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* *See Appendix 1*

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Energy Balance in Tearing Modes

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1. Introduction

The dynamical behaviour of the tearing mode in a highly conducting fluid was fully described by Furth, Killeen and Rosenbluth [1]. Over almost all of the fluid the eigenfunctions have the form of neighbouring equilibrium solutions [2]. The governing equation for slab geometry is

$$\psi'' - k^2\psi - \frac{F''}{F}\psi = 0 \qquad (1)$$

where ψ is the perturbed flux function, k is the wave-number and $F = \underline{k}.\underline{B}$. The solution of this equation gives the quantity

$$\Delta' = \frac{\psi'\Big|_{-\varepsilon}^{+\varepsilon}}{\psi(0)} \quad (\varepsilon \to 0)$$

where ε is a small distance from the resonant surface at which F = 0 and $\psi(0)$ is the value of ψ at this surface. The eigenvalues are determined by matching the solutions of equation (1) to a solution of the full equations in a narrow layer around the resonant surface. The resulting relation takes the form

$$\Delta' = \Delta'_{in}(\gamma) \ .$$

where γ is the growth rate. Since $\Delta'(0) = 0$, the sign of Δ' determines stability, $\Delta' > 0$, giving instability. Thus stability is determined by the outer solutions and the growth rate is determined by the response of the inner layer.

Furth [3] pointed out that minimisation of the integral

$$V = \tfrac{1}{4}\int (\psi'^2 + k^2\psi^2 + \frac{F''}{F}\psi)\, dx \qquad (2)$$

over the outer region leads to equation (1) and substitution of this equation into equation (2) gives

$$V_m = \tfrac{1}{4}\psi\psi'\Big|_{-\varepsilon}^{+\varepsilon} = -\tfrac{1}{4}\psi^2(0)\Delta' \ .$$

Stability is therefore determined by the sign of the minimum of V. Furth states that the driving energy for the tearing mode comes from the gross configuration, which is able to lower its magnetic energy in fluids with finite resistivity. Under this description the energy released from the outer region provides a Poynting flux proportional to $\psi\psi'|$ and this is released in kinetic energy and Joule heating in the inner layer.

The question of the driving energy of the tearing mode was re-addressed by Adler, White and Kulsrud [4]. They found that indeed the magnetic energy released is

$$-M = \tfrac{1}{4}\psi^2(0)\Delta' \ . \qquad (3)$$

However from a detailed analysis they concluded that the driving energy comes entirely from the region inside the tearing layer.

Bondeson and Sobel [5] pointed out that equation (3) is only valid for the symmetric case. More generally

$$-M = \tfrac{1}{4}\psi^2(0)\left(\Delta' + \frac{(\pi F''/F')^2}{\Delta'}\right)$$

where F″ and F′ take their values at F = 0. They conclude that the quadratic form given by Furth must be completed by the inclusion of the additional term leading to

$$V = \tfrac{1}{4}\int(\psi'^2 + k^2\psi + \frac{F''}{F}\psi)dx \ + \tfrac{1}{4}\psi^2(0)\frac{(\pi F''/F')^2}{\Delta'} \ .$$

We shall here reconsider this problem and describe the energy balance in a way which clarifies the underlying physics.

The energy balance in tearing modes is not as straightforward as might be imagined. Consider the outer region where the current gradients which determine stability lie. In this region inertia is negligible. Consequently the force on each element of the plasma is zero and the work done in displacing each element is therefore also zero. However the underlying process is that the driving force of the instability induces a magnetic perturbation and the force arising from the resulting bending of the magnetic field lines balances this driving force. Thus we see that although the work done by each element of the plasma is zero, there is a change in the magnetic energy density.

Another way of describing this behaviour is to regard the zero work done as the balance between the change in the local magnetic energy density and an equal transfer of energy described by the divergence of the energy flux. That is

$$\delta W = 0 = \delta M + \nabla \cdot \mathbf{F} \quad .$$

When we consider the whole outer region there will be an overall magnetic energy change and an associated energy flux to the resistive layer.

It is tempting to look for the source of the energy for the instability by examining the distribution of the change in magnetic energy density. However there is no unique procedure. In general there will be regions of the positive and negative changes in magnetic energy but it is not possible to attribute the overall energy source to any particular region since the energy change at any point is a composite of stabilising the destabilising contributions. Furthermore the destabilising forces can be transmitted across the plasma to produce an energy change in another region.

We shall here adopt a different approach, based on the energy balance between recognisable physical contributions. It might be expected that this would just involve the derivation of the appropriate form of the equation for the conservation of energy. However it turns out that there are subsidiary energy balance equations relating well defined physical quantities. The sum of these equations gives the equation of energy conservation. One of the subsidiary energy balance equations is similar to the relation proposed by Furth and another describes the role of the extra term introduced by Bondeson and Sobel.

In the following sections we shall first review the tearing mode model and the basic stability theory, and derive expressions for the kinetic and magnetic energy densities. We then proceed to the analysis of the energy balance.

2. Model

The coordinate system is illustrated in Fig. 1

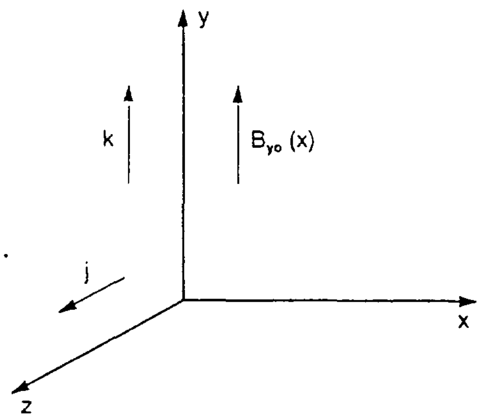

Figure 1. Showing the co-ordinate system and the directions of the equilibrium magnetic field $B_{yo}(x)$, the current density j, and the wave vector k.

The y coordinate is chosen to be in the direction of k. The perturbations are taken to be incompressible and so only the B_x and B_y components of the magnetic field enter the calculation, any equilibrium component B_{zo} playing no role.

There is a choice of perturbation variables with which to describe the problem. Here we use the magnetic flux ψ and the x-component of the displacement vector, ξ. We shall use physical quantities throughout, avoiding the use of complex variables.

3. Basic Equations

Tearing modes are described by the equation for resistive diffusion of the flux together with the equation of motion. Defining the flux function ψ by

$$B_x = \frac{\partial\psi}{\partial y} \qquad\qquad B_y = -\frac{\partial\psi}{\partial x} \quad ,$$

then Maxwell's equations

$$\underline{\nabla} \times \underline{B} = \underline{j} \qquad\qquad \underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

combine with Ohm's law

$$\underline{E} + \underline{v} \times \underline{B} = \eta \underline{j} \ .$$

and $\underline{\nabla}.\underline{v} = 0$ to give

$$\frac{\partial \psi}{\partial t} + \underline{v}.\underline{\nabla}\psi = \eta \underline{\nabla}^2 \psi \ . \qquad (4)$$

The curl of the equation of motion

$$\rho \frac{d\underline{v}}{dt} = j \times B - \nabla p$$

gives the other required equation

$$\rho \frac{\partial}{\partial t}(\underline{\nabla} \times \underline{v}) = \underline{\nabla}\psi \times \underline{\nabla}(\underline{\nabla}^2 \psi) \qquad (5)$$

where the term $\underline{v}.\underline{\nabla}\ \underline{v}$ has been dropped because it does not enter the analysis of small perturbations.

4. Linearised Equations

Linearising equation (4), using the definition

$$\underline{v} = \frac{\partial \xi}{\partial t}$$

with $\qquad \underline{\xi} = (\hat{\underline{x}}\ \xi_x \cos ky + \hat{\underline{y}}\ \xi_y \sin ky)e^{\gamma t}$

and $\qquad \psi = (\psi_o + \psi_1 \cos ky)e^{\gamma t}$

leads to

$$\gamma(\psi_1 - \xi_x B_{yo}) = \eta(\psi_1'' - k^2\psi_1) \tag{6}$$

where the primes indicate derivatives with respect to x.

Similarly, using $\nabla.\xi = 0$, that is

$$\xi_y = -\frac{1}{k}\xi_x' \quad ,$$

the linearised form of the z component of equation (5) is

$$\rho\gamma^2(\xi_x'' - k^2\xi_x) = -k^2 B_{yo}(\psi_1'' - k^2\psi_1) + k^2 j_{20}'\psi_1 \quad . \tag{7}$$

It is usual in this subject to use dimensionless variables but in the present treatment the basic physical quantities will be retained.

Equations (6) and (7) are the required linearised equations. To simplify the subsequent equations, the subscripts on ψ_1, ξ_x, B_{yo} and j_{zo} will now be dropped, that is

$$\psi_1 \to \psi, \quad \xi_x \to \xi, \quad B_{yo} \to B \quad \text{and} \quad j_{zo} \to j,$$

and we note the relationship to the usual notation

$$\frac{j'}{B} = \frac{F''}{F} \quad .$$

Equations (6) and (7) then become

$$\gamma(\psi - \xi B) = \eta(\psi'' - k^2\psi) \tag{8}$$

$$\rho\gamma^2(\xi'' - k^2\xi) = -k^2 B(\psi'' - k^2\psi) + k^2 j'\psi \quad . \tag{9}$$

5. Eigen-solution

We summarise here the usual stability calculation [1]. The analysis is based on the assumption that the ratio

$$S = \frac{\tau_R}{\tau_A}$$

is very large, where

$$\tau_R = \frac{1}{\eta k^2} \quad \text{and} \quad \tau_A = \frac{\rho^{1/2}}{B'}$$

In the outer regions inertia is negligible and ψ is given by the resulting form of equation (9)

$$\psi'' - k^2\psi - \frac{j'}{B}\psi = 0 \quad . \qquad (10)$$

The resitivity is also negligible in this region and, from equation (8), ξ is given by

$$\xi = \frac{\psi}{B} \quad .$$

The solution of equation (10) which satisfies the chosen boundary conditions has a discontinuous derivative at the surface on which B = 0. Taking x = 0 to be at this surface, the magnitude of the discontinuity is measured by the quantity

$$\Delta' = \left.\frac{\psi'}{\psi}\right|_{-\varepsilon}^{+\varepsilon} \quad (\varepsilon \to 0) \quad .$$

In the inner region the full equations are solved but, because of the narrowness of the layer, ρ, η and B' can be taken as constants. Furthermore in this region the k^2 terms are small compared to the second derivatives, and the last term in equation (9) can be neglected. Equations (8) and (9) can then be arranged in the form

$$\psi'' = \frac{\gamma}{\eta}(\psi - \xi B' x) \qquad (11)$$

$$\xi'' = -\frac{k^2 B'}{\eta\rho\gamma}(\psi - \xi B' x)x + \frac{k^2 j'}{\rho\gamma^2}\psi \qquad (12)$$

where B', η and ρ take their values at x = 0.

The solution of equations (11) and (12) is such that ψ is approximately constant, that is

$$\psi = \psi(0) + \text{small terms}$$

The small terms play a crucial role but equation (12) can be solved for ξ keeping only $\psi(0)$. Thus equation (12) can be written

$$\frac{\eta\rho\gamma}{k^2B'^2}\xi'' + x^2\xi = -\frac{1}{B'}x\psi(0) + \frac{\eta j'}{\gamma B'^2}\psi(0) \qquad (13)$$

This is an inhomogeneous equation for ξ. The first term on the right-hand side of the equation is odd in x and the second is even. Thus these terms produce odd and even parts of the solution for ξ. It turns out that only the odd part of ξ enters the stability analysis. This is an interesting feature, the significance of which will become apparent later. The equation to be solved is therefore

$$\frac{\eta\rho\gamma}{k^2B'^2}\xi_o'' + x^2\xi_o = -\frac{1}{B'}x\psi(0) \quad ,$$

where ξ_0 is the odd part of ξ. This gives a solution of the form

$$\xi_0 = f\left[(B',\eta,\rho,k,\gamma)\,,\,x\right]\psi(0) \quad ,$$

the characteristic width of the solution being

$$d = \left(\frac{\eta\gamma\rho}{kB'}\right)^{1/4} \quad . \qquad (14)$$

The solution for ξ_0 is substituted into equation (11), again putting $\psi = \psi(0)$ on the right-hand side, to obtain the change in ψ'/ψ across the inner layer

$$\Delta'_{in} = \left.\frac{\psi'}{\psi}\right| = \frac{\gamma}{\eta}\int\frac{\xi_o}{\psi(0)}B'x\,dx \quad .$$

The dispersion relation is then given by

$$\Delta' = \Delta'_{in}(B',\eta,\rho,k,\gamma)$$

and the specific form of Δ'_{in} leads to the growth rate

$$\gamma = 0.55\frac{(\Delta'/k)^{4/5}}{\tau_A{}^{2/5}\tau_R{}^{3/5}} \quad .$$

To illustrate the behaviour of the functions ψ and ξ_o, examples taken from reference [6] are shown in Fig. 2. In the case shown, the resonant surface is at the mid-plane of a symmetric equilibrium but no such symmetry is assumed in the present general analysis.

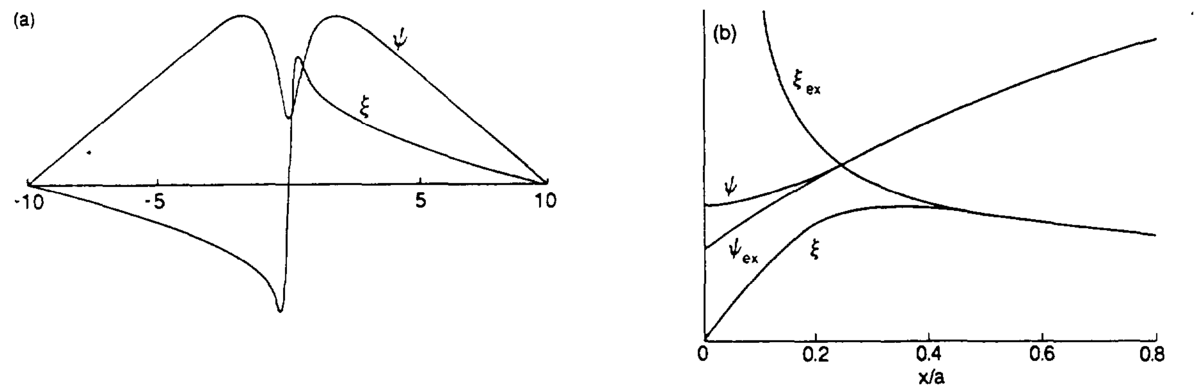

Figure 2 (a) Eigenfunctions ψ and ξ for the symmetric mode of a current configuration $j = j_o/(1 + x^2/a^2)$ with ka = 0.1, the ratio of the characteristic resistive time to mhd time, $S = 10^3$ and a conducting wall at x = 10a. (b) Expanded forms of the same eigenfunctions in the inner region together with the extensions, ψ_{ex} and ξ_{ex}, of the outer solutions into this region.

Before deriving energy balance equations we shall derive equations for the kinetic and magnetic energies.

6. Kinetic Energy

In calculating energies we shall follow convention and use the average energy per unit area in the (y,z) plane. Then the kinetic energy, $\int \frac{1}{2} \rho v^2 dx$, over any regions of x is

$$K = \frac{k}{\pi}\int_o^{\pi/k} \int \tfrac{1}{2} \rho \gamma^2 \left(\xi^2 \cos^2 ky + \xi_y^2 \sin^2 ky\right) \, \mathrm{dxdy} \quad .$$

Thus

$$K = \tfrac{1}{4}\gamma^2 \int \rho\left(\xi^2 + \xi_y^2\right) dx$$

and using $\underline{\nabla}.\underline{\xi} = 0$

$$K = \tfrac{1}{4}\frac{\gamma^2}{k^2}\int \rho\left(\xi'^2 + k^2\xi^2\right) \mathrm{dx} \quad . \qquad (15)$$

7. Magnetic Energy

The magnetic energy is $\int \frac{1}{2} B^2 dx$ and the change in magnetic energy is therefore

$$M = \frac{k}{\pi}\int_o^{\pi/k}\int \tfrac{1}{2}\left(B_{x1}^2 + B_{y1}^2 + 2B_{yo}B_{y2}\right) \mathrm{dxdy} \quad . \qquad (16)$$

The required y independent, second order B_{y2} field is given by the y independent, second order flux, ψ_2, and following Adler et al this is determined from the second order part of equation (4). Thus, noting that the second order terms have an exp $(2\gamma t)$ dependence,

$$\int_o^{\pi/k}\left[2\gamma\psi_2 + \gamma\left(\xi\psi'\cos^2 ky - \xi_y k\psi\sin^2 ky\right)\right] \mathrm{dy} = \int_o^{\pi/k} \eta\psi_2'' \, \mathrm{dy} \quad .$$

Thus, using $\underline{\nabla}.\underline{\xi} = 0$

$$2\gamma\psi_2 + \frac{1}{2}\gamma(\xi\psi' + \xi'\psi) = \eta\psi_2''$$

and hence ψ_2 is determined by the second order inhomogeneous equation

$$\eta\psi_2'' - 2\gamma\psi_2 = \frac{1}{2}\gamma(\xi\psi)' \quad . \qquad (17)$$

Writing the perturbed magnetic fields in equation (15) in terms of the first and second order fluxes now gives

$$M = \frac{k}{\pi}\int_o^{\pi/k}\int \tfrac{1}{2}\left(\left(\psi'^2 + k^2\psi^2\right)\cos^2 ky \; - 2B\psi_2'\right) \mathrm{dxdy}$$

so that

$$M = \tfrac{1}{4}\int\left(\psi'^2 + k^2\psi^2 - 4B\psi_2'\right)\,\mathrm{dx} \tag{18}$$

where ψ_2 is determined by equation (17).

8. Energy Balance Equations

In addition to the equation describing the conservation of energy there are separate equations which describe the energy balance between the various physical effects. We shall first derive two such equations. The sum of these two equations gives the equation of energy conservation. The equations will initially be for an arbitrary region of x.

9. First Energy Balance Equation

We first write equations (5) and (6) in the form

$$\gamma\xi = -\frac{\eta}{B}\left(\psi'' - k^2\psi\right) + \frac{\gamma}{B}\psi \tag{19}$$

$$\rho\frac{\gamma}{k^2}\left(\xi'' - k^2\xi\right) - \mathrm{j}'\psi = -\mathrm{B}\left(\psi'' - \mathrm{k}^2\psi\right) \tag{20}$$

Equating the products of the left-hand sides of these equations to the product of the right hand sides gives

$$\rho\frac{\gamma^3}{k^2}\xi\left(\xi'' - k^2\xi\right) - \gamma\mathrm{j}'\xi\psi = \eta\left(\psi'' - \mathrm{k}^2\psi\right)^2 - \gamma\psi\left(\psi'' - k^2\psi\right)$$

Integrating over x, and integrating the second derivative terms in the first and last terms by parts

$$\tfrac{1}{2}\gamma\int(\psi'^2 + k^2\psi^2 + j'\xi\psi + \frac{\rho\gamma^2}{\mathrm{k}^2}(\xi'^2 + k^2\xi^2))dx = -\tfrac{1}{2}\int\eta(\psi'' - k^2\psi)^2 dx$$

$$+(\tfrac{1}{2}\gamma\,\psi\psi' + \tfrac{1}{2}\frac{\rho\gamma^3}{k^2}\xi\xi')\Big|$$

and noting that

$$\psi'' - k^2\psi = -j_1$$

we have

$$\tfrac{1}{2}\gamma\int(\psi'^2 + k^2\psi^2 + j'\xi\psi)dx + \frac{dK}{dt} = \int\tfrac{1}{2}\eta j_1^2 dx + \left(\tfrac{1}{2}\psi\psi' + \tfrac{1}{2}\frac{\rho\gamma^3}{k^2}\xi\xi'\right)\Big| \quad . \qquad (21)$$

Writing for the ohmic heating term

$$\Omega_1 = \tfrac{1}{2}\int\eta j_1^2 dx$$

and considering the whole region, equation (21) becomes

$$\tfrac{1}{4}\gamma\int\left(\psi'^2 + k^2\psi^2 + j'\xi\psi\right)dx = -\frac{dK}{dt} + \Omega_1 \quad .$$

This equation was given by Furth. It is an exact equation but at this stage it does not allow an attribution of the left hand side to the outer region and the right hand side to the inner region. We shall return to this question shortly but first we shall analyse the energy balance more generally.

10. Second Energy Balance Equation

The second energy balance equation is obtained by multiplying equation (17) by j and integrating over x. Thus using partial integration on each term, and noting that

$$\psi_2'' = -j_2$$

we obtain

$$\tfrac{1}{4}\gamma\int(4B\psi_2' + j'\xi\psi)\,dx = -\int\eta jj_2\,dx - \left(\tfrac{1}{2}\gamma j\xi\psi + 2\gamma B\psi_2\right)\Big| \qquad (22)$$

11. Equation Energy Conservation

Adding equations (21) and (22) together we obtain the equation of energy conservation

$$\frac{d}{dt}(M+K) = -\int (\tfrac{1}{2}\eta j_1^2 + \eta j j_2)dx + F \qquad (23)$$

with

$$F = \left(-\tfrac{1}{2}\gamma j\xi\psi + \tfrac{1}{2}\left(\rho\gamma^3/k^2\right)\xi\xi' - 2\gamma B\psi_2 + \tfrac{1}{2}\gamma\psi\psi'\right)\Big| \quad .$$

The integral represents the ohmic heating and F is the energy flux represented by the boundary term. The factor $\frac{1}{2}$ in the j_1^2 term results from averaging $\cos^2 ky$ over a wavelength. The first term in F is the flux, due to the displacement ξ, of the linear energy transfer to the plasma $E_1 j$, the second term is the kinetic energy flux and the last two terms constitute the Poynting flux.

We shall now proceed to analyse the energy transfers involved in the instability. First however it is necessary to identify the term representing the "driving" energy for the instability.

12. The Driving Energy

The energy for the instability of the incompressible fluid clearly has to come from the magnetic field. Equation (18) for the magnetic energy change contains the familiar B_{x1}^2 and B_{y1}^2 stabilising terms together with the rather opaque $B\psi_2'$ term. However, we know that stability is determined by the outer equation

$$\psi'' - k^2\psi - \frac{j'}{B}\psi = 0 \quad .$$

If this equation is multiplied by ψ and integrated it gives Furth's relation

$$\int (B_{x1}^2 + B_{y1}^2 + \frac{j'}{B}\psi^2)dx = \psi\psi'\Big| \quad . \qquad (24)$$

This equation suggests that the free energy comes from the current gradient term. More explicitly the rate of energy transfer to the plasma is $E_1 j_1$, and part of j_1 arises from this current gradient. This part of j_1 is $-\xi j'$, and using $E_1 = \gamma\psi$, the resulting rate of release of energy is

$$D = -\tfrac{1}{2}\gamma\int j'\xi\psi \, dx \quad . \qquad (25)$$

In the outer region where $\xi = \psi B$, D has the form of the driving term in equation (24).

In what follows we shall explore the consequences of assuming that equation (25) does in fact represent the rate of release of free energy.

13. Overall Energy Balance

From equation (18) the rate of change of magnetic energy is

$$\frac{dM}{dt} = \tfrac{1}{2}\gamma\int\left(\psi'^2 + k^2\psi^2\right)dx - S - D, \tag{26}$$

where

$$S + D = 2\gamma\int B\psi_2'\, dx \quad .$$

The first two terms in equation (26) are stabilising. Taking the second energy balance equation (22) for the whole region we find

$$S = \int \eta\, j\, j_2\, dx \quad . \tag{27}$$

and it is seen that the magnetic energy source term S balances the ohmic heating term involving j_2. The role of the remainder of the energy balance, as represented by the driving term D, can be determined using the first energy balance equation (21). For the whole region this equation can be written

$$D = \tfrac{1}{2}\gamma\int\left(\psi'^2 + k^2\psi^2\right)dx + \tfrac{1}{2}\frac{\gamma^3}{k^2}\int\rho\left(\xi'^2 + k^2\xi^2\right)dx + \tfrac{1}{2}\int\eta\, j_1^2 dx\ . \tag{28}$$

Writing

$$M_1 = \tfrac{1}{4}\int\left(\psi'^2 + k^2\psi^2\right)dx$$

and

$$\Omega_1 = \tfrac{1}{2}\int \eta\, j_1^2\, dx$$

equation (28) becomes

$$D = \frac{dM_1}{dt} + \frac{dK}{dt} + \Omega_1$$

Thus, summarising, the source term S balances the y independent ohmic heating term and the driving term D gives the increase in magnetic energy arising from B_1^2, plus the kinetic energy, plus the Joule heat arising from ηj_1^2.

We shall now investigate the energy flow between the outer and inner regions. This will lead to a natural splitting of the first energy balance equation into two parts.

14. Energy Flow Between the Two Regions and Splitting of First Energy Balance Equation

Equation (28) gives the first energy balance equation over the whole region. Now the kinetic and resistive terms are negligible in the outer region, and the contribution of the magnetic energy integral in the inner region is negligible because of its narrowness. So, without affecting the values of the integrals, equation (28) can be written

$$D = \left(\tfrac{1}{2}\gamma\int(\psi'^2 + k^2\psi^2)dx\right)_{out.} + \left(\tfrac{1}{2}(\gamma^3/k^2)\int\rho(\xi'^2 + k^2\xi^2)dx + \tfrac{1}{2}\int\eta j_1^2 dx\right)_{in} \cdot \quad (29)$$

In order to make further progress it is necessary to separate D into its contributions in the two regions. In the outer region $\xi = \psi/B$ and the definition of D given by equation (25) becomes

$$D_{out.} = \left(\tfrac{1}{2}\gamma\int\frac{j'}{B}\psi^2 dx\right)_{out.} \cdot$$

In the inner region D becomes

$$D_{in} = \left(\tfrac{1}{2}\gamma j'\int(\xi_e\psi(0) + \xi_o\psi_o)dx\right)_{in} . \quad (30)$$

where the subscripts e and o refer to the even and odd parts of the solutions for ξ and ψ in the inner region. First we shall determine the relative order of the two terms appearing in the integral of equation (30). To determine the relative ordering of ξ_e to ξ_o we need the ordering of the ratio of the even and odd terms on the right-hand side of equation (13) which determines ξ. This ratio is $\eta j'/\gamma B'd$ where d is the characteristic width of the layer as given by equation (14). Thus, since

$$d \sim \frac{1}{kS^{2/5}} \qquad and \qquad \gamma \sim \frac{1}{\tau_A{}^{2/5}\tau_R{}^{3/5}}$$

we have for the required ratio,

$$\frac{\xi_e}{\xi_o} \sim \frac{\eta j'}{\gamma B' d} \sim S^0$$

and so $\xi_e \sim \xi_o$.

We now need the ordering of the other factors, ψ_o and $\psi(0)$, appearing in equation (30). The order of $\psi_o/\psi(0)$ is given by the form of ψ in the outer region as the layer is approached. The small x expansion for ψ is [7]

$$\psi = \psi(0)\left[1 + \frac{j'}{B'} x \ \ell n|x| + Ax\right]$$

and so

$$\psi_o \sim \psi(0)\frac{j'}{B'} d \ \ell n d \sim \frac{\psi(0)}{S^{2/5}} \ell n S^{2/5} << \psi(0)$$

Thus in equation (30) the second term is negligible and

$$D_{in} = \tfrac{1}{2}\gamma j' \psi(0) \int \xi_e \, dx \quad .$$

An energy balance equation for D_{in} can now be obtained using the even parts of equations (19) and (20) in the inner region, that is

$$\gamma \xi_e = \frac{\eta}{B}\left(\psi_o'' - k^2 \psi_o\right) + \frac{\gamma}{B}\psi_o$$

$$\rho \frac{\gamma^2}{k^2}\left(\xi_e'' - k^2 \xi_e\right) - j'\psi_e = -B\left(\psi_o'' - k^2\psi_o\right) \quad .$$

Then following the same procedures as used following equations (19) and (20) we obtain

$$D_{in} = \frac{1}{2}\frac{\gamma^3}{k^2}\int\left(\xi_e'^2 + k^2 \xi_e^2\right)dx + \frac{1}{2}\int \eta \, j_{1e}^2 \, dx \qquad (31)$$

where the integrals are over the inner region and the magnetic energy term involving $(\psi_o'^2 + k^2\psi_o^2)$ is small and has therefore been dropped. Subtracting equation (31) from equation (28) we obtain

$$D_{out} = \tfrac{1}{2}\gamma\int\left(\psi'^2 + k^2\psi^2\right)dx + \tfrac{1}{2}\frac{\gamma^3}{k^2}\int\rho\left(\xi_o'^2 + k^2\xi_o^2\right)dx + \tfrac{1}{2}\int\eta\, j_{1o}^2\, dx \qquad (32)$$

Equation (32) expresses the energy balance associated with the stability calculation which was outlined in Sections (4) and (5). Rewriting equation (32) in the form

$$\left(-\tfrac{1}{4}\int(\psi'^2 + k^2\psi^2 + \frac{j'}{B}\psi^2)dx\right)_{out} = \tfrac{1}{4}\psi\psi'\Big|_{-\varepsilon}^{+\varepsilon} = \left(\tfrac{1}{4}(\frac{\gamma^2}{k^2}\int\rho(\xi_o'^2 + k^2\xi_o^2) + \tfrac{1}{4}\frac{1}{\gamma}\int\eta j_{1o}^2 dx\right)_{in},$$

we see that magnetic energy from the outer region flows via the Poynting flux to provide kinetic energy and ohmic heating in the inner layer. The magnetic energy term is just that which results from the terms in the stability equation for the outer region, and this provides the energy associated with the odd parts of the inner solution, ξ_o and j_{1o} ($= -\psi_o''$), which were the only parts involved in the stability calculation.

Equation (31) is seen to be an independent energy balance equation for the even functions ξ_e and j_{ie}. The kinetic and Joule energies associated with these terms arise solely in the inner region and are balanced by a driving energy, D_{in}, which is also localised within this region. Thus, although equation (31) describes part of the energy balance, it has a subsidiary role and the processes involved in this energy balance occur "in parallel" with the principal processes described by equation (32).

15. Summary of Energy Balance

It is perhaps helpful to bring together the energy balance equations and to show how they lead to the equation of conservation of energy. Thus writing the kinetic energy

$$K = K_e + K_o$$

and the ohmic heating

$$\Omega = \Omega_e + \Omega_o + \Omega_2$$

with

$$K_e = \tfrac{1}{4}\frac{\gamma^2}{k^2}\int\rho\left(\xi_e'^2 + k^2\xi_e^2\right)dx \qquad \Omega_e = \tfrac{1}{2}\int\eta\, j_{1e}^2\, dx$$

$$K_o = \tfrac{1}{4}\frac{\gamma^2}{k^2}\int\rho\left(\xi_o'^2 + k^2\xi_o^2\right)dx \qquad \Omega_o = \tfrac{1}{2}\int\eta\, j_{1o}^2\, dx$$

$$\Omega_2 = \int\eta\, j\, j_2\, dx$$

the energy balance equations (32), (31) and (27) may be written

$$D_{out} = \frac{dM_1}{dt} + \frac{dK_o}{dt} + \Omega_o$$
$$D_{in} = \frac{dK_e}{dt} + \Omega_e$$
$$S = \Omega_2$$

Adding these three equations together and recalling that D = D_{out} + D_{in} and that from equation (26)

$$\frac{dM}{dt} = \frac{dM_1}{dt} - S - D,$$

we obtain the equation for the conservation of energy

$$\frac{dM}{dt} + \frac{dK}{dt} + \Omega = O \quad ,$$

16. Potential Energy, δW

Furth writes for the infinite conductivity energy principle integral

$$\delta W_\infty = \int B^2\left(\xi'^2 + k^2\xi^2\right) dx \quad .$$

However the potential energy change is

$$\delta W = \int \xi \cdot F(\xi) dx$$

where F(ξ) is the perturbed force. Now in the outer region of the plasma F(ξ) = 0, and so

$$\delta W_{out} = O .$$

This is at first sight surprising since the description we have arrived at is one in which the driving term for the instability arises in the outer region. However there is no conflict between these results. The force which provides the driving torque for the instability is transmitted across the plasma through the magnetic field deformation. It exerts its effect in the resistive layer through the resulting magnetic field deformation at the edge of the layer, this deformation providing the boundary condition for the layer dynamics.

17. Summary

From our analysis we see that we can describe the exact energy balance in the tearing mode in a way which is consistent with intuition. This corresponds quite closely to Furth's description. The only clarification which was needed arises from the recognition that the energy flow arising from Furth's energy integral V is dissipated solely in the kinetic energy association with the odd part of ξ and the ohmic heating associated with the odd j_1. This has a satisfying self-consistency in that it is only these variables which arise in the stability calculation, since the minimisation of V gives Δ', and Δ'_{in} is obtained by solving for the odd ξ and using this to integrate the odd ψ''.

This energy balance forms part of what we have called the first energy balance equation. The remaining part of this equation involves only layer quantities and shows how the internal driving term D_{in} is complementary to D_{out}. Thus whereas D_{out} drives the "odd" energies, D_{in} drives the kinetic energy involving the even part of ξ and the ohmic heating associated with the even ψ''. This latter energy transfer corresponds to the extra term introduced by Bondeson and Sobel. The other energy balance is between the residual source term S and the y-independent ohmic heating term involving $\eta j j_2$.

Adler et al., attribute the driving energy to the magnetic energy decrease in the resistive layer. Their results show that there are energy contributions of varying signs from different regions and that the total magnetic energy change is approximately equal to that in the inner layer. However there is no unique accounting procedure and the attribution of the driving energy to the inner layer is somewhat arbitrary. We should also recognise that the lowering of the magnetic energy in a region does not mean that the region necessarily provides the energy source for the instability. The magnetic field energy can be lowered by the transmission of forces exerted elsewhere.

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Appendix I

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