}\{K_0(\kappa r_0)I_0(\kappa r)-K_0(\kappa a)I_0(\kappa r_0)I_0(\kappa r)/I_0(\kappa a)\} \qquad r<r_0 \qquad [4a]$$

$$=\frac{S_0 e^{i\omega t}}{4\pi^2 RD}\{K_0(\kappa r)I_0(\kappa r_0)-K_0(\kappa a)I_0(\kappa r_0)I_0(\kappa r)/I_0(\kappa a)\} \qquad r>r_0 \qquad [4b]$$

where $\kappa\equiv(i\omega/D)^{1/2}$ and I and K are modified Bessel functions of a complex argument. We construct the solution to the matrix equation 2 by looking for particular solutions which satisfy the condition

$$(\tilde{n}/n_0)=\varepsilon\,(\tilde{T}/T_0) \qquad [5]$$

where ε is a constant in both space and time. Substituting equation 5 into equation 2 leads to:

$$\begin{pmatrix}\varepsilon\\ 1\end{pmatrix}\frac{\partial(\tilde{T}/T_0)}{\partial t}=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\bullet\begin{pmatrix}\varepsilon\\ 1\end{pmatrix}\bullet\nabla^2(\tilde{T}/T_0)+\begin{pmatrix}S_n\\ S_T\end{pmatrix}\bullet\frac{\delta(r-r_0)}{4\pi^2 rR}e^{i\omega t} \qquad [6]$$

If $S_n/S_T=\varepsilon$ and ε satisfies the eigenvalue equation:

$$\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\bullet\begin{pmatrix}\varepsilon\\ 1\end{pmatrix}=\lambda\bullet\begin{pmatrix}\varepsilon\\ 1\end{pmatrix} \qquad [7]$$

(there are, in general 2 solutions ε_1 and ε_2 corresponding to 2 eigenvalues λ_1 and λ_2) equation 6 will be solved by $\tilde{T}/T_0=(S_T/S_0)\,y(r,t,r_0,D)$ with $D=\lambda$. The solution for an arbitrary source vector

$$\begin{pmatrix}S_n\\ S_T\end{pmatrix}$$

can be obtained by decomposing it into the eigenvectors

$$\begin{pmatrix}\varepsilon_1\\1\end{pmatrix}, \begin{pmatrix}\varepsilon_2\\1\end{pmatrix}$$

and, for an arbitrarily distributed source the solution is obtained by making this decomposition at each radial point:

$$\begin{pmatrix}S_n(r)\\S_T(r)\end{pmatrix} \bullet e^{i\omega t} = \left\{F_1(r)\begin{pmatrix}\varepsilon_1\\1\end{pmatrix} + F_2(r)\begin{pmatrix}\varepsilon_2\\1\end{pmatrix}\right\} \bullet e^{i\omega t} \qquad [8]$$

and integrating the Green's function corresponding to each eigenvalue:

$$\tilde{T}/T_0 = \frac{4\pi^2 R}{S_0}\int (F_1(r_0)\, y(r, t, r_0, \lambda_1) + F_2(r_0)\, y(r, t, r_0, \lambda_2))\, r_0\, dr_0 \qquad [9a]$$

$$\tilde{n}/n_0 = \frac{4\pi^2 R}{S_0}\int (\varepsilon_1\, F_1(r_0)\, y(r, t, r_0, \lambda_1) + \varepsilon_2\, F_2(r_0)\, y(r, t, r_0, \lambda_2))\, r_0\, dr_0 \qquad [9b]$$

3. VALIDITY OF FAR-FIELD APPROXIMATIONS

Equation 9 gives a solution which is valid everywhere. Thus it allows us to estimate the errors introduced by assuming that the observed density and temperature modulations are made far from the particle and heat sources which generate them.

We take as an example a transport matrix similar to the one determined in a recent analysis of JET ICRF power modulation experiments /12/:

$$A = \begin{pmatrix} 0.3 & -0.6 \\ 0.2 & 1.6 \end{pmatrix} \quad (m^2 \sec^{-1}) \qquad [10]$$

Note that the definition of the transport matrix used here differs somewhat from that used in /12/. The matrix coefficients have been transformed accordingly /10/.

The eigenvalues of the transport matrix are $\lambda_1 = 1.5 m^2 \sec^{-1}$ and $\lambda_2 = 0.4 m^2 \sec^{-1}$. Although the eigenvalues are rather close to the diagonal transport coefficients A_{22} and A_{11}, the corresponding eigenvectors exhibit considerable coupling of the temperature and the density with $\varepsilon_1 = -0.5$ and $\varepsilon_2 = -6$.

The heat source is assumed to be of unit amplitude on axis and to have the form:

$$S_T(r) = e^{-(r/\alpha a)^2} \quad \sec^{-1} \qquad [11]$$

Initially we consider modulations at 4 Hz, with a = 1 m. and $\alpha = 0.2$. Figure 1 shows the amplitude and phase of the temperature and density modulations when the ratio of the particle source to the heat source, $S_n(r)/S_T(r)$, is taken to be ε_1. Figure 2 shows the corresponding results for $S_n(r)/S_T(r) = \varepsilon_2$. These cases correspond to the launching of pure eigenmodes. As expected, the ratio of the amplitudes of the density and temperature modulations, $|\tilde{n}/n_0|/|\tilde{T}/T_0|$, is radially constant (and equal to ε_1 and ε_2 respectively) and the phase difference between the density and temperature modulations, $|\Phi_n - \Phi_T|$, is 180 degrees. This also corresponds to the analytic far-field results /8,9/.

For the case corresponding to the higher eigenvalue (figure 1), the amplitude decreases more slowly with radius and the phase increases more slowly. The central temperature amplitude scales as $\lambda^{-1/2}$. These features reflect the larger transport rates associated with the higher eigenvalue.

Figure 3 shows the results for pure power modulations ($S_n = 0$). In this case, the temperature modulations generate density modulations but there is no simple relation between the amplitudes or the phases. At large radii, the relationship of figure 1 is recovered: $|\tilde{n}/n_0|/|\tilde{T}/T_0| \sim \varepsilon_1 = 0.5$, and $|\Phi_n - \Phi_T| \sim 180°$. This is because, as noted above, the amplitude of the modulations corresponding to the slower eigenmode decay more quickly with radius. Near the centre, however, $|\tilde{n}/n_0|$ approaches $|\tilde{T}/T_0|$ and Φ_n approaches Φ_T.

For the far-field approximation to be valid it is necessary that the distance from the centre of the modulated source to the point of observation, Δr, be much greater than both the source width, αa, and the characteristic transport distance, $\gamma a \equiv (\lambda/\omega)^{1/2}$. Figures 4 and 5 show the sensitivity of $|\tilde{n}/n_0|/|\tilde{T}/T_0|$ and

$|\Phi_n - \Phi_T|$ to α and γ. From these curves an estimate of the error made by using the far-field results can be obtained. If we require that the error in $|\tilde{n}/n_0|/|T/T_0|$, be less than 20% and the error in $|\Phi_n - \Phi_T|$ be less than 30°, we obtain the approximate criteria: $\Delta r/a > 2.5\,\alpha$ and $\Delta r/a > 2.5\,\gamma$. (If we relax the requirements on the error to 40% and 60° respectively, the criteria change little: $\Delta r/a > 2.\,\alpha$ and $\Delta r/a > 2.\,\gamma$). If $f = \omega/2\pi > \lambda_1/a^2$, then $\Delta r > a$ and the far-field results are not applicable anywhere. It should be noted that these criteria for the determination of coupling characteristics are more stringent than the corresponding criteria for the determination of diagonal transport coefficients in modulation experiments./13/.

REFERENCES

/1/ Lopes Cardozo N.J., de Haas J.C.M., Hogeweij G.M.D., et al.. Plasma Physics and Controlled Fusion 32(1990)983.

/2/ Gondhalekar A., Cheetham A.D., de Haas J.C.M., et al.. Plasma Physics and Controlled Fusion 31(1989)805.

/3/ Bishop C. M. et al.. Proceedings 16th European Conf. on Controlled Fusion and Plasma Physics, Venice, 1989, Vol III, p 1131.

/4/ Brower D. L., et al.. Proceedings 17th European Conf. on Controlled Fusion and Plasma Heating, Amsterdam, 1990, Vol I, p 150.

/5/ Hogeweij G.M.D., O'Rourke J. and Sips A.C.C.. Plasma Physics and Controlled Fusion 33(1991)189.

/6/ O'Rourke, J.. Nuclear Fusion 27(1987)2075.

/7/ Hossain M., et al.. Phys. Fluids 31(1988)2165.

/8/ Gentle K. W.. Phys. Fluids 31(1988)1105.

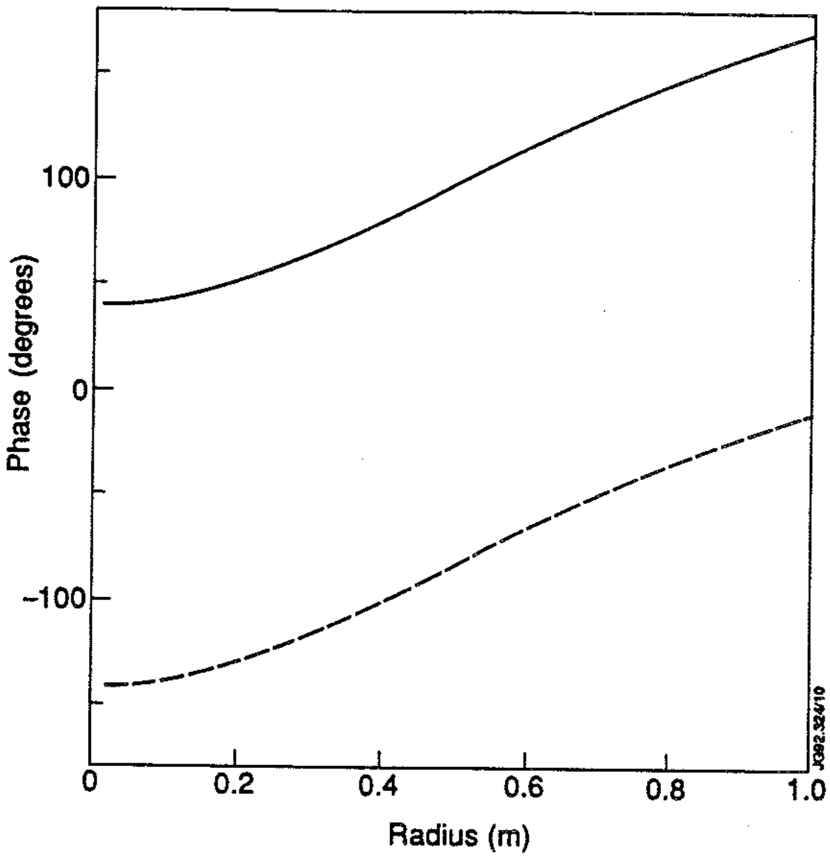
/9/ Bishop C. M. and Connor J. W.. Plasma Physics and Controlled Fusion 32(1990)203.

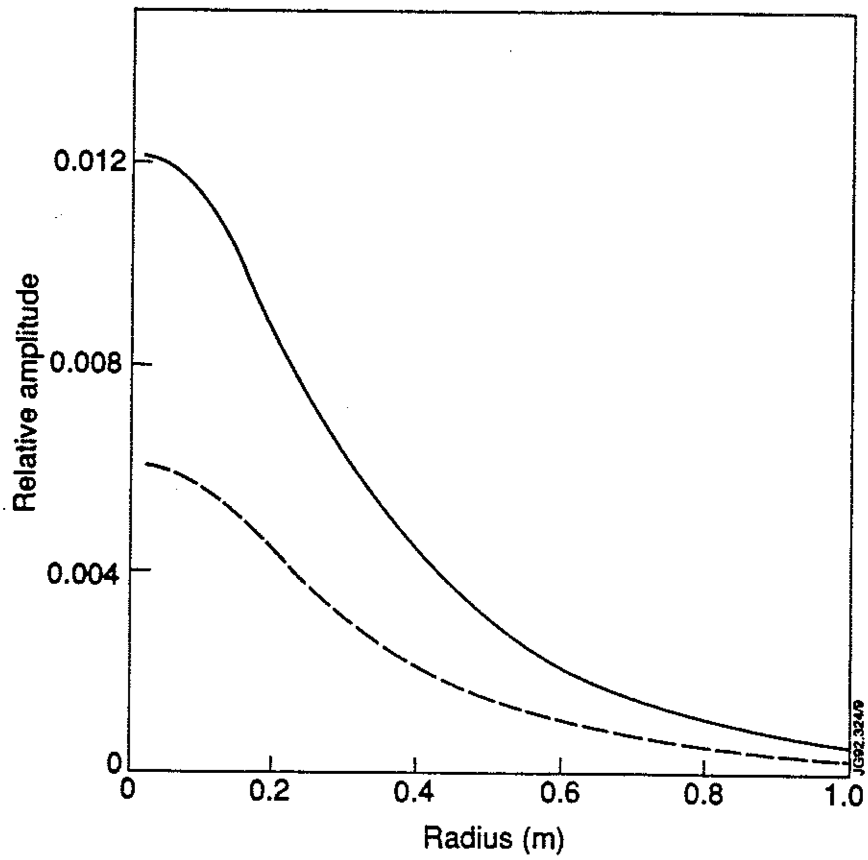
/10/ de Haas J.C.M., O'Rourke J., Sips A.C.C., et al.. Nuclear Fusion 31(1991)1261.

/11/ Hogeweij G. M. D., Lopes Cardozo N. J., De Luca F., et al.. Submitted for publication in Plasma Physics and Controlled Fusion.

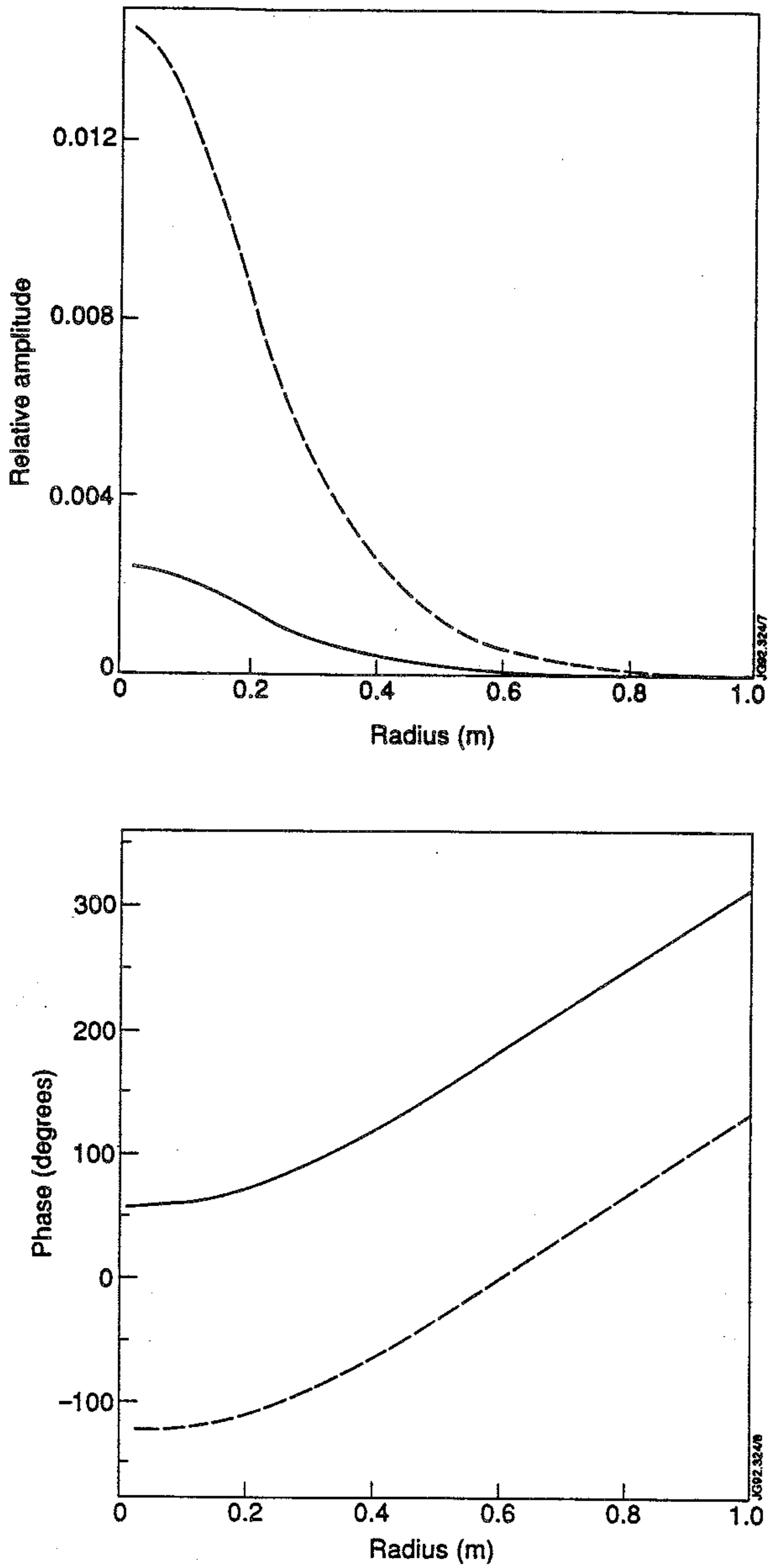
/12/ O'Rourke J, Rimini F. and Start D. F. H., submitted for publication in Nuclear Fusion. JET Report JET-P(91)47.

/13/ De Luca F., et al.. Proceedings 18th European Conf. on Controlled Fusion and Plasma Physics, Berlin, 1991, Vol I, p 277.

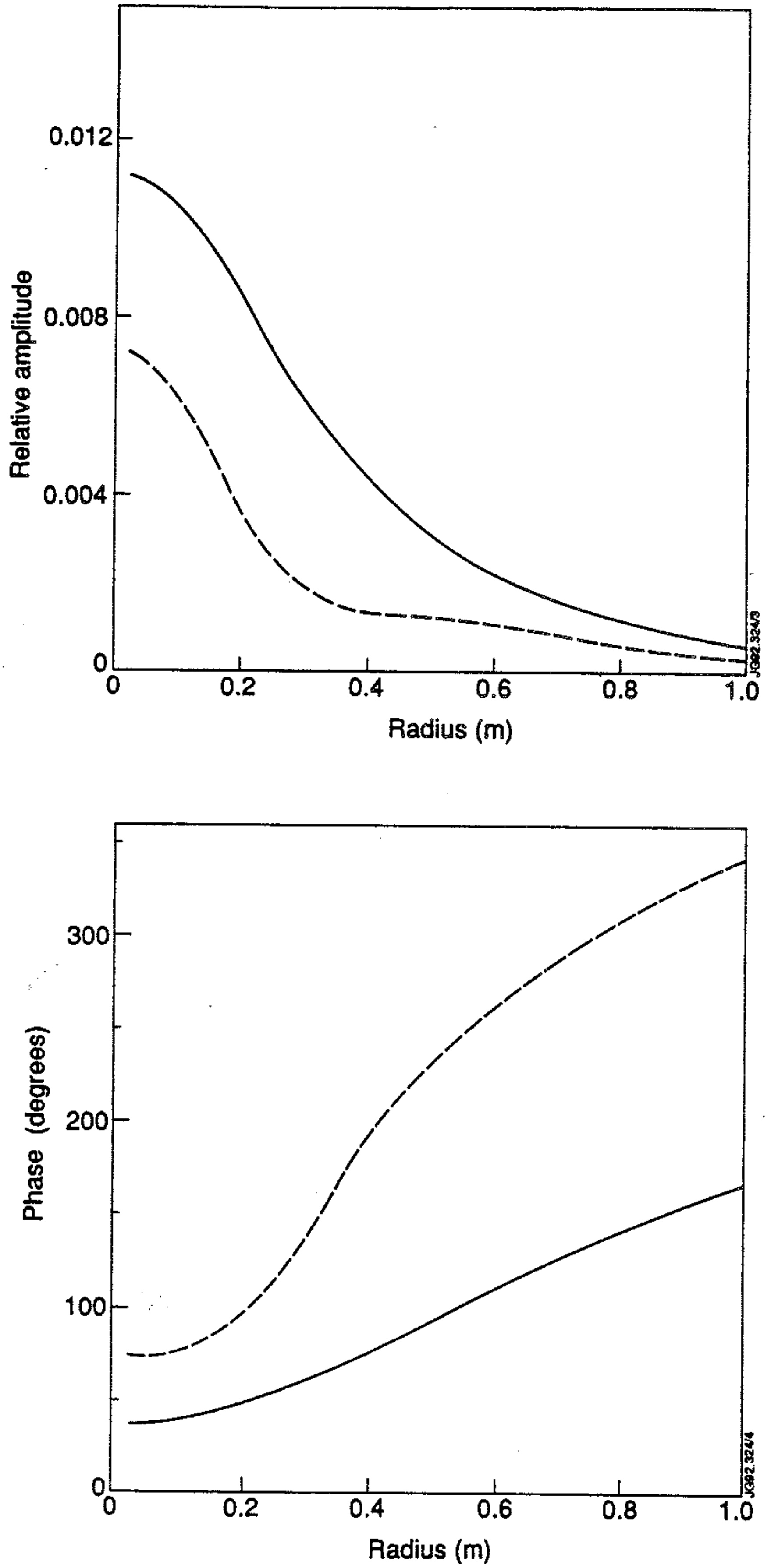




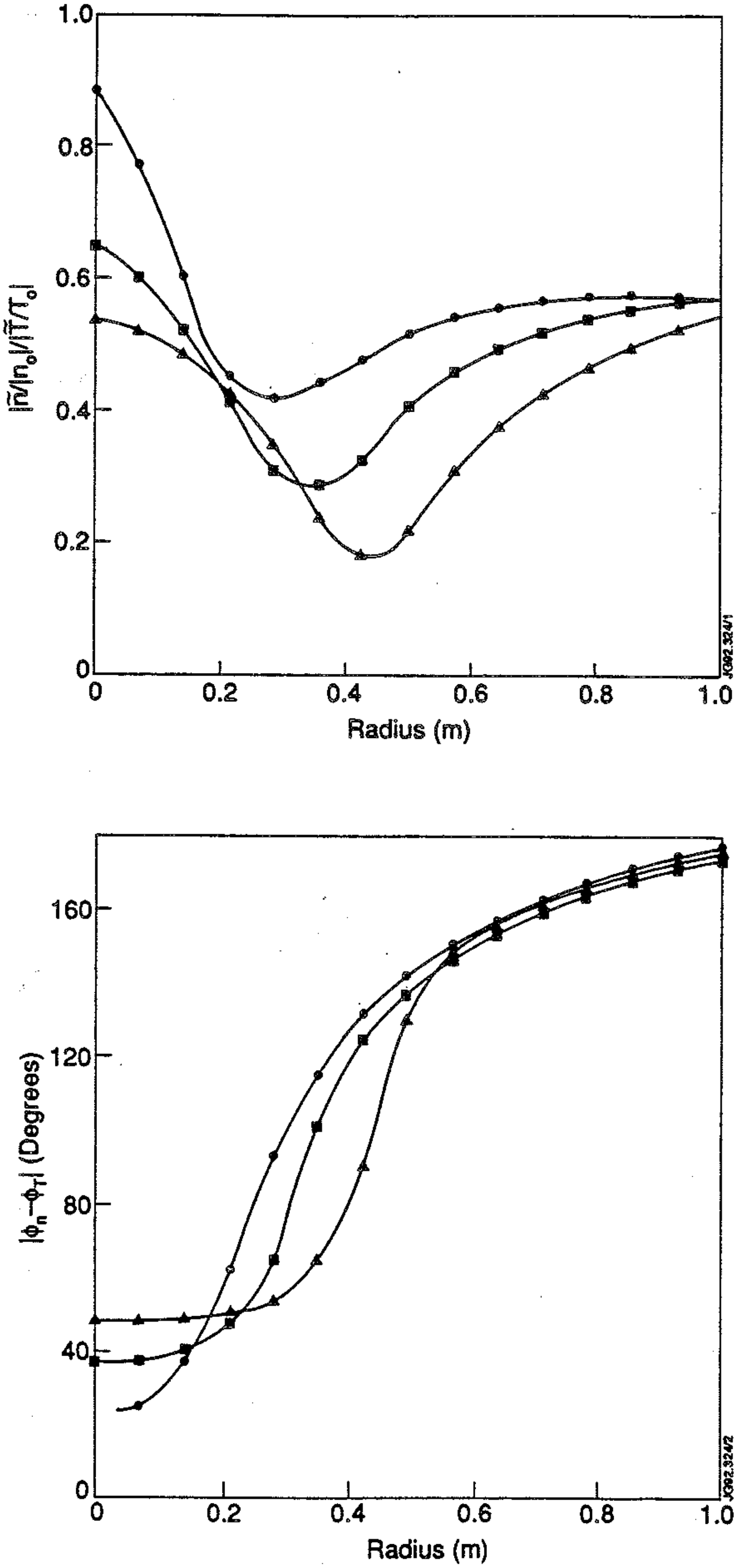
1. Modulations resulting from $S_n(r)/S_T(r) = \varepsilon_1$
(a) relative amplitude of the temperature (solid line) and density (dashed line) versus minor radius.
(b) phase of the temperature (solid line) and density (dashed line) versus minor radius.



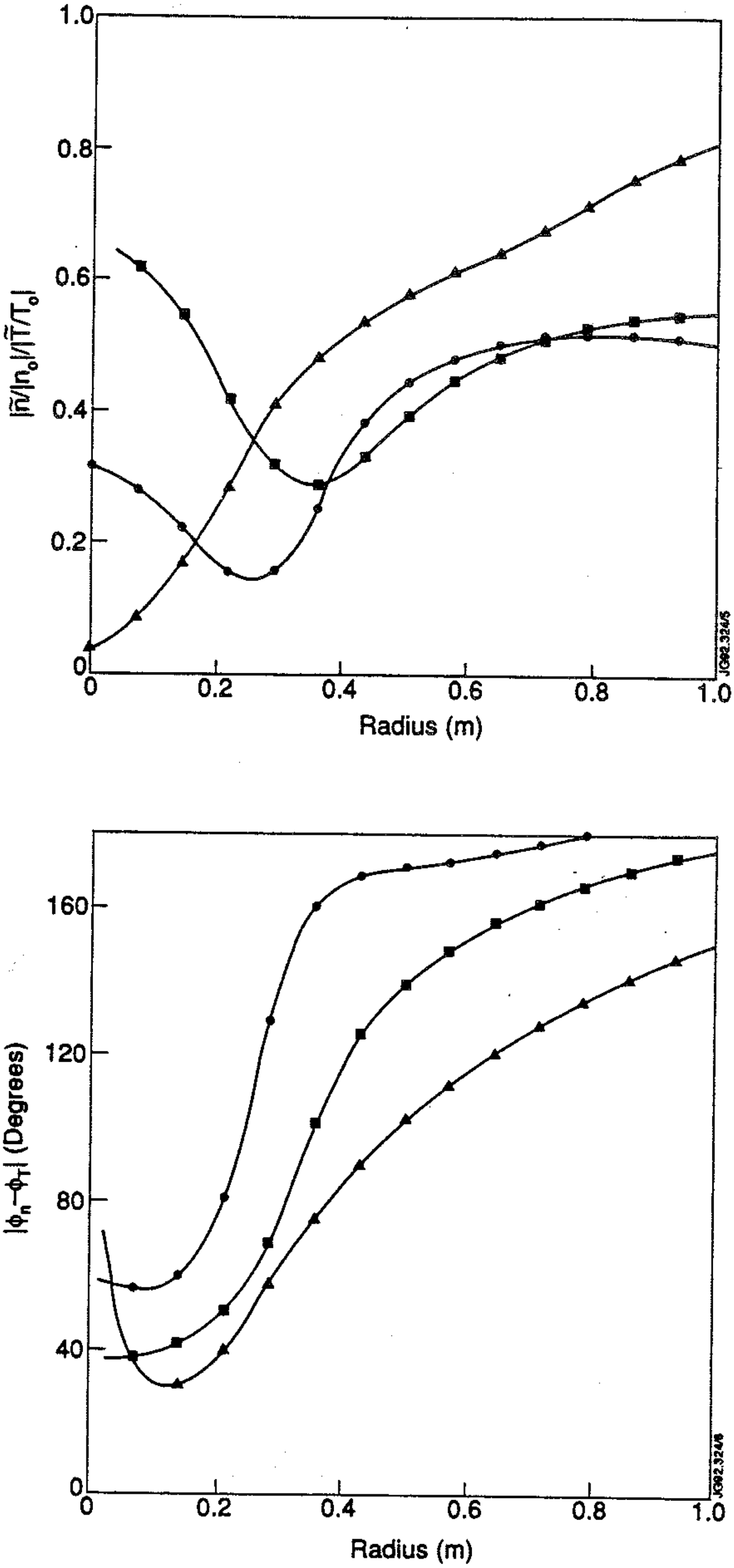
2. Modulations resulting from $S_n(r)/S_T(r) = \varepsilon_2$.
(a) relative amplitude of the temperature (solid line) and density (dashed line) versus minor radius.
(b) phase of the temperature (solid line) and density (dashed line) versus minor radius.



3. Modulations resulting from $S_n(r)/S_T(r) = 0$.
(a) relative amplitude of the temperature (solid line) and density (dashed line) versus minor radius.
(b) phase of the temperature (solid line) and density (dashed line) versus minor radius.



4. (a) ratio of the amplitudes of the density and temperature modulations for $S_n(r) = 0$. $\alpha = 0.1$ (circles), $\alpha = 0.2$ (squares), $\alpha = 0.3$ (triangles).
(b) phase difference between the density and temperature modulations. Legend as for (a).



5. (a) ratio of the amplitudes of the density and temperature modulations for $S_n(r) = 0$. $\gamma = 0.12(f = 16Hz)$ (circles), $\gamma = 0.24(f = 4Hz)$ (squares), $\gamma = 0.48(f = 1Hz)$ (triangles).
(b) phase difference between the density and temperature modulations. Legend as for (a).

ANNEX

P.-H. REBUT, A. GIBSON, M. HUGUET, J.M. ADAMS[1], B. ALPER, H. ALTMANN, A. ANDERSEN[2], P. ANDREW[3], M. ANGELONE[4], S. ALI-ARSHAD, P. BAIGGER, W. BAILEY, B. BALET, P. BARABASCHI, P. BARKER, R. BARNSLEY[5], M. BARONIAN, D.V. BARTLETT, L. BAYLOR[6], A.C. BELL, G. BENALI, P. BERTOLDI, E. BERTOLINI, V. BHATNAGAR, A.J. BICKLEY, D. BINDER, H. BINDSLEV[2], T. BONICELLI, S.J. BOOTH, G. BOSIA, M. BOTMAN, D. BOUCHER, P. BOUCQUEY, P. BREGER, H. BRELEN, H. BRINKSCHULTE, D. BROOKS, A. BROWN, T. BROWN, M. BRUSATI, S. BRYAN, J. BRZOZOWSKI[7], R. BUCHSE[22], T. BUDD, M. BURES, T. BUSINARO, P. BUTCHER, H. BUTTGEREIT, C. CALDWELL-NICHOLS, D.J. CAMPBELL, P. CARD, G. CELENTANO, C.D. CHALLIS, A.V. CHANKIN[8], A. CHERUBINI, D. CHIRON, J. CHRISTIANSEN, P. CHUILON, R. CLAESEN, S. CLEMENT, E. CLIPSHAM, J.P. COAD, I.H. COFFEY[9], A. COLTON, M. COMISKEY[10], S. CONROY, M. COOKE, D. COOPER, S. COOPER, J.G. CORDEY, W. CORE, G. CORRIGAN, S. CORTI, A.E. COSTLEY, G. COTTRELL, M. COX[11], P. CRIPWELL[12], O. Da COSTA, J. DAVIES, N. DAVIES, H. de BLANK, H. de ESCH, L. de KOCK, E. DEKSNIS, F. DELVART, G.B. DENNE-HINNOV, G. DESCHAMPS, W.J. DICKSON[13], K.J. DIETZ, S.L. DMITRENKO, M. DMITRIEVA[14], J. DOBBING, A. DOGLIO, N. DOLGETTA, S.E. DORLING. P.G. DOYLE, D.F. DÜCHS, H. DUQUENOY, A. EDWARDS, J. EHRENBERG, A. EKEDAHL, T. ELEVANT[7], S.K. ERENTS[11], L.G. ERIKSSON, H. FAJEMIROKUN[12], H. FALTER, J. FREILING[15], F. FREVILLE, C. FROGER, P. FROISSARD, K. FULLARD, M. GADEBERG, A. GALETSAS, T. GALLAGHER, D. GAMBIER, M. GARRIBBA, P. GAZE, R. GIANNELLA, R.D. GILL, A GIRARD, A. GONDHALEKAR, D. GOODALL[11], C. GORMEZANO, N.A. GOTTARDI, C. GOWERS, B.J. GREEN, B. GRIEVSON, R. HAANGE, A. HAIGH, C.J. HANCOCK, P.J. HARBOUR, T. HARTRAMPF, N.C. HAWKES[11], P. HAYNES[11], J.L. HEMMERICH, T. HENDER[11], J. HOEKZEMA, D. HOLLAND, M. HONE, L. HORTON, J. HOW, M. HUART, I. HUGHES, T.P. HUGHES[10], M. HUGON, Y. HUO[16], K. IDA[17], B. INGRAM, M. IRVING, J. JACQUINOT, H. JAECKEL, J.F. JAEGER, G. JANESCHITZ, Z. JANKOVICZ[18], O.N. JARVIS, F. JENSEN, E.M. JONES, H.D. JONES, L.P.D.F. JONES, S. JONES[19], T.T.C. JONES, J.-F. JUNGER, F. JUNIQUE, A. KAYE, B.E. KEEN, M. KEILHACKER, G.J. KELLY, W. KERNER, A. KHUDOLEEV[21], R. KONIG, A. KONSTANTELLOS, M. KOVANEN[20], G. KRAMER[15], P. KUPSCHUS, R. LÄSSER, J.R. LAST, B. LAUNDY, L. LAURO-TARONI, M. LAVEYRY, K. LAWSON[11], M. LENNHOLM, J. LINGERTAT[22], R.N. LITUNOVSKI, A. LOARTE, R. LOBEL, P. LOMAS, M. LOUGHLIN, C. LOWRY, J. LUPO, A.C. MAAS[15], J. MACHUZAK[19], B. MACKLIN, G. MADDISON[11], C.F. MAGGI[23], G. MAGYAR, W. MANDL[22], V. MARCHESE, G. MARCON, F. MARCUS, J. MART, D. MARTIN, E. MARTIN, R. MARTIN-SOLIS[24], P. MASSMANN, G. MATTHEWS, H. McBRYAN, G. McCRACKEN[11], J. McKIVITT, P. MERIGUET, P. MIELE, A. MILLER, J. MILLS, S.F. MILLS, P. MILLWARD, P. MILVERTON, E. MINARDI[4], R. MOHANTI[25], P.L. MONDINO, D. MONTGOMERY[26], A. MONTVAI[27], P. MORGAN, H. MORSI, D. MUIR, G. MURPHY, R. MYRNÄS[28], F. NAVE[29], G. NEWBERT, M. NEWMAN, P. NIELSEN, P. NOLL, W. OBERT, D. O'BRIEN, J. ORCHARD, J. O'ROURKE, R. OSTROM, M. OTTAVIANI, M. PAIN, F. PAOLETTI, S. PAPASTERGIOU, W. PARSONS, D. PASINI, D. PATEL, A. PEACOCK, N. PEACOCK[11], R.J.M. PEARCE, D. PEARSON[12], J.F. PENG[16], R. PEPE DE SILVA, G. PERINIC, C. PERRY, M. PETROV[21], M.A. PICK, J. PLANCOULAINE, J.-P POFFÉ, R. PÖHLCHEN, F. PORCELLI, L. PORTE[13], R. PRENTICE, S. PUPPIN, S. PUTVINSKII[8], G. RADFORD[30], T. RAIMONDI, M.C. RAMOS DE ANDRADE, R. REICHLE, J. REID, S. RICHARDS, E. RIGHI, F. RIMINI, D. ROBINSON[11], A. ROLFE, R.T. ROSS, L. ROSSI, R. RUSS, P. RUTTER, H.C. SACK, G. SADLER, G. SAIBENE, J.L. SALANAVE, G. SANAZZARO, A. SANTAGIUSTINA, R. SARTORI, C. SBORCHIA, P. SCHILD, M. SCHMID, G. SCHMIDT[31], B. SCHUNKE, S.M. SCOTT, L. SERIO, A SIBLEY, R. SIMONINI, A.C.C. SIPS, P. SMEULDERS, R. SMITH, R. STAGG, M. STAMP, P. STANGEBY[3], R. STANKIEWICZ[32], D.F. START, C.A. STEED, D. STORK, P.E. STOTT, P. STUBBERFIELD, D. SUMMERS, H. SUMMERS[13], L. SVENSSON, J.A. TAGLE[33], M. TALBOT, A. TANGA, A. TARONI, C. TERELLA, A TERRINGTON, A TESINI, P.R. THOMAS, E. THOMPSON, K. THOMSEN, F. TIBONE, A. TISCORNIA, P. TREVALION, B. TUBBING, P. VAN BELLE, H. VAN DER BEKEN, G. VLASES, M. VON HELLERMANN, T. WADE, C. WALKER, R. WALTON[31], D. WARD, M.L. WATKINS, N. WATKINS, M.J. WATSON, S. WEBER[34], J. WESSON, T.J. WIJNANDS, J. WILKS, D. WILSON, T. WINKEL, R. WOLF, D. WONG, C. WOODWARD, Y. WU[35], M. WYKES, D. YOUNG, I.D. YOUNG, L. ZANNELLI, A. ZOLFAGHARI[19], W. ZWINGMANN

[1] Harwell Laboratory, UKAEA, Harwell, Didcot, Oxfordshire, UK.
[2] Risø National Laboratory, Roskilde, Denmark.
[3] Institute for Aerospace Studies, University of Toronto, Downsview, Ontario, Canada.
[4] ENEA Frascati Energy Research Centre, Frascati, Rome, Italy.
[5] University of Leicester, Leicester, UK.
[6] Oak Ridge National Laboratory, Oak Ridge, TN, USA.
[7] Royal Institute of Technology, Stockholm, Sweden.
[8] I.V. Kurchatov Institute of Atomic Energy, Moscow, Russian Federation.
[9] Queens University, Belfast, UK.
[10] University of Essex, Colchester, UK.
[11] Culham Laboratory, UKAEA, Abingdon, Oxfordshire, UK.
[12] Imperial College of Science, Technology and Medicine, University of London, London, UK.
[13] University of Strathclyde, Glasgow, UK.
[14] Keldysh Institute of Applied Mathematics, Moscow, Russian Federation.
[15] FOM-Institute for Plasma Physics "Rijnhuizen", Nieuwegein, Netherlands.
[16] Institute of Plasma Physics, Academia Sinica, Hefei, Anhui Province, China.
[17] National Institute for Fusion Science, Nagoya, Japan.
[18] Soltan Institute for Nuclear Studies, Otwock/Świerk, Poland.
[19] Plasma Fusion Center, Massachusetts Institute of Technology, Boston, MA, USA.
[20] Nuclear Engineering Laboratory, Lappeenranta University, Finland.
[21] A.F. Ioffe Physico-Technical Institute, St. Petersburg, Russian Federation.
[22] Max-Planck-Institut für Plasmaphysik, Garching, Germany.
[23] Department of Physics, University of Milan, Milan, Italy.
[24] Universidad Complutense de Madrid, Madrid, Spain.
[25] North Carolina State University, Raleigh, NC, USA.
[26] Dartmouth College, Hanover, NH, USA.
[27] Central Research Institute for Physics, Budapest, Hungary.
[28] University of Lund, Lund, Sweden.
[29] Laboratorio Nacional de Engenharia e Tecnologia Industrial, Sacavem, Portugal.
[30] Institute of Mathematics, University of Oxford, Oxford, UK.
[31] Princeton Plasma Physics Laboratory, Princeton University, Princeton, NJ, USA.
[32] RCC Cyfronet, Otwock/Świerk, Poland,
[33] Centro de Investigaciones Energéticas, Medioambientales y Tecnológicas, Madrid, Spain.
[34] Freie Universität, Berlin, Germany.
[35] Insitute for Mechanics, Academia Sinica, Beijing, China.