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** See Appendix 1*

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1. Introduction

Considerable evidence is emerging from experiments covering a wide range of physical situations, from the reversed field pinches and the spheromaks to the medium and large size tokamaks, which indicates the tendency of the plasma to assume preferential configurations, largely independent of external conditions. This behaviour suggests that, beyond the complexity of the various modes and instabilities, some global constraint holds, which expresses a basic physical property of the plasma.

In the case of a collisional plasma a privileged equilibrium notoriously exists, that is the Maxwellian equilibrium associated with a maximum of the entropy. The idea that privileged equilibria could exist also in the case of a collisionless plasma, such as to be expressible, under certain conditions, as the extremum of an entropy, suitably extended to the collisionless collective (static) equilibria (henceforth called "Vlasov equilibria") has been the driving motive of our research.

By formulating a statistical model, whose basic objects are not particles, but volume elements (containing many particles) forming a statistical system to be considered in a configurational space of the electric charge or of the current density, an entropy functional $S = - \int P \lg P \, d\Gamma$ of the collective quantities pertaining to a suitably defined Vlasov configuration was introduced, where P is probability distribution in the space Γ above. The variational properties of S are related to relevant equilibrium and stability properties of the electrostatic (Minardi 1972, 1973) and of the magnetic Vlasov equilibria (the pinch, Minardi 1981, 1989; the tokamak, 1990, 1992).

On the light of the more recent applications, one gains a more mature outlook of the physical context and of the conceptual basis of the original model. The purpose of the present paper is to retake the model ex novo, so as to take profit of the new perspective for clarifying, exemplifying (and in some cases for amending) the previous arguments, by showing at the same time how the entropy of the Vlasov equilibria and the related variational procedure can be inserted in the conceptual context of classical thermodynamics.

Sections 2 to 5 contain the formulation of the model and the derivation of the entropy functional. The calculations are presented in detail. In section 2 the statistical model is formulated. The basic constraint in the statistical procedure expresses the absence of energy interaction between the collective configuration in Vlasov equilibrium and an underlying neutral medium of individual fluctuating particles. In fact this constraint constitutes the very definition of Vlasov equilibrium in our phenomenological scheme.

In section 3 the model is shown to lead to a thermodynamic formalism similar to that based on the canonical ensemble. In section 4 we introduce the so called "thermodynamic limit" in which the number of objects of the statistics (the volume elements) as well as the total volume of the system tend to infinity. This procedure will allow the separation of the collective effects from the effects of the infinite background of individual particles. In the absence of collective fields the background is electrically neutral and currentless in the average, but it is the site of fluctuations of the charge and of the current density, due to its particle structure. The concept of background will allow (section 5) the introduction, in the thermodynamic limit, of the concept of "isothermal variation" and fulfils, in a certain sense, the same function as the "heat bath" in the statistics of Gibbs. This point of view is developed further in section 6, where we study in detail the energy interaction due to charge exchange between the collective system and the background and show that the interaction process is reversible. The reversibility enables us to define (section 7) in the conventional way an entropy attached to the reversibly accessible states and to relate the variations of the collective entropy to the transfer of heat, so that the entropy functional acquires the same meaning as in classical thermodynamics of macroscopic systems.

Once the entropy of the collective system is defined through the reversible interaction with the background, we can study entropy increasing irreversible variations of the collective system considered as isolated. This allows the introduction of the concept of "thermodynamic instability": an isolated collective system in Vlasov equilibrium is declared to be unstable when it admits neighbouring accessible states with higher entropy.

In the next sections we apply our thermodynamic concepts to practical systems. In section 8 we discuss cases of Vlasov electrostatic equilibria, both in unstable situations and in the Maxwellian limit. In section 9 we study examples of magnetostatic equilibria namely the case of the reversed field pinch, considered as isolated from the external world by a closed conductive shell, and the case of the tokamak, which is energetically open to the external world. One aim of our discussion is the illustration of the relationship between the thermodynamic instability and the instabilities resulting from the dynamic treatment. Another point is the illustration of the thermodynamic meaning of the boundary conditions to be applied in the variational treatment of the entropy. When the collective system is isolated, the boundary conditions should express the absence of energy exchanges with the external world represented by the background. In the case of an open system, the boundary conditions should describe properly the energy balance between the plasma and the external systems.

Finally, in section 10 we extend the concept of collective entropy in order to include the description of those properties of the Vlasov equilibria which depend on the local structure of the magnetic configuration (for instance the interchange modes) and apply the extended concept to the discussion of the thermodynamic properties of the tearing modes.

2. The Statistics of the Plasma Volume Elements in Information Space

The electrostatic or magnetic Vlasov equilibria of a plasma with given boundary conditions are completely described when the macroscopic distribution of the collective charge and current density is known. Any collective quantity pertaining to the equilibrium should then be expressed as some functional of these density variables. This is, of course, true also for those quantities, as for instance the entropy, whose basic definition involves the particle structure of the plasma and the statistical behaviour of the system in phase space.

It is then natural to ask whether a kind of short circuiting procedure can be found for constructing the entropy of a Vlasov configuration, such as to involve from the beginning only the information which is relevant for the problem at hand.

In order to proceed with this task we shall take the point of view of the information theory (see e.g. Jaynes 1957). We start from the information that a collective charge or current density distribution is given and that the energy of the corresponding electrostatic or magnetostatic configuration is uncorrelated, in the average, to the fluctuations of the quantities arising from the particle structure of the system, that is the energy of a Vlasov equilibrium can be defined independently of the effects of this structure.

We shall illustrate our procedure by considering first the electrostatic case which is formally simpler than the magnetic case to which the formalism can be easily extended afterwards.

We shall consider a plasma of volume V and a large assembly of N volume elements $\Delta V = V/N$, where ΔV is large enough to contain many particles. The N volume elements are the basic objects of our statistics. Let us introduce a four-dimensional space for the values of the charge density σ and for the position $\vec{x}$ in the volume V. Now take one volume element from the assembly, call it ΔV_j and throw it at random into a copy of the space above, called S_j. The volume element will take a certain position denoted by $(\vec{x}_j, \sigma_j)$ ($\vec{x}_j$ is for instance the centre of mass of ΔV_j). Do the same for another volume element ΔV_i, in another space S_i, taking care that ΔV_i can only occupy in S_i a position $\vec{x}_i \neq \vec{x}_j$ because the volume elements cannot overlap in V. Do this for all N volume elements. At the end one will get a single point in the 4N dimensional space Γ_I which is the product of all S_j. This point represents a particular electrostatic configuration of the plasma, reconstructed in the volume V with a coarse-graining ΔV.

The space Γ_I is called "information space" and the σ_j are the "information variables". Our purpose is to calculate the probability $P(\sigma_1,.., \sigma_N; \vec{x}_1,.., \vec{x}_N)$ for the assembly of the N volume elements to occupy at random any given volume element $d\Gamma_I$ in information space. The form of $d\Gamma_I$ is easily derived. Indeed the volume accessible to the volume element j = 1 is $V = N\Delta V$, that accessible to the volume elements j = 2, j = 3, ... is $V - \Delta V = (N-1)\,\Delta V$, $(V-2\,\Delta V) = (N-2)\,\Delta V$, ... Then the volume accessible to the assembly of N volume elements is

$$[N(N-1)(N-2)...1]\,(\Delta V)^N = N!\ \Delta V^N \qquad (2.1)$$

This must be divided by N! because a permutation among the N volume elements has no physical effect and must not contribute to the total volume element in information space. Hence $d\Gamma_I$ is given by the expression

$$d\Gamma_I = \Delta V^N\, d\sigma_1 .... d\sigma_N \qquad (2.2)$$

In order to proceed to the calculation of $P(\sigma_i; \vec{x}_i)$ we must introduce the constraints on the random process considered above. One constraint should express the information characterizing the Vlasov equilibrium. To discuss this, let us look more closely at the form of the random charge density σ. The collective electrostatic configuration, as we said, is characterized by a collective charge density distribution $\sigma(\vec{x})$ which we suppose to be given. This collective distribution is, however, always superimposed on a fluctuating charge density $\tilde{\sigma}$ related to the particle structure which can never be completely eliminated. It is just this fluctuating part which introduces the random character of the problem, so that in fact the charge density in the volume element ΔV_j is a random variable $\sigma_j = \sigma(\vec{x}_j) + \tilde{\sigma}_j$.

Even in the absence of any collective excitation, fluctuations related to a population of individual particles are always present. It is only after suitable averaging over these fluctuations that the macroscopic properties of the system emerge. On the other hand, the collective equilibrium is physically quite independent of the background of individual fluctuating particles. In particular, let us consider the average interaction energy ϕ_{int} between the charge density fluctuations $\tilde{\sigma}$ and the potential $\varphi(\vec{x})$ created by the collective charge density $\sigma(\vec{x})$:

$$\phi_{int} \equiv \tfrac{1}{2}\int_{\Gamma_I} P(\sigma_i;\vec{x}_i)\sum_{1}^{N}{}_j\,\tilde{\sigma}_j\,\varphi(\vec{x}_j)\Delta V_j d\Gamma_I \qquad (1 \le i \le N) \tag{2.3}$$

(One can express ϕ_{int} with the form $\sigma(\vec{x})\,\tilde{\varphi}$ rather than with $\tilde{\sigma}\,\varphi(\vec{x})$; under reasonable boundary conditions the result would be the same, but the latter form is more convenient). The fact that in a Vlasov equilibrium the fluctuating background should not interact with the collective electrostatic energy, so that this energy can be specified independently of any effect related to the random fluctuations, is expressed by the condition $\phi_{int} = 0$. Indeed let us write

$$\phi_{int} + \phi = \tfrac{1}{2}\int_{\Gamma_I} P(\sigma_i;\vec{x}_i)\sum_{1}^{N}{}_j(\tilde{\sigma}_j + \sigma(\vec{x}_j))\varphi(\vec{x}_j)\Delta V_j\, d\Gamma_I \tag{2.4}$$

where, in view of the normalization condition

$$\int_{\Gamma_I} P(\sigma_i;\vec{x}_i)d\Gamma_I = 1 \tag{2.5}$$

ϕ is given by the expression

$$\phi = \tfrac{1}{2}\sum_{1}^{N}{}_j\,\sigma(\vec{x}_j)\varphi(\vec{x}_j)\Delta V_j \tag{2.6}$$

and is the part of the electrostatic energy which belongs to the collective configuration. Thus the condition $\phi_{int} = 0$ characterizes an equilibrium which is purely collective in the sense that the energy interaction with the electric charge fluctuations of the individual particles of the background is absent. So $\phi_{int} = 0$ constitutes the very definition of "Vlasov equilibrium" in our phenomenological picture. Naturally one can consider in general variations such that $\delta\,\phi_{int} \neq 0$ around a Vlasov equilibrium with $\phi_{int} = 0$. This involves an energy exchange between the varied collective configuration and the background so that the system is no longer in a "pure" Vlasov equilibrium. In practice, as we shall see, $\delta\phi_{int} \neq 0$ will simulate the energy change of the collective configuration in interaction with the external world.

A further constraint on P arises naturally from the fact that σ, in view of its component $\tilde{\sigma}$ is a random variable whose values are then subject to dispersion. It is convenient to characterize the dispersion by fixing the value $\overline{\Delta\sigma^2}$ of the following average quadratic form

$$\overline{\Delta\sigma^2} = \frac{1}{N}\int_{\Gamma_I} P(\sigma_i;\vec{x}_i)\sum_{1}^{N}{}_j\sigma_j^2 d\Gamma_I \tag{2.7}$$

We shall see later (section 4) that this form is indeed identical in the proper limit (the thermodynamic limit) to the conventional definition of the variance, as $\overline{\sigma^2} - \overline{\sigma}^2$. Although $\overline{\Delta\sigma^2}$ is to be ascribed physically to the particle structure, it will not be necessary to know its specific form in terms of the particle fluctuations and it will be treated merely as a parameter.

The probability P $(\sigma_i;\vec{x}_i)$ can now be determined from the requirement that the entropy

$$S = -\int_{\Gamma_I} P\ln P\, d\Gamma_I \tag{2.8}$$

is stationary with respect to variations of P under the three constraints (2.7), (2.5) and (2.4) with $\phi_{int} = 0$. These constraints and the variational condition above on the entropy constitute the formal postulational basis of our model. A further specification of the model is a matter of physical interpretation rather than of new basic assumptions.

3. The Thermodynamic Formalism

We shall apply the constraint (2.4) in its general form with $\phi_{int} \neq 0$ and only at the end of the calculation we shall specify to the case $\phi_{int} = 0$. This will give us the possibility of including in the description the variations $\delta\,\phi_{int} \neq 0$ around a Vlasov equilibrium with $\phi_{int} = 0$.

Applying the technique of the Lagrange multipliers, one is led to find the extremum of the following functional

$$\hat{S} = -\int_{\Gamma_I} P\lg P d\Gamma_I - \alpha\int_{\Gamma_I} P\sum{}_j\sigma_j{}^2 d\Gamma_I - \frac{1}{\tau}\int_{\Gamma_I} P\sum{}_j\sigma_j\varphi(\vec{x}_j)\Delta V_j d\Gamma_I - \gamma\int_{\Gamma_I} P d\Gamma_I \tag{3.1}$$

where α, γ, τ are constants to be determined later in order to satisfy the constraints. The vanishing of the first variation of $\hat{S}$ with respect to arbitrary variations of P gives

$$\mathrm{P} = \mathrm{P}_o \exp -\left[\alpha\sum{}_j\sigma_j{}^2 + (1/\tau)\sum{}_j\sigma_j\varphi(\vec{x}_j)\Delta V_j\right] \tag{3.2}$$

when $P_o = \exp - (\gamma+1)$. Let us put

$$\alpha\sigma_j^2 + \frac{1}{\tau}\sigma_j\varphi(\vec{x}_j) = \alpha(\sigma_j - \overline{\sigma_j})^2 + C_j \tag{3.3}$$

where $\overline{\sigma}_j$ and C_j are independent of σ_j and given by the expressions

$$\overline{\sigma}_j = -\frac{1}{2\alpha\tau}\varphi(\vec{x}_j)\Delta V_j \,, C_j = -\frac{1}{4\alpha\tau^2}\varphi^2(\vec{x}_j)\Delta V_j^2 \tag{3.4}$$

The Γ_I integrations then reduce to integrations over gaussian distributions. Applying the normalization condition (2.5) one obtains

$$P = \frac{\exp - \alpha\sum_j(\sigma_j - \overline{\sigma}_j)^2}{\Delta V^N(\frac{\pi}{\alpha})^{N/2}} = \frac{\exp - \alpha(\tilde{\sigma}_j - \overline{\tilde{\sigma}}_j)^2}{\Delta V^N(\frac{\pi}{\alpha})^{N/2}} \tag{3.5}$$

One can verify the relations

$$\overline{\sigma}_i = \sigma(\vec{x}_i) + \overline{\tilde{\sigma}}_i = \int P\sigma_i d\Gamma_I = -\frac{1}{2\alpha\tau}\varphi(\vec{x}_i)\Delta V_i \tag{3.6a}$$

$$\phi_{int} = \frac{1}{2}\sum_j \overline{\sigma}_j \varphi(\vec{x}_j)\Delta V_j = -\phi - \frac{\Delta V^2}{4\alpha\tau}\sum_j \varphi^2(\vec{x}_j) \tag{3.6b}$$

$$\overline{\sigma_i^2} = \int P\sigma_i^2 d\Gamma_I = \frac{1}{2\alpha} + \overline{\sigma}_i^2 = \frac{1}{2\alpha} + \frac{\Delta V_i^{\;2}}{4\alpha^2\tau^2}\varphi^2(\vec{x}_i) \tag{3.6c}$$

$$\overline{\Delta\sigma^2} = \frac{1}{N}\sum_j \int P\sigma_j^2 d\Gamma_I = \frac{1}{2\alpha} + \frac{\Delta V^2}{4\alpha^2\tau^2 N}\sum_j \varphi^2(\vec{x}_j) \tag{3.6d}$$

Combining (3.6d) with (3.6b) one obtains

$$\begin{gathered} \alpha = \frac{1}{2\overline{\Delta\sigma^2}}\left[1 - \frac{2}{N\tau}(\phi + \phi_{int})\right] \\ \tau = -\frac{\Delta V}{4\alpha}\cdot\frac{\sum_j \varphi^2(\vec{x}_j)\Delta V_j}{\phi_{int} + \phi} \end{gathered} \tag{3.7}$$

We can now proceed to the calculation of the entropy

$$S = -\int P \lg P d\Gamma_I = N\lg\left[\Delta V\left(\frac{\pi}{\alpha}\right)^{N/2}\right] + \alpha\int P\sum_j(\sigma_j - \overline{\sigma}_j)^2 d\Gamma_I =$$

$$= N\lg\left[\Delta V\left(\frac{\pi}{\alpha}\right)^{N/2}\right] + \alpha N\overline{\Delta\sigma^2} - \frac{\Delta V^2}{4\alpha\tau^2}\sum_j \varphi^2(\vec{x}_j) = \tag{3.8}$$

$$= N\lg\left[\Delta V\left(\frac{\pi}{\alpha}\right)^{N/2}\right] + \frac{N}{2} = -\frac{N}{2}\lg\alpha + N\lg(\Delta V\pi^{1/2}) + \frac{N}{2} =$$

$$= -\frac{N}{2}\lg\left[\frac{1}{2\Delta\sigma^2}(1-\frac{2}{N\tau}(\phi+\phi_{int}))\right] + N\lg(\Delta V\pi^{1/2}) + \frac{N}{2} =$$

$$= N\lg\left[\Delta V(2\pi\overline{\Delta\sigma^2})^{1/2}\right] + \frac{N}{2} - \frac{N}{2}\lg(1-\frac{2}{N\tau}(\phi+\phi_{int}))$$

Let us calculate the free energy, in accordance with the usual definition F = $\tau \lg P_0$. From (2.5) and (3.2) one has

$$P_o = \frac{\exp\Sigma_j C_j}{\Delta V^N (\frac{\pi}{\alpha})^{N/2}} \tag{3.9}$$

Thus recalling (3.4), (3.6b) and (3.6d) on obtains

$$F = \tau\lg P_o = \tau\Sigma_j C_j - N\tau\lg(\Delta V\pi^{1/2}) + \frac{N\tau}{2}\lg\alpha =$$

(3.10)

$$= \phi + \phi_{int} - N\tau\lg\left[\Delta V(2\pi\overline{\Delta\sigma^2})^{1/2}\right] + \frac{N\tau}{2}\lg(1-\frac{2}{N\tau}(\phi+\phi_{int}))$$

Comparison with (3.8) gives the following relation between F and S.

$$F = -\tau S + \phi + \phi_{int} + \frac{N}{2}\tau \tag{3.11}$$

4. Thermodynamic Limit

A more expressive form for the entropy is obtained by separating the part corresponding to the collective configuration from the part related to the fluctuating background. Both effects are contained in $\overline{\Delta\sigma^2}$ which can be split as follows:

$$\overline{\Delta\sigma^2} = \frac{1}{N}\int P\Sigma_j(\sigma(\vec{x}_j)+\tilde{\sigma}_j)^2 d\Gamma_l = \frac{1}{N}\int P\Sigma_j(\sigma^2(\vec{x}_j)+2\tilde{\sigma}\sigma(\vec{x}_j)+\tilde{\sigma}_j^2)d\Gamma_l =$$

$$= \frac{1}{N}\Sigma_j\sigma^2(\vec{x}_j) + \frac{2}{N}\Sigma_j\overline{\tilde{\sigma}}_j\sigma(\vec{x}_j) + \frac{1}{N}\Sigma_j\overline{\tilde{\sigma}_j^2} \tag{4.1}$$

Here the first term is a purely collective part, the second term describes a spatial correlation between the collective charge density and the average charge density of the fluctuating background in the presence of the collective field, and the last term is the mean square of the fluctuations of the background.

Let us put

$$\overline{\Delta\tilde{\sigma}^2} \equiv \frac{1}{N}\Sigma_j \overline{\tilde{\sigma}_j^2}$$

$$\Delta\sigma_c \equiv \frac{2}{N}\Sigma_j \overline{\tilde{\sigma}_j}\sigma(\vec{x}_j) = -\frac{2}{V}\Sigma_j\left[\sigma^2(\vec{x}_j) + \frac{1}{2\alpha\tau}\sigma(\vec{x}_j)\varphi(\vec{x}_j)\right]\Delta V_j = \tag{4.2}$$

$$= -\frac{2}{V}\left\{\int\sigma^2(\vec{x})dV + \frac{\Delta V}{2\alpha\tau}\int\sigma(\vec{x})\varphi(\vec{x})dV\right\}$$

where we have replaced the sums $\sum_j \Delta V_j$ with integrals $\int dV$. After inserting (4.2) into (4.1) one obtains

$$\overline{\Delta\sigma^2} = \overline{\Delta\tilde{\sigma}^2}\left\{1 - \frac{1}{V\Delta\tilde{\sigma}^2}\left[\int\sigma^2(\vec{x})dV + \frac{\Delta V}{\alpha\tau}\int\sigma(\vec{x})\varphi(\vec{x})dV\right]\right\} \tag{4.3}$$

Let us suppose that the collective configuration is localized in a volume Ω much smaller than than the volume V of our statistical system. We imagine that the part of V outside Ω is filled only by the background of fluctuating particles. In the limit $V\to\infty$ the second term in (4.3) is much smaller than unity and lg $\overline{\Delta\sigma^2}$ in (3.8) can be approximated by an expansion up to first order in Ω/V. Moreover, the limit $V\to\infty$ implies the limit $N\to\infty$, in order that ΔV and τ (given by (3.7) remain finite. It follows that $(\phi + \phi_{int})/N\tau$ is much smaller than unity, so that also the last term of (3.8) can be approximated by expanding the logarithm up to first order in $(\phi + \phi_{int})/N\tau$. In this limit (thermodynamic limit) S can be expressed as the sum of two parts

$$S = S_b + S_c \tag{4.4}$$

where S_b contains the effects of the background

$$S_b = \frac{N}{2} + N\,\mathrm{lg}\left[\Delta V(2\pi\overline{\Delta\tilde{\sigma}^2})^{1/2}\right] + \frac{\phi_{int}}{\tau} \tag{4.5}$$

and S_c is a purely collective part

$$S_c = -\frac{1}{2\Delta V\overline{\Delta\tilde{\sigma}^2}}\int_\Omega\sigma^2(\vec{x})dV - \frac{\phi}{\tau} = \frac{-1}{2\Delta V\overline{\Delta\tilde{\sigma}^2}}\left[\int_\Omega\sigma^2(\vec{x})dV + \frac{\Delta V\overline{\Delta\tilde{\sigma}^2}}{\tau}\int_\Omega\sigma(\vec{x})\varphi(\vec{x})dV\right] \tag{4.6}$$

The thermodynamic limit of the various quantities introduced in section 3 is the following

$$\overline{\sigma_i^2} - \overline{\sigma_i}^2 = \frac{1}{2\alpha} = \overline{\Delta\sigma^2} = \overline{\Delta\tilde{\sigma}^2}$$

$$\tau = -\frac{\Delta V \overline{\Delta\tilde{\sigma}^2}}{2} \frac{\int \varphi^2(\vec{x})dV}{\phi + \phi_{int}} \tag{4.7}$$

$$\Delta\sigma_c = -\frac{2}{V}\left[\int_\Omega \sigma^2(\vec{x})dV + \frac{\Delta V \overline{\tilde{\sigma}^2}}{\tau}\int_\Omega \sigma(\vec{x})\varphi(\vec{x})dV\right]$$

$$F = -N\tau \lg\left[\Delta V(2\pi\overline{\Delta\sigma^2})^{1/2}\right]$$

Comparing the expression (4.7) for $\Delta\sigma_C$ with (4.6) one can write

$$S_c = \frac{N}{4\overline{\Delta\tilde{\sigma}^2}}\Delta\sigma_c \tag{4.8}$$

The collective entropy is then proportional to the spatial correlation between the collective charge density and the fluctuating charge density of the background.

5. The Vlasov Equilibrium and the Isothermal Variations

As we said, in our phenomenological picture the Vlasov equilibrium is defined by the absence of energy interaction with the background, that is by $\phi_{int} = 0$. In this case the collective entropy has the form

$$S_c = \frac{1}{2\Delta V \overline{\Delta\tilde{\sigma}^2}}\left[-\int \sigma^2(\vec{x})dV + \frac{\int \sigma(\vec{x})\varphi(\vec{x})dV}{\int \varphi^2(\vec{x})dV}\right] \tag{5.1}$$

Applying the Shwartz inequality one obtains that $S_C \leq 0$. Thus, recalling (4.8), the absolute maximum $S_C = 0$ of the collective entropy at Vlasov equilibrium is associated with the absence of correlations between the collective and the fluctuating charge densities. One sees that when $\sigma(\vec{x})$ is equal to the canonical average (3.6a)

$$\sigma(\vec{x}) = \bar{\sigma}(\vec{x}) = -\frac{1}{2\alpha\tau}\varphi(\vec{x})\Delta V \tag{5.2}$$

the collective entropy has an absolute maximum. In general, the collective entropy is maximum within the family of the Vlasov equilibria with $\phi_{int} = 0$ when $\sigma(\vec{x})$ is proportional to $\varphi(\vec{x})$. This is the case of static solutions of the Vlasov equation for quasi-homogeneous systems when the charge density $\sigma(\varphi)$ is a linear homogeneous function $\sigma(\varphi) = k^2\varphi$.

It is worthwhile to study the variations of the entropy when the Vlasov equilibrium is perturbed by an energy interaction $\delta\phi_{int} \neq 0$. Let us consider a specific Vlasov equilibrium with a potential $\varphi_m(\vec{x})$ created by a charge distribution $\sigma_m(\vec{x})$. The parameter τ is given by the equality

$$\tau = -\Delta V \overline{\Delta\tilde{\sigma}^2} \frac{\int \varphi_m^2 dV}{\int \sigma_m \varphi_m dV} \tag{5.3}$$

We now keep τ fixed and change φ_m, σ_m according to $\varphi = \varphi_m + \delta\varphi$, $\sigma = \sigma_m + \delta\sigma$ (where $\delta\varphi$ and $\delta\sigma$ are related by Poisson's equation, $4\pi\delta\sigma = k^2\delta\varphi$). According to the expression (4.7) of τ this variation must be accompanied by a variation $\delta\phi_{int}$ given by the relations

$$\delta\phi_{int} = -\delta\phi - \frac{\Delta V \overline{\Delta\tilde{\sigma}^2}}{\tau} \int \varphi \delta\varphi dV \tag{5.4}$$

Thus the variations with τ fixed describe the interaction of the collective configuration with the background which brings the system outside the Vlasov equilibrium. The parameter τ plays the role of a (generalized) temperature and the background, on which the collective equilibrium is immersed isothermally, is formally similar to the heat bath of the statistics of Gibbs. We shall call "isothermal" the variations with τ fixed and $\delta\phi_{int} \neq 0$. The value of $\delta\phi_{int}$ will be calculated explicitly in the next sections in a number of cases and we shall see that $\delta\phi_{int}$ describes the energy interaction of the collective system with the external world.

When isothermal variations are considered, the collective entropy (henceforth simply called entropy and denoted by S) remembering (4.6), has the form

$$S = \frac{1}{2\Delta V \overline{\Delta\tilde{\sigma}^2}} \left[-\int_\Omega \sigma^2 dV + \frac{k_m^2}{4\pi} \int_\Omega \sigma\varphi dV \right] \tag{5.5}$$

when k^2_m is a parameter (to be kept fixed during the variations) related to τ by the equality

$$k_m^2 = -\frac{4\pi\Delta V \overline{\Delta\tilde{\sigma}^2}}{\tau} \tag{5.6}$$

The functional (5.5) is the entropy which will constitute the basis for the discussion of the thermodynamic properties of the Vlasov equilibria.

6. The Interaction Between the Collective System and the Surrounding Medium

As we have seen, the thermodynamic limit implies a scheme in which a subsystem representing the inhomogeneous collective equilibrium is immersed in a much larger system constituted by a fluctuating medium of individual particles. The only property of the large system which is essential to us is the ability to exchange energy with the collective subsystem. In this way, as we shall see in the next sections, one can simulate the complicated interactions of the collective configuration with the external world without any necessity to consider the detailed mechanisms of the external interaction, because the very large system remains practically unaffected by the energy exchange (the heat bath of the statistics of Gibbs). The properties of the collective subsystem in interaction with the background can then be extrapolated to the realistic situations in which the background extending to infinity does not exist but nevertheless energy is exchanged between the collective system and the external world.

In the electrostatic case the exchange of energy is related to an exchange of electric charges between the collective system and the fluctuating medium. Let us consider the isothermal variations around a quasihomogeneous Vlasov equilibrium $4\pi\sigma_m = k^2{}_m\varphi_m$. The reaction of the background to the presence of the collective charge σ_m is described by $\overline{\overline{\sigma}}_m$ and this quantity vanishes at Vlasov equilibrium, according to (3.6a)

$$\overline{\overline{\sigma}} = -\sigma_m + \frac{k_m^2}{4\pi}\varphi_m = 0 \tag{6.1}$$

At the contrary, when considering variations with τ fixed, the reaction of the background is given, according to (3.6a) by the expression

$$\overline{\overline{\sigma}} = -\sigma_m - \delta\sigma + \frac{k_m^2}{4\pi}(\varphi_m + \delta\varphi) = -\delta\sigma\left(1 - \frac{k_m^2}{k^2}\right) \tag{6.2}$$

which describes the exchange of electric charge and the transfer of energy $\delta\phi_{int} = (1/2)\int \varphi\overline{\overline{\sigma}}dV$ between the background and the collective system, when the latter is not in Vlasov equilibrium ($k^2 \neq k^2{}_m$).

As a consequence of this interaction the entropy of the background, given by (4.5), varies as follows:

$$\delta S_b = \frac{\delta\phi_{int}}{\tau} = -\frac{1}{2\tau}\int_\Omega \varphi_m\left(1 - \frac{k_m^2}{k^2}\right)\delta\sigma dV \tag{6.3}$$

The change of entropy of the collective system, remembering (5.5), is given by the relation

$$\delta S_c = \frac{1}{2\Delta V \Delta\tilde{\sigma}^2}\left[-2\int_\Omega \sigma_m \delta\sigma dV + \frac{k_m^2}{4\pi}\int_\Omega (\sigma_m \delta\varphi + \varphi_m \delta\sigma) dV\right] =$$

$$= -\frac{1}{2\Delta V \Delta\tilde{\sigma}^2}\int_\Omega \sigma_m (1 - \frac{k_m^2}{k^2})\delta\sigma dV = \frac{1}{2\tau}\int_\Omega \varphi_m \left(1 - \frac{k_m^2}{k^2}\right)\delta\sigma dV = -\delta S_b \qquad (6.4)$$

One then concludes that in the interaction process the total entropy of the system formed by the collective configuration and by the background is conserved. Since the total system is isolated and the entropy is conserved, the interaction process is reversible.

7. The Connection with the Classical Thermodynamics and the Concept of Thermodynamic Instability

The fact that the interaction process is reversible and that the total system is isolated makes it possible to apply the conventional procedure (the Charathéodory axiomatics; see e.g. Adkins 1975; Redlich 1981) for defining the entropy of the collective system and of the background, according to which a reversible transfer of heat δQ between the two systems is accompanied by the entropy variations

$$\delta S_c = \frac{\delta Q}{\tau}, \quad \delta S_b = -\frac{\delta Q}{\tau} \qquad (7.1)$$

where $1/\tau$ is an integration factor. The heat transfer δQ can be assimilated, after comparison with (6.3) and (6.4), to the energy interaction $\delta\phi_{int} = -\delta Q$ and in this way one attributes to the variations δS_b, δS_c the meaning of entropy variations in the same sense of classical thermodynamics. We assume that <u>a positive</u> δQ means that energy is transferred <u>from</u> the background <u>to</u> the collective system.

Let us now discuss the isothermal variation of the free energy in the interaction process. From (3.11) one obtains

$$\delta F = \delta\phi + \delta\phi_{int} = \delta\phi - \delta Q \qquad (7.2)$$

Comparing this relation with the energy balance between the collective system and the external world

$$\delta Q = \delta\phi + \delta L \qquad (7.3)$$

where δL is the work performed in the interaction process (see the examples in the next section) one recovers the usual meaning of the free energy in an isothermal reversible process, that is $\delta F = -\delta L$. Using (5.4) and (5.6) one obtains

$$\delta F = \int_{\Omega} \sigma_m \delta\varphi dV \tag{7.4}$$

The heat transfer can be expressed as a surface integral

$$\delta Q = \frac{1}{2}\int_{\Omega}(\varphi_m \delta\sigma - \sigma_m \delta\varphi)dV = \frac{1}{8\pi}\int_{\Sigma}(\varphi_m \delta\vec{E} - \vec{E}_m \delta\varphi)d\vec{\Sigma} \tag{7.5}$$

where Σ is the surface enclosing Ω. When Σ is an equipotential surface, that is φ_m = const, $\delta\varphi = 0$ on Σ, (7.5) becomes

$$\delta Q = \frac{\varphi_m}{2}\int_{\Omega} \delta\sigma dV \tag{7.6}$$

and δQ is proportional to the variation of the total charge of the collective system.

Once the entropy of the collective configuration is defined in a reversible transformation, one can proceed to consider irreversible variations of an isolated collective system. Indeed, let us suppose that the collective system in contact with the background can accede reversibly to a neighbouring state of increased entropy. In the realistic situation, in which the infinite background of neutral plasma does not exist and the collective system is isolated, the same neighbouring state of increased entropy can be reached with an irreversible transformation. If this transformation is in accordance with the physical and the geometrical constraints acting on the system, the isolated collective equilibrium is declared to be thermodynamically unstable. The limitations in the accessibility of the neighbouring equilibria imposed by the constraints are, of course, of primary importance for deciding about the thermodynamic stability of the collective system. This also results from a closer examination of the conditions under which the entropy can increase according to the thermodynamic relation (7.1). Indeed, as we shall see (next section) τ can be positive or negative. If τ is positive, then $\delta S_c > 0$ implies $\delta Q > 0$ according to (7.1). This corresponds to a situation in which the collective system is in contact with an external source of heat simulated by the background. But if $\tau < 0$ the collective system yields heat out to the background while its entropy increases. This simulates an equivalent irreversible process which may spontaneously occur in the collective system. At the same time, the energy $W = \phi + L$ decays and is transformed into heat, because $\delta W = \delta Q < 0$. Hence the condition $\tau < 0$ is indicative of thermodynamic instability provided that accessible neighbouring states of the collective system with higher entropy exist. This condition, however, is not sufficient for thermodynamic instability as it follows from the simple observation that when the system is conservative and W is minimum, accessible states with $\delta W < 0$ cannot exist.

The existence of neighbouring states with higher entropy is insured by the following general property of entropy functionals of the form (5.5): If the Vlasov equilibrium is not unique (that is, if it admits neighbouring static modes satisfying the same boundary conditions and such that $\delta S = 0$) then, on one hand, the inequality $\tau < 0$ holds, and on the other, the entropy is not at a maximum. (For the proof in a magnetic case see Minardi 1988; see also the discussion of the next section on the electrostatic case.)

The relation between a negative τ and the thermodynamic instability expresses the fact, known in ordinary thermodynamics, that a system with negative temperature is "hotter" than one with positive temperature: the collective system with $\tau < 0$ yields out heat to the ambience while ϕ decays and the entropy increases.

The concept of thermodynamic instability based on the existence of neighbouring accessible states with higher entropy is broader than the concept based on the energy principle because it can be applied to situations not amenable to a dynamical treatment. The extremum of the energy is always defined with respect to a limited physical family of variations depending on the dynamical model within which the energy and its variations are considered. By refining the physical model one can include a larger family of perturbations with respect to which the minimum of the energy can be assessed. However, a point is always reached in practice such that the family of physical variations to be considered is so large that it is impossible, for practical reasons, to construct an energy able to encompass all of them in a dynamical treatment. A typical example is given by a so-called non-conservative system for which a minimum of the energy does not exist or is not related to the stability. It is just when a large global set of perturbations is considered that the thermodynamic method shows its full power and the concept of thermodynamic stability supersedes the concept of dynamic stability.

8. Thermodynamic Stability of Electrostatic Equilibria and the Maxwellian Limit

As a simple illustration of our thermodynamic concepts we consider an electrostatic Vlasov equilibrium described by distribution functions $f_s(\varepsilon_s)$ where $\varepsilon_s = (1/2)\, m_s v^2 + q_s\varphi$ and s denotes different particle species. Up to first order in φ the charge density is given by the relation

$$\sigma_m = \sum_s q_s \int f_s(\varepsilon_s) d^3v = \varphi_m \Sigma_s q_s^2 \int \frac{\partial f_s}{\partial \varepsilon_s} d^3v \qquad (8.1)$$

Applying (5.3) and (5.6), one obtains the following expression for the temperature:

$$\tau = -\frac{4\pi\Delta V\overline{\Delta\tilde{\sigma}^2}}{k_m^2} = -\frac{\Delta V\overline{\Delta\tilde{\sigma}^2}}{\Sigma_s q_s^2 \int \frac{\partial f_s}{\partial \varepsilon_s} d^3 v} \tag{8.2}$$

The first variation of the entropy functional (5.5) is the following

$$\delta S = \frac{1}{2\Delta V\overline{\Delta\tilde{\sigma}^2}}\left[-2\int_\Omega \sigma_m \delta\sigma dV + \frac{k_m^2}{4\pi}\int_\Omega (\varphi_m \delta\sigma + \sigma_m \delta\varphi) dV\right] \tag{8.3}$$

Using the relation

$$\int_\Omega \sigma_m \delta\varphi dV = -\frac{1}{4\pi}\int_\Omega \delta\varphi \nabla^2 \varphi_m dV = -\frac{1}{4\pi}\int_\Omega \varphi_m \nabla^2 \delta\varphi dV - \frac{1}{4\pi}\int_\Sigma (\delta\varphi\nabla\varphi_m - \varphi_m \nabla\delta\varphi).d\vec{\Sigma} \tag{8.4}$$

The first variation becomes

$$\delta S = \frac{1}{\Delta V\overline{\Delta\tilde{\sigma}^2}}\left[-\int_\Omega \sigma_m \delta\sigma dV + \frac{k_m^2}{4\pi}\int_\Omega \varphi_m \delta\sigma dV - \left(\frac{k_m}{4\pi}\right)^2 \int_\Sigma (\varphi_m \delta\vec{E} - \delta\varphi \vec{E}_m)\cdot d\vec{\Sigma}\right] \tag{8.5}$$

Recalling (7.5) the surface integral is proportional to the energy transfer δQ between the collective system and the background. In an isolated system this integral must vanish. The entropy is then extremum in an isolated system when the following relation holds:

$$\sigma_m = \frac{k_m^2}{4\pi}\varphi_m \tag{8.6}$$

It follows that the electric field $\vec{E}_m = -\nabla\varphi_m$ of the extremal equilibrium must satisfy the equation

$$\nabla^2 \vec{E}_m + k_m^2 \vec{E}_m = 0 \tag{8.7}$$

Let us assume k^2_m positive (τ negative) and sufficiently large, so that (8.7) admits a regular solution with at least one zero in Ω (including the boundary). In this case a solution of class C^o of (8.7), with prescribed boundary conditions, is not unique. Moreover, the existence of a zero would violate the so-called Jacobi condition of the variational calculus (Goursat 1964) which requires a solution of (8.7) not vanishing in Ω in order that the sign of the second variation of S be definite. Indeed, according to the variational calculus, if zeros exist, a smooth variation $\delta\vec{E}$ (of class C^1) can always be constructed arbitrarily close to a solution of (8.7) of class C^o (so that δS is arbitrarily close to zero) and such that the configuration $\vec{E}_m + \delta\vec{E}$ describes a neighbouring equilibrium with increased entropy ($\delta^2 S > 0$). At the contrary, when k^2_m is negative (τ positive) a non-vanishing solution in Ω exists, (unique for given boundary conditions) and S can be shown to be maximum. The condition $k^2_m > 0$ ($\tau < 0$) is then necessary for the thermodynamic instability and in this case the distribution function of at least one of the species must satisfy, in some domain of velocity space, to the inequality

$$\frac{\partial f_s}{\partial \varepsilon_s} > 0 \qquad (8.8)$$

which is then necessary for thermodynamic instability.

We now consider the case when f_s is Maxwellian. In this case one has, up to first order

$$\sigma_m = \Sigma_s n_s q_s \exp-(q_s \varphi_m / T) = -\varphi_m \Sigma_s q_s^2 n_s / T_s \qquad (8.9)$$

Thus

$$k_m^2 = -4\pi \Sigma_s q_s^2 n_s / T_s < 0 \qquad (8.10)$$

and S is maximum. The temperature is given by the expression

$$\tau = \frac{\Delta V \overline{\Delta\tilde{\sigma}^2}}{\Sigma_s q_s^2 n_s / T_s} > 0 \qquad (8.11)$$

Let us assume that the mean square deviation of the number of particles in the volume element ΔV is given by Poisson's statistics. Then

$$\overline{\Delta\tilde{\sigma}^2} = \Sigma_s q_s^2 \overline{(\tilde{n} - n_s)^2} = \frac{\Sigma_s q_s^2 n_s}{\Delta V} \qquad (8.12)$$

and

$$\tau = \frac{\Sigma_s q_s^2 n_s}{\Sigma_s \frac{q_s^2 n_s}{T_s}} \qquad (8.13)$$

It follows that when all species have the same Maxwellian temperature T, the generalized temperature τ is identical to T. Moreover, $- k^{-2}_m$ is identical to the square of the Debye length.

We conclude this section with two remarks. The first remark is that the relation between the maximum of the entropy and the uniqueness of the equilibrium is consistent with the meaning of the entropy principle as an evolution principle. The uniqueness of the equilibrium is indeed necessary in order that the maximum of the entropy could indicate unambiguously the direction followed globally by an evolving system.

The second remark is that the thermodynamic instability can be interpreted in terms of a negative sign of a dielectric constant describing the reaction of the background to the presence of a collective charge $\delta\sigma$ immersed in it. The modification of the charge density of the background, as we know, is described by $\bar{\tilde{\sigma}}$ given by (6.2). Let us call $\varepsilon\delta\sigma$ the modification of the collective charge density in the presence of the reacting background. The local conservation of the real charges gives

$$\varepsilon \, \delta \sigma + \bar{\bar{\sigma}} = 0 \tag{8.14}$$

or, remembering (6.2)

$$\varepsilon = 1 - \frac{k_m^2}{k^2} \tag{8.15}$$

It is easy to verify that the second variation of S is positive for $\varepsilon < 0$. In this situation the entropy is minimum, the collective system is unstable and the total system formed by the background and the collective system is not in thermodynamic equilibrium. Starting from this observation, one can develop a method for calculating the saturation levels of unstable collective modes by taking into account the nonlinear reaction of a medium, formed by a certain population of individual particles, to the buildup of the unstable collective charge (Minardi 1985). Taking into account higher orders in ϕ, one finds that a neighbouring equilibrium of maximum entropy can be reached which corresponds to a saturated level of the electric charges fluctuating between the medium and the collective system. The saturation of the instability is then related to the nonlinear reaction of a background of individual particles, namely particles which, in the linear limit, do not participate to the formation of the collective field.

9. The Vlasov Magnetostatic Equilibria

The formalism developed above for the electrostatic equilibria can be immediately extended to collective equilibria of gravitational or magnetic nature introducing the appropriate potentials and information variables. In the magnetic case the information variable is the current density and the potential is the corresponding vector potential. The entropy functional related to isothermal variations is given by the equality (Minardi 1981).

$$S = \frac{3}{2\Delta V \overline{\Delta \vec{j}^2}} \left[-\int_\Omega \vec{j}^2 dV + \frac{\mu^2 c}{4\pi} \int_\Omega \vec{j} \cdot \vec{A} dV \right] \tag{9.1}$$

Here μ^2 is related to the generalized temperature and to the magnetic configuration at equilibrium by the relations (analogous to (5.3) and (5.6)):

$$\tau = -\frac{4\pi \Delta V \overline{\Delta \vec{j}^2}}{3\mu^2 c^2}$$

$$\frac{1}{\mu^2}=\frac{c}{4\pi}\frac{\int\limits_{\Omega}\vec{A}_m^2 dV}{\int\limits_{\Omega}\vec{j}_m\cdot\vec{A}_m dV} \tag{9.2}$$

It follows from these relations that the arbitrarity of the gauge is reflected in the freedom of μ^2 as well as of τ. The parameter μ^2 will act merely as a label of the equilibria which belong to the family characterized by the extremum of the entropy or by the vanishing of the entropy production defined by dS/dt (see the examples later). The magnetic configurations which follow from the stationary properties of S were examined in recent papers both for collective systems of the pinch and of the tokamak type. Here we shall discuss some aspects of these systems which are related to the interaction with the external world and are relevant for the thermodynamic interpretation of the boundary conditions to be applied in the variational treatment of the functional S.

9.1 Example of a closed system: the reversed field pinch.

A closed system can be schematized as a system formed by a cylindrical or a toroidal plasma and a perfectly conductive shell enclosing the plasma and screening it completely from the external world. The plasma carries a current density $\vec{j}_p$ and the shell a current $\vec{j}_s$ which screens the magnetic field $\vec{B}$ created by $\vec{j}_p$ so that the total magnetic field vanishes outside the shell:

$$\frac{4\pi}{c}\vec{j}_p=\nabla\times\vec{B}=\nabla\times\nabla\times\vec{A}_p,\ \frac{4\pi}{c}\vec{j}_s=-\delta(\rho-\rho_s)\vec{e}_n\times\vec{B} \tag{9.3}$$

Here $\vec{e}_n$ is the unit vector normal to the shell directed outwardly. The function $\delta(\rho\text{-}\rho_S)$ describes the localization of $\vec{j}_s$ on the conductive shell with effective minor radius ρ_S and will be approximated by a δ-function. The integration volume Ω in the entropy functional (9.1) includes the shell and $\vec{j}=\vec{j}_s+\vec{j}_p$.

We shall calculate the energy transfer $\delta Q = -\delta\phi_{int}$ between the collective system and the background. The variation of ϕ_{int} is given by the expression (analogous to (5.4)):

$$\delta\phi_{\rm int}=-\delta\left[\phi+\frac{\Delta V\overline{\Delta\vec{j}^2}}{6\pi c^2}\int\limits_{\Omega}\vec{A}^2 d\Omega\right]=-\delta\left[\frac{1}{2c}\int\limits_{\Omega}(\vec{j}_p+\vec{j}_s)\cdot\vec{A}_p dV-\frac{\mu^2}{8\pi}\int\limits_{\Omega}\vec{A}_p^2 dV\right] \tag{9.4}$$

Note that the screening field created by $\vec{j}_s$ exists only outside Ω and the same holds for the corresponding vector potential $\vec{A}_S$.

A field reversed force-free configuration is an extremum of (9.1) (Minardi 1989). With a suitable choice of the gauge one has

$$\vec{A}_{pm} = \frac{4\pi}{\mu^2 c}\vec{j}_{pm} = \frac{1}{\mu}\vec{B}_m = \frac{1}{\mu}\nabla\times\vec{A}_{pm} = \frac{1}{\mu^2}\nabla\times\vec{B}_m \qquad (9.5)$$

Note that the first equality is consistent with the supplementary condition (9.2). We calculate the variation δQ arising from a variation $\delta\vec{A}$ of the vector potential inside Ω, assuming that the corresponding magnetic field variation in Ω is screened by a variation of $\vec{j}_s$:

$$\delta Q = -\delta\phi_{\text{int}} = \frac{1}{8\pi}\int_\Omega \delta\vec{A}\cdot\nabla\times\vec{B}_m dV + \frac{1}{8\pi}\int_\Omega \vec{A}_{pm}\cdot\nabla\times\delta\vec{B}dV -$$

$$(9.6)$$

$$-\frac{1}{8\pi}\int_\Sigma d\vec{S}\times\delta\vec{B}\cdot\vec{A}_{pm} - \frac{1}{8\pi}\int_\Sigma d\vec{S}\times\vec{B}_m\cdot\delta\vec{A} - \frac{\mu^2}{4\pi}\int_\Omega \vec{A}_{pm}\cdot\delta\vec{A}dV =$$

$$= \frac{1}{4\pi}\int_\Sigma \delta\vec{A}\times\vec{B}_m\cdot d\vec{S}$$

Considering that $\delta\vec{A}$ arises from a time dependence, $\delta\vec{A} = (\partial\vec{A}/\partial t)\delta t = -c\,\vec{E}\,\delta t$, (9.6) becomes

$$\frac{\delta Q}{\delta t} = -\frac{c}{4\pi}\int_\Sigma \vec{E}\times\vec{B}_m\cdot d\vec{S} \qquad (9.7)$$

and one sees that $\delta\phi_{int} = -\delta Q$ simulates a radiation energy emitted by the collective system. In a closed system the variations $\delta\vec{A} = -c\vec{E}\delta t$ must be chosen in order that δQ = 0.

The defining relation (9.4) of $\delta\phi_{int}$ is nothing else than the energy conservation

$$\frac{\delta\phi}{\delta t} = \frac{1}{2c}\frac{\delta}{\delta t}\int_\Omega (\vec{j}_p + \vec{j}_s)\cdot\vec{A}_p dV = -\frac{\delta\phi_{\text{int}}}{\delta t} + \frac{\mu^2}{4\pi}\int_\Omega \vec{A}_{pm}\frac{\delta\vec{A}}{\delta t}dV =$$

$$= -\frac{c}{4\pi}\int_\Sigma \vec{E}\times\vec{B}\cdot d\vec{S} - \int_\Omega \vec{j}_{pm}\cdot\vec{E}dV = -\frac{1}{4\pi}\int_\Omega \vec{B}_m\frac{\delta\vec{B}}{\delta t}dV \qquad (9.8)$$

Using (7.2) and (7.3) one obtains the free energy and the work per unit time

$$\frac{\delta F}{\delta t} = -\frac{\delta L}{\delta t} = -\int_\Omega \vec{E}\cdot\vec{j}_{pm}dV \qquad (9.9)$$

9.2 Example of an open system : the tokamak

A typical example of an open collective system is given by the plasma in a tokamak, which is externally coupled to the primary of the ohmic transformer and is subject to an auxiliary injection of power. The states of an open system cannot be characterized by the maximum entropy principle, which has a meaning for isolated systems only, but one can assume that the entropy remains stationary in time under the external action of the ohmic and the auxiliary heating and investigate the properties of the states which satisfy this requirement. Applying the condition to the confinement region of the tokamak $q(s\lambda) \leq q \leq q(s)$, where $q(s\lambda) \approx 1$, $q(s) \approx 2$, one is led (Minardi and Lampis 1990) to the following expression for the time derivative of S (or entropy production) in the absence of auxiliary heating

$$\frac{dS}{dt} = \frac{1}{\mu^2\tau}\int_{\Omega} \vec{E}\cdot(\nabla^2\vec{j}_p + \mu^2\vec{j}_p)dV \qquad (9.10)$$

A family of isoentropic states (labelled by μ^2) is then obtained whose axial current density distribution, in the confinement zone, satisfies the equation

$$\nabla^2 j_p + \mu^2 j_p = 0 \qquad (s\lambda \leq r \leq s) \qquad (9.11)$$

Comparing this equation with D'Alembert equation $-\nabla^2 A_p = (4\pi / c) j_p$, one has the relation (apart from a constant related to the gauge)

$$j_p = \frac{c\mu^2}{4\pi} A_p \qquad (9.12)$$

which shows that the solution of (9.11) is consistent with the supplementary condition (9.2) of the statistical model.

We shall now proceed to calculating explicitly the time derivative of the interaction energy ϕ_{int}. Following the same scheme of previous work, the plasma is considered as surrounded by a thin conductive shell with narrow cuts through which an inductive axial electric field $\vec{E}$ with $\nabla \times \vec{E} = 0$ is created in the plasma by a magnetic field $\vec{B}_e$ changing in time outside the shell and vanishing inside it. The electric field vanishes on the perfectly conductive shell so that the shell acts as a surface of discontinuity for $\vec{B}_e$ and $\vec{E}$. It follows from this schematization that a time dependent current $\vec{j}_s$

$$\frac{4\pi}{c}\frac{\partial \vec{j}_s}{\partial t} = \delta(\rho - \rho_s)\vec{e}_n \times \frac{\partial \vec{B}_e}{\partial t} \qquad (9.13)$$

exists on the shell, while $\vec{E}$, the plasma current density $\vec{j}_p$, and $\vec{A}_p$ are assumed as stationary. We also assume that the field created by $\vec{j}_p$ is screened by the conductive shell so that $\vec{A}$ vanishes outside the shell.

For our calculation we shall need the change of $\nabla \times \vec{E}$ and $\nabla \times \vec{A}_p$ across the discontinuity surface. Considering an infinitesmal volume element $\Delta\tau$ with vanishing thickness and base surfaces situated on opposite sides of the discontinuity surface and parallel to it, and applying the general coordinate-free definition of the curl, one obtains the following expression for quantities localised on the surface

$$-\frac{1}{c}\frac{\partial \vec{B}_e}{\partial t} = \nabla \times \vec{E} = \frac{1}{\Delta\tau}\oint d\vec{S} \times \vec{E} = \frac{1}{\Delta\tau} d\vec{S}_{in} \times \vec{E}$$

$$\nabla \times \vec{A}_p = \frac{1}{\Delta\tau}\oint d\vec{S} \times \vec{A}_p = \frac{1}{\Delta\tau} d\vec{S}_{in} \times \vec{A}_p \tag{9.14}$$

where $d\vec{S}_{in} = -d\vec{S}$.

In the calculation of $d\phi_{int}/dt$ we shall consider the time dependent quantities (denoted by the subscript s) as small perturbations and will retain only first order terms. One can then write

$$\frac{d\phi_{int}}{dt} = -\frac{1}{2c}\frac{d}{dt}\int_\Omega (\vec{j}_p + \vec{j}_s)\cdot(\vec{A}_p + \vec{A}_s)dV + \frac{\mu^2}{8\pi}\frac{d}{dt}\int_\Omega (\vec{A}_p + \vec{A}_s)^2 dV =$$

$$= -\frac{1}{2c}\int_\Omega \left(\frac{\partial \vec{j}_s}{\partial t}\cdot\vec{A}_p + \frac{\partial \vec{A}_s}{\partial t}\cdot\vec{j}_p\right)dV + \frac{\mu^2}{4\pi}\int_\Omega \frac{\partial \vec{A}_s}{\partial t}\cdot\vec{A}_p dV \tag{9.15}$$

Recalling (9.13) and noting that $(1/c)\ \partial\vec{A}_s/\partial t = -\vec{E}$, one has

$$\frac{d\phi_{int}}{dt} = -\frac{1}{8\pi}\int_\Sigma d\vec{S} \times \frac{\partial \vec{B}_e}{dt}\vec{A}_p + \frac{1}{2}\int_\Omega \vec{E}.\vec{j}_p dV - \frac{\mu^2 c}{4\pi}\int_\Omega \vec{E}\cdot\vec{A}_p dV \tag{9.16}$$

We now use (9.14) and write

$$-\int d\vec{S} \times \frac{\partial \vec{B}_e}{\partial t}\cdot\vec{A}_p = \int d\vec{S}\cdot\vec{A}_p \times \frac{\partial \vec{B}_e}{dt} = -\int d\vec{S}\cdot\vec{A}_p \times \left(\frac{1}{\Delta\tau}d\vec{S}_{in} \times \vec{E}\right) =$$

$$= -\int d\vec{S}\cdot\left(\vec{A}_p \times \frac{1}{\Delta\tau}d\vec{S}_{in}\right)\times\vec{E} = \int d\vec{S}\cdot(\nabla \times \vec{A}_p)\times\vec{E} \tag{9.17}$$

where in the transition before the last the conditions $\vec{A}_p \cdot d\vec{S} = \vec{E}\cdot d\vec{S} = 0$ were taken into account.

Applying the relation

$$\int_\Sigma d\vec{S}\cdot(\nabla \times \vec{A}_p)\times\vec{E} = \int_\Omega \nabla\cdot(\vec{B}\times\vec{E})dV = \frac{4\pi}{c}\int_\Omega \vec{E}\cdot\vec{j}_p dV \tag{9.18}$$

one finally obtains from (9.16), remembering (9.12),

$$\frac{d\phi}{dt} = -\int_\Omega \vec{E}\cdot\vec{j}_p dV, \qquad -\frac{d\phi_{int}}{dt} = \frac{dQ}{dt} = -\int_\Omega \vec{E}\left(\vec{j}_p - \frac{\mu^2 c}{4\pi}\vec{A}_p\right)dV = 0 \tag{9.19}$$

From these relations and the energy balance (7.3) one has the equality

$$\frac{dL}{dt} = -\frac{d\phi}{dt} = \int_{\Omega} \vec{E} \cdot \vec{j}_p dV \tag{9.20}$$

and dL/dt is the rate of work performed by the generator driving the plasma current. From the second relation (9.19) one concludes that the states with vanishing entropy production remain in Vlasov equilibrium during the ohmic heating, in accordance with our definition of Vlasov equilibria.

The auxiliary power can now be easily inserted in this scheme. Let p_A be the power density, supposed as uniform for simplicity. Expressing the axial current density as $j = j_p - p_A/E$, the entropy production (9.10) becomes

$$\frac{dS}{dt} = \frac{1}{\mu^2 \tau} \int_{\Omega} E(\nabla^2 j + \mu^2 j) dV + \frac{1}{\tau} \int_{\Omega} p_A dV \tag{9.21}$$

where E is the axial electric field.

The first term in the r.h.s describes the production of magnetic entropy pertaining to the collective configuration while the second term is the entropy production due to the external heating rate dQ/dt = $\int p_A dV$. Thus the family of "isoentropic states" satisfies the equation

$$\nabla^2 j + \mu^2 j = -\frac{\mu^2 p_A}{E} \tag{9.22}$$

which was studied in detail in our previous work. The work per unit time performed by the external system is now

$$\frac{dL}{dt} = \int_{\Omega} p_A dV + \int_{\Omega} E j dV \tag{9.23}$$

This work is subsequently transformed into heat inside the plasma. We refer to our recent work (1990,1992) for the discussion of the relation between the entropy production (9.21) and the energy balance inside the plasma.

10. Local Magnetic Entropy of Inhomogeneous Systems: the Example of the Tearing Modes

10.1 The Local Entropy

The magnetic entropy functional (9.1) is expressed with an integral over the whole volume Ω of the collective system so that its properties are related only to the global aspects of the system. In order to include in our thermodynamic point of view those physical aspects which depend on the local structure and inhomogeneities of the magnetic configuration, as for instance, instabilities with respect to interchange modes, one must define an entropy functional which is sensitive to the local properties. A natural procedure consists in dividing the system in small tube elements Ω_n of the magnetic flux and in calculating the entropy S_n in each Ω_n, by treating them as statistically independent systems. The entropy of the total system is then the sum of all entropies S_n. Of course, the Ω_n should be large enough to preserve the macroscopic aspects implicit in our description (for instance $\Omega >> V/N$).

The local structure is taken into account considering that both the current density and the vector potential deviate from the average value $\vec{j}_n$ and $\vec{A}_n$ in Ω_n by a small amount $\vec{j}(\vec{x})-\vec{j}_n$, $\vec{A}(\vec{x})-\vec{A}_n$, where $\vec{x}$ varies inside Ω_n. Thus we can proceed taking $\vec{j}-\vec{j}_n$ as information variable and $\vec{A}-\vec{A}_n$ as vector potential and construct the local entropy functional S_n following the same formal lines as in section 2:

$$S=\sum_n S_n=\frac{3N}{2\Delta V\overline{\Delta\vec{j}^2}}\sum_n\left[-\int_{\Omega_n}(\vec{j}-\vec{j}_n)^2 d\Omega_n+\frac{\mu_n^2 c}{4\pi}\int_{\Omega_n}(\vec{j}-\vec{j}_n)\cdot(\vec{A}\cdot\vec{A}_n)d\Omega_n\right] \qquad (10.1)$$

where

$$\mu_n^2=-\frac{4\pi\Delta V\overline{\Delta\vec{j}^2}}{3c^2\tau_n}=\frac{4\pi}{c}\frac{\int_{\Omega_n}(\vec{j}-\vec{j}_n)\cdot(\vec{A}-\vec{A}_n)d\Omega_n}{\int(\vec{A}-\vec{A}_n)^2 d\Omega_n} \qquad (10.2)$$

The extremum properties of S are connected with the uniqueness of the magnetic equilibrium just in the same way as discussed above. If the isothermal (τ_n fixed) equilibrium associated with $\delta S = 0$ is not unique, which implies a domain where the τ_n are negative, and if this domain is sufficiently large and the n-dependence of τ_n sufficiently smooth, then variations can be constructed which increase S. For the proof see Minardi 1988. The thermodynamic instability is then related to the negative sign of τ_n or, recalling (10.2), to the positive sign of the quadratic form

$$(\vec{j}-\vec{j}_n)\cdot(\vec{A}-\vec{A}_n)=(\vec{\xi}\cdot\nabla\vec{j})(\vec{\xi}\cdot\nabla\vec{A})=\frac{1}{2}(\vec{\xi}\cdot\nabla\vec{j})\cdot(\vec{B}(\vec{x}_n)\times\vec{\xi})>0 \qquad (10.3)$$

where $\vec{\xi}=\vec{x}-\vec{x}_n$; $\vec{B}$ was assumed as practically uniform in Ω_n. For the relation between the condition above and various kinds of instabilities, see Minardi and Santini 1967, Santini 1967, Minardi 1981. By expressing $\vec{j}$ in terms of the distribution function one can relate the thermodynamic instability to the form of this function, just as we did in the electrostatic case, section 8. In this way one can see (Santini 1969) that the sufficient conditions of stability of the low β interchange modes (including trapped particle modes) derived by Rutherford and Frieman 1968 by means of an energy principle, imply $\tau_n > 0$ and a maximum of S. One can also prove (Minardi 1988) that the integral on Ω_n of the quadratic form (10.3) divided by $|\vec{\xi}|^2$ is locally identical in value and sign, in the limit of an infinitesimal Ω_n, to the form $\vec{B}\cdot\nabla^2\vec{B}$ and is then related to the local structure of the magnetic configuration, to the minimum of $|\vec{B}|$, and to a diamagnetic reaction of the low-β plasma (Minardi 1981).

The criterion above was investigated numerically in the case of inhomogeneous electrostatic or gravitational one-dimensional equilibria by verifying the relation between the condition

$$\int_{\Omega}(\sigma-\sigma_o)(\varphi-\varphi_o)dx>0 \tag{10.4}$$

(where σ_o, φ_o are averages in Ω) and the global instability (Cuperman and Tzur 1973, Finzi et al. 1974, Schwarzmeier et al. 1979).

In the following we shall apply the concept of local magnetic entropy to the non-ideal MHD equilibria by discussing the example of the tearing modes.

10.2 Thermodynamic Treatment of the Tearing Modes

We consider the case of a cylindrical plasma with helical symmetry carrying an axial current density j_z which creates a vector potential A_z. The plasma is situated in a uniform large axial magnetic field B_z which remains practically unperturbed. As known from the conventional treatment of the tearing modes, one distinguishes in the plasma two physical regions, namely a resistive resonant layer where $q = m/n$ inside which the dissipation is taking place and a dissipationless outer region. Correspondingly the calculation of the entropy follows different lines in the two regions and one must resort to the local formulation of the entropy presented above.

10.2.1 The Entropy in the Outer Region

In this region the dissipation is neglected and the magnetic flux is conserved. We assume that the magnetic configuration is slightly helically perturbed. As is well known, in view of the helical symmetry the equilibrium current density depends on space through the helical flux $\chi = - mA_z - nrA_\theta/R$, that is

$j_z(\vec{r}) = j_z(\chi(\vec{r}))$.

We shall consider a set of tube elements Ω_n of the unperturbed magnetic flux with finite lengths and infinitesimal thickness. Let be $\vec{x}_n \equiv (r_n, \theta_n, \varphi_n)$ the cylindrical coordinates of a suitably chosen point inside Ω_n. The coordinates of a point $\vec{x}$ varying along the tube Ω_n are denoted $\vec{x} \equiv (r_n, \theta, \varphi)$.. We assume that B_Z is so large that it is not sensitive to the small helical deformations, so that the θ, φ dependence of A_θ can be neglected. The variation $A_z(\vec{x}) - A_z(\vec{x}_n)$ arising from the helical dependence inside Ω_n is then equal to $-(1/m)(\chi(\vec{x}) - \chi(\vec{x}_n))$. One can also write (dropping henceforth the subscript z)

$$j(\vec{x}) - j_n = \frac{dj}{d\chi_n}(\chi - \chi_n) = -\frac{dj}{d\chi_n} m(A - A_n) \tag{10.5}$$

Inserting this expression into (10.2) one obtains, for sufficiently small integration domains Ω_n

$$\mu_n^2 = -\frac{4\pi m}{c}\frac{dj}{d\chi_n} \tag{10.6}$$

The first variation of the entropy (10.1) is then

$$\delta S = \frac{1}{2\Delta V \overline{\Delta j^2}} \sum_n \left\{ -2\int_{\Omega_n} (j - j_n)\delta j d\Omega_n - m\frac{dj}{d\chi_n}\int_{\Omega_n} [\delta j(A - A_n) + \delta A(j - j_n)] d\Omega_n \right\} =$$

$$= \frac{1}{2\Delta V \overline{\Delta j^2}} \sum_n \int_{\Omega_n} \left\{ -(j - j_n)(\delta j + m\delta A\frac{dj}{d\chi_n}) - \left[(j - j_n) + m\frac{dj}{d\chi_n}(A - A_n)\right]\delta j \right\} d\Omega_n = \tag{10.7}$$

$$= -\frac{1}{2\Delta V \overline{\Delta j^2}} \int \left[(j - j_n)(\delta j + m\delta A\frac{dj}{d\chi_n})\right] d\Omega_n$$

where δj and δA are related by $\nabla^2 \delta A = -(4\pi/c)\delta j$ (note that a factor 3 is missing in the expression above with respect to the expression (10.1) of S; this depends on the fact that here the information variable $j - j_n$ has only one component instead of three and the dimension of the information space is correspondingly reduced). As for the higher variations of S one has

$$\delta^2 S = -\frac{1}{\Delta V \overline{\Delta j^2}} \sum_n \int_{\Omega_n} \delta j(\delta j + m\frac{dj}{d\chi_n}\delta A) d\Omega_n, \quad \delta^n S = 0 \; for \; n > 2 \tag{10.8}$$

Let us now discuss the interaction energy ϕ_{int} and its variations:

$$\phi_{int} = -\frac{1}{2c}\sum_n \left[\int_{\Omega_n} (j - j_n)(A - A_n) d\Omega_n + m\frac{dj}{d\chi_n}\int (A - A_n)^2 d\Omega_n \right] = 0$$

$$\delta\phi_{int} = -\frac{1}{2c}\sum_n \int_{\Omega_n} (\delta j + m\frac{dj}{d\chi_n}\delta A)(A - A_n)d\Omega_n \tag{10.9}$$

$$\delta^2\phi_{int} = -\frac{1}{c}\sum_n \int_{\Omega_n} (\delta j + m\frac{dj}{d\chi_n}\delta A)\delta A d\Omega_n\,, \quad \delta^n\phi_{int} = 0 \; for \; n > 2$$

The condition $\phi_{int} = 0$, which follows from (10.5), insures that the plasma is in Vlasov equilibrium in the outer region, in accordance with our basic definition.

At this point we introduce the assumption that the system remains in Vlasov equilibrium in the outer region also in the varied state. This is a reasonable assumption because the dissipation is neglected in the outer region. We must then restrict the variations δj to those for which ϕ_{int} vanishes locally at all orders. Then we must have

$$-\nabla^2\delta A + \frac{4\pi m}{c}\frac{dj}{d\chi_n}\delta A = 0 \tag{10.10}$$

This is just the conventional equation of the tearing modes in the outer region. Taking into account this equation one sees that the variations (10.6) and (10.7) of the entropy vanish. The entropy is then stationary in the outer region with respect to tearing modes.

10.2.2 The Entropy in the Inner Region

Thus an entropy change can only result from the dissipative process in the resonant layer. We consider a resonant layer around r = s with finite thickness 2ε and volume Ω_s. The magnetic flux is not conserved in Ω_s and the unperturbed current is not a function of the helical flux. For the radial variation of the current density and of the potential inside Ω_s one can write simply

$$j - j_s = \left(\frac{dj}{dr}\right)_s (r - s)\,, \quad A - A_s = \left(\frac{dA}{dr}\right)_s (r - s) \tag{10.11}$$

where $s - \varepsilon \le r \le s + \varepsilon$. The generalized temperature in Ω_s is then given by the equality

$$\tau_s = -\frac{\overline{\Delta\tilde{j}^2}\Delta V}{c}\frac{\int_{\Omega_s}(A - A_s)^2 d\Omega_s}{\int_{\Omega_s}(j - j_s)(A - A_s)d\Omega_s} = \frac{\overline{\Delta\tilde{j}^2}\Delta V}{c}(B_\theta\frac{dr}{dj})_s < 0 \tag{10.12}$$

Note that τ is discontinuous across the resonant layer; τ_S is negative because the current density was assumed to decrease at r = s.

We now calculate the entropy on the assumption that the only process which is taking place in the resonant layer is a decay of the local plasma magnetic energy ϕ_S into heat. One can write

$$\delta\phi_s = \frac{1}{2c}\int_{\Omega_s}(A\delta j + j\delta A)d\Omega_s\,,\ \delta^2\phi_s = \frac{1}{c}\int \delta j\delta A d\Omega_s\,,\ \delta^n\phi_s = 0\ for\ n > 2 \qquad (10.13)$$

and in the limit of an infinitesimal thickness 2ε of the layer

$$\delta\phi_s = \frac{RsA(s)}{2c}\int\int d\varphi d\vartheta \int_{s-\varepsilon}^{s+\varepsilon}\delta j dr$$

$$\delta^2\phi_s = \frac{Rs}{c}\int\int d\varphi d\vartheta \int_{s-\varepsilon}^{s+\varepsilon}\delta j\delta A dr = -\frac{R}{4\pi}\int\int d\varphi d\vartheta \int_{s-\varepsilon}^{s+\varepsilon}\delta A\left(\frac{d}{dr}r\frac{d}{dr}\delta A\right)dr = \qquad (10.14)$$

$$= -\frac{Rs}{4\pi}\int\int \Delta'(\delta A)^2 d\varphi d\vartheta$$

where

$$\Delta' = \frac{\left|\frac{d}{dr}\delta A\right|_{s-\varepsilon}^{s+\varepsilon}}{\delta A(s)}$$

Note that the first variation $\delta\phi_S$ can always be put to zero by choosing A(s) = 0. The variation of the entropy is calculated, in accordance with the procedure of section 7, by considering an equivalent reversible process in which a quantity of heat $\delta Q = \delta^2\phi_S$ is absorbed by the background:

$$\delta^2 S = \frac{\delta Q}{\tau_s} = \frac{\delta^2\phi_s}{\tau_s} = -\frac{\delta^2\phi_s}{|\tau_s|} \qquad (10.15)$$

It follows from this relation that the entropy is maximum with respect to tearing modes when $\delta^2\phi_S > 0$. The thermodynamic stability depends on the sign of Δ'. The entropy is minimum and the system is unstable for $\Delta' > 0$. This is the same result as the conventional dynamical theory. We have found the stability properties of the tearing modes on purely thermodynamic grounds.

10.2.3 Free Energy and Interaction Energy of the Tearing Mode

It is instructive to calculate the other thermodynamic quantities involved in the tearing process. Let us calculate the variation of ϕ_{int} in the resonant layer. Since A and δA must be continuous across the layer, there is no contribution from the term $\int_{\Omega_s}(A-A_n)^2 d\Omega_s$ in the limit of an infinitesimal thickness.

One has then that

$$\delta^2\phi_{\rm int} = -\delta^2\phi_s = -\delta Q \qquad (10.18)$$

So $\delta^2\phi_{int}$ is the heat supplied to the background by the decaying magnetic energy ϕ_s. Clearly the system is not in Vlasov equilibrium in the inner region.

In the outer region, where $\delta Q = 0$, the energy conservation is expressed by $\delta^2 \phi_{out} + \delta L = 0$, where $\delta^2 \phi_{out}$ is the outer variation of the plasma magnetic energy and δL is the work performed on the external circuits by the collective plasma configuration adjusting itself to the varied magnetic configuration. If the process in the outer region is reversible, the work δL is expressed, as usual, (see section 7) by $\delta L = - \delta^2 F$, where $\delta^2 F$ is the global variation of the free energy. The variation $\delta^2 F$ vanishes in the resonant layer, as is seen applying the relation (7.3), and in the outer region is given by the expression

$$\delta^2 F = \sum_n \delta^2 F_n = \sum_n \left[(\delta^2\phi_{\rm int})_n + \delta^2\phi_n\right] = \sum_n \delta^2\phi_n = \delta^2\phi_{out} \qquad (10.17)$$

So the free energy available to the instability is the magnetic energy ϕ_{out} in the outer region, but the instability can develop only when this energy is allowed to sink locally to the background in the resonant layer, while the entropy is increasing. The condition for this to happen is the linear instability criterion of the tearing modes. The energy balance of the global process is expressed by the equation

$$\delta Q = \frac{1}{8\pi}\int_\Omega (\delta B_\theta)^2 d\Omega + \delta L \qquad (10.18)$$

where

$$\frac{1}{8\pi}\int_\Omega (\delta B_\theta)^2 d\Omega = \delta^2\phi_{out} + \delta^2\phi_s \qquad (10.19)$$

Here δB_θ is the variation of the poloidal field.

We now go further and look for the nonlinear evolution of the instability according to the entropy principle. We observe that all variations of order $n > 2$ vanish identically. This fact allows us to write the relation (10.13) in finite terms

$$\Delta S = \frac{\Delta Q}{\tau_s} = \frac{\Delta\phi_s}{\tau_s} \qquad (10.20)$$

where $\Delta\phi_s$ is is still given by the same equation (10.14) for $\delta^2\phi_s$. We can then apply our considerations to the nonlinear deviations from an unstable state. So when one starts from an unstable state with $\Delta' > 0$, the nonlinear evolution must be directed towards a state with a negative Δ' in order that the entropy reaches a maximum. But as soon as Δ' becomes negative the entropy is maximum and the system is thermodynamically stable with respect to the tearing modes. At this point there is no mechanism for a further evolution towards more negative Δ' values on the short time scale of the tearing modes (the evolution can, however, proceed on the longer time scale of the ordinary resistive dissipation). We then reach the conclusion that, as far as only tearing modes are considered, the system should stabilize at the marginal state with $\Delta' \approx 0$ and consequently the current profile must consistently adapt to this state. Arguments substantiating the evolution towards a state with $\Delta' \approx 0$ were given earlier by Furth (1985).

11. Conclusion

We calculated the probability distribution P of a statistical assembly of volume elements in a configurational space of the electric charge density or of the current density under a constraint which expresses the existence of Vlasov equilibrium. The Vlasov equilibrium is defined as a smeared out or collective charge and current density distribution whose energy is uncorrelated to the fluctuations of these quantities arising from the single particle structure of an underlying medium. We have shown that the statistical procedure leads to a characterization of the Vlasov equilibria in the frame of a thermodynamic formalism similar to that of the canonical ensemble. In this frame one can define an entropy $S = -\int P \lg P d\Gamma$ which is a functional of the collective quantities. We have investigated the consequences of the thermodynamic formalism when there is an energy interaction between the collective system and the external world and in particular we studied in detail the variation of the entropy of the collective system when the exchange of energy is reversible. One can then apply the conventional procedures of classical thermodynamics for defining the entropy of reversibly accessible states and through this way one can connect the variations of S, as defined above, to the heat transfer in a reversible transformation. The entropy functional is therefore inserted in the traditional context of classical thermodynamics of the macroscopic systems.

The variational properties of the functional S are connected with the equilibria and the stability of the collective system. We have explicitly illustrated this connection taking examples of electrostatic

Vlasov equilibria, in unstable situations and in the stable Maxwellian limit.

The description of the interaction of a magnetostatic collective system with the external world, provided by the thermodynamic formalism, allows a precise characterization of an open or a closed system. The former is, for instance, a plasma in a tokamak, subject to ohmic and auxiliary heating, and the latter is a plasma completely isolated from the external world by a closed perfectly conductive shell. The thermodynamically privileged states of a closed system are a maximum of the magnetic entropy. For an open system only the weaker condition of vanishing entropy production, $\frac{dS}{dt} = 0$, can be assumed.

While we consider that the present coarse-grained statistical model of the Vlasov equilibria (collisionless and in general not Maxwellian) can be formally and physically interpreted in the conceptual frame of classical thermodynamics, the problem remains to see how this model can be justified on the more fundamental grounds of the microscopic phase space description of statistical mechanics. The result of the present paper, that the <u>same</u> formalism can be scaled from the level of a system of particles to that of a system of volume elements with finite size, seems to indicate that the techniques of the renormalization group might be helpful for attacking the problem. Needless to say, the solution would be of significance for a better understanding of the interpretative basis of statistical mechanics.

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Appendix I

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