JOINT EUROPEAN TORUS

JET–P(91)14

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* *See Appendix 1*

Preprint of Paper to be submitted for publication in
Plasma Physics and Controlled Fusion

Dielectric effects on Thomson scattering in a relativistic magnetized plasma

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24/4-1991

Abstract

The effects of the dielectric properties of a relativistic magnetized plasma on the scattering of electromagnetic radiation by fluctuations in electron density are investigated. The origin of the density fluctuations is not considered. Expressions for the scattering cross section and the scattered power accepted by the receiving antenna are derived for a plasma with spatial dispersion. The resulting expressions allow thermal motion to be included in the description of the plasma and remain valid for frequencies of the probing radiation in the region of ω_p and ω_{ce}, provided the absorption is small. Symmetry between variables relating to incident and scattered fields is demonstrated and shown to be in agreement with the reciprocity relation. Earlier results are confirmed in the cold plasma limit. Significant relativistic effects, of practical importance to the scattering of millimeter waves in large tokamaks, are predicted.

1 Introduction

The modelling of Thomson scattering is traditionally split into two major parts: (A), the determination of the fluctuations in electron density and other quantities which give rise to scattering and (B), relations between incident and scattered fields, given the fluctuations. (A) typically deals with the spectral density function $S(\mathbf{k},\omega)$ while (B), the subject of this paper, is concerned with scattering cross sections and equations of transfer between launcher and receiver in a scattering diagnostic system. If vacuum propagation is assumed the task is a relatively straightforward summation of Lienard–Wiechert fields (see e.g. Hutchinson, 1989). Dielectric effects in a cold plasma have been investigated by a number of authors (Akhiezer *et al.*, 1958, 1962 and 1967; Sitenko, 1967; Simonich, 1971; Simonich and Yeh, 1972; Bretz, 1987; Hughes and Smith, 1989). The present work extends the theory of dielectric effects to hot and relativistic plasmas.

A major motivation for the present work is the fact that collective Thomson scattering diagnostics, intended to measure alpha particle and other fast ion velocity distributions in tokamak plasmas, are presently under development (Costley *et al.*, 1988, 1989a and 1989b; Woskov *et al.*, 1988; Machuzak *et al.*, 1990). In these diagnostics the frequency of the probing radiation is in general not large relative to the electron plasma and cyclotron frequencies. This means that a number of simplifying assumptions, which are well satisfied for the majority of laser scattering experiments, are not valid. Most notably, vacuum propagation cannot be assumed. In the work by Hughes and Smith (1989), the Thomson scattering cross section and the scattered power accepted by the receiving antenna were investigated for a cold plasma. It was shown that, in the parameter ranges relevant for the planned alpha and fast ion Thomson scattering diagnostic at JET, it is necessary to take the dielectric properties of the plasma into account. The dielectric effects manifest themselves in the term referred to as the *geometrical factor*, G. Relative to predictions based on vacuum propagation, large increases in the scattered power were found for X to X mode scattering, with a singularity at the R-cutoff (ordinary and extraordinary mode are referred to as O and X mode respectively). X to X scattering may therefore be an attractive option. At the high temperatures found in large tokamaks, the scattering theory based on the cold plasma model is, however, not reliable in this regime: relativistic effects must be taken into account.

In this paper the theory of Thomson scattering in a magnetized plasma with spatial dispersion is developed *ab initio*. Spatial dispersion, which is caused by thermal motion, must be taken into account when describing the plasma by hot or relativistic models. Expressions for the differential scattering cross section and the geometrical factor G are derived. Earlier results are confirmed in the limit of no spatial dispersion (cold plasma). In the transparency regime (when the anti-Hermitian part of the dielectric tensor can be neglected) the expression for G can be written in a compact form which is symmetrical with respect to incident and scattered fields. It is shown that this symmetry follows from the reciprocity relation. The earlier result by Hughes and Smith (1989) lacks this symmetry. It does, however, become symmetrical if care is taken to preserve invariance under time reversal when introducing the perturbations to the dielectric tensor caused by the density fluctuations.

The paper is organized as follows. In Section 2 a homogeneous plasma is first assumed and the field resulting from an embedded current is found. The far field is identified with the propagating modes. An inhomogeneous plasma is then considered and the field at the detector (e.g. the mixer for heterodyne detection), resulting from a point current source embedded in the plasma, is derived. In Section 3 the source currents for a scattered field are identified as the currents arising from the interaction of an incident field with density perturbations. The equation of transfer is derived for a complete scattering system and the geometrical factor, G, is identified. The differential scattering cross section is derived separately. In the transparency regime, G is put into a compact symmetrical form. Section 4 contains the results of numerical computations of G for cold, hot and weakly relativistic plasmas. In Section 5 the impact on scattering diagnostics is discussed.

Some material has been placed in appendices to ease the flow of the development in the main body of the paper. In Appendix A the Fourier–Laplace transformation used throughout the paper is defined. In Appendix B the energy flux associated with a broad–band field in a spatially dispersive medium is derived from Poynting's theorem and presented in a convenient form. A useful relation between the energy flux and the derivative of the Maxwell operator with respect to the wave vector is also shown. Appendix C discusses the étendue of an antenna which is looking at an anisotropic plasma. Appendix D discusses preservation of invariance under time reversal, the symmetry of the equation of transfer and the reciprocity relation for a scattering system. A list of symbols is given in Appendix E.

2 Field due to current source in plasma

In a medium with spatial and temporal dispersion Maxwell's equations can be written in the form (Landau, Lifshitz and Pitaevskii, 1984, §103):

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad , \tag{1}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j} \quad , \tag{2}$$

where $\mathbf{j}$ refers to externally induced currents and

$$\mathbf{B}(\mathbf{r},t) = \mu_0 \mathbf{H}(\mathbf{r},t) \quad , \tag{3}$$

$$\frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t} = \varepsilon_0 \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} + \int \boldsymbol{\sigma}'(\mathbf{r},t,\mathbf{r}',t')\mathbf{E}(\mathbf{r}',t')\, d\mathbf{r}'\, dt'. \tag{4}$$

The rank two tensor, $\boldsymbol{\sigma}(\mathbf{r}, t, \mathbf{r}', t')$, is the kernel of the integral operator which describes the current response of a general inhomogeneous non–stationary medium.

2.1 Homogeneous magnetized plasma

In a stationary and homogeneous medium the kernel of the integral in (4) simplifies:

$$\frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t} = \varepsilon_0 \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} + \int \boldsymbol{\sigma}(\mathbf{r}-\mathbf{r}', t-t')\mathbf{E}(\mathbf{r}',t')\, d\mathbf{r}'\, dt'. \tag{5}$$

Taking the curl of (1) and eliminating $\mathbf{B}$ gives the wave equation:

$$\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2}\frac{\partial^2 \mathbf{D}}{\partial t^2} = -\frac{1}{\varepsilon_0 c^2}\frac{\partial \mathbf{j}}{\partial t} \quad . \tag{6}$$

Fourier transforming (see Appendix A) yields:

$$\mu^2\, \hat{\mathbf{k}} \times \left(\hat{\mathbf{k}} \times \mathbf{E}(\mathbf{k},\omega)\right) + \mathbf{K}(\mathbf{k},\omega)\mathbf{E}(\mathbf{k},\omega) = \frac{-i}{\omega\varepsilon_0}\mathbf{j}(\mathbf{k},\omega) \quad . \tag{7}$$

μ is the scalar refractive index, $\mu = kc/\omega$. $\mathbf{K}$ is the dielectric tensor:

$$\mathbf{K}(\mathbf{k},\omega) = \mathbf{I} + \frac{i\boldsymbol{\sigma}(\mathbf{k},\omega)}{\omega\varepsilon_0} \tag{8}$$

where $\mathbf{I}$ is the identity tensor and $\boldsymbol{\sigma}(\mathbf{k},\omega)$, the conductivity tensor, is the Fourier transform of the kernel of the integral in (5). It is convenient to introduce also the susceptibility tensor, $\mathbf{Q}(\mathbf{k},\omega)$:

$$\begin{aligned} \mathbf{Q}(\mathbf{k},\omega) &= \frac{i\boldsymbol{\sigma}(\mathbf{k},\omega)}{\omega\varepsilon_0} \\ &= \mathbf{K}(\mathbf{k},\omega) - \mathbf{I} \quad . \end{aligned} \tag{9}$$

The tensor acting on $\mathbf{E}$ on the left hand side in (7) will be referred to as the Maxwell tensor, $\boldsymbol{\Lambda}$:

$$\mathbf{\Lambda}(\mathbf{k},\omega) = \mu^2 \left\{\hat{\mathbf{k}} \times \hat{\mathbf{k}}\times\right\} + \mathbf{K}(\mathbf{k},\omega) \tag{10}$$

With this tensor the wave equation can be written compactly:

$$\mathbf{\Lambda E} = \frac{-i}{\omega\varepsilon_0}\mathbf{j} \quad . \tag{11}$$

The fields resulting from a set of externally induced currents can be found by solving for $\mathbf{E}$ in equation (11). To this end it is useful to introduce the eigenvalues, λ_j, and unit eigenvectors, $\hat{\mathbf{g}}_j$, of the Maxwell tensor. Let the columns of the tensor g_{ij} be the unit eigenvectors (index j identifies the j^{th} eigenvector). Let λ_{ij} be a diagonal tensor containing the eigenvalues:

$$\lambda_{ij} = \left\{ \begin{array}{ccc} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{array} \right\} \quad .$$

The eigenvalue equation for $\mathbf{\Lambda}$ can be expressed as

$$\Lambda_{ij}\, g_{jk} = g_{il}\, \lambda_{lk} \quad . \tag{12}$$

Solving for $\mathbf{\Lambda}$,

$$\Lambda_{ij} = g_{il}\, \lambda_{lk}\, g_{kj}^{-1} \quad , \tag{13}$$

where

$$g_{ij}^{-1} g_{jk} = \delta_{ik} \quad .$$

Note that $\mathbf{\Lambda}$ is in general not Hermitian and therefore $g_{ij}^{-1} \neq g_{ji}^*$.

The inverse of $\mathbf{\Lambda}$ is

$$\Lambda_{ij}^{-1} = g_{il}\, \lambda_{lk}^{-1}\, g_{kj}^{-1} \quad , \tag{14}$$

where

$$\lambda_{ij}^{-1} = \left\{ \begin{array}{ccc} \lambda_1^{-1} & 0 & 0 \\ 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & \lambda_3^{-1} \end{array} \right\} \quad .$$

Solving for $\mathbf{E}$ in (11) and inserting (14) we obtain

$$E_i = \frac{-i}{\omega\varepsilon_0} \sum_{jk} \frac{g_{ij}\, g_{jk}^{-1}\, j_k}{\lambda_j} \quad , \tag{15}$$

where there is no implicit summation over repeated indices.

$\mathbf{E}$ is analytic in the four complex arguments $(\mathbf{k}, \omega)$, except on a number of singular surfaces, which will play an important role in the Fourier inversion of (15).

Assuming $\mathbf{j}(\mathbf{r}, t)$ is physically reasonable then $\mathbf{j}(\mathbf{k}, \omega)$ will have no singularities for finite values of $(\mathbf{k}, \omega)$. g_{ij} and g_{ij}^{-1} can also be assumed to be analytic for all finite values of $(\mathbf{k}, \omega)$. The only singularities of $\mathbf{E}$ are therefore the poles where one of the eigenvalues, λ_ν, equal zero. When $\lambda_\nu = 0$, the eigenvalue equation for $\hat{\mathbf{g}}_\nu$ (ref. equation 12) is identical to the homogeneous field equation

$$\mathbf{\Lambda}\hat{\mathbf{e}} = 0 \quad . \tag{16}$$

A solution to (16) only exists when $\mathrm{Det}\{\mathbf{\Lambda}\} = 0$, which is the dispersion equation for electromagnetic waves in the medium. This implies that, for values of ω and $\mathbf{k}$ which satisfy the dispersion equation $\mathrm{Det}\{\mathbf{\Lambda}(\mathbf{k}, \omega)\} = 0$, there will be at least one eigenvalue $\lambda_\nu(\mathbf{k}, \omega)$ which is equal to zero. In this case $\hat{\mathbf{g}}_\nu$ is identical to the polarization vector, $\hat{\mathbf{e}}$, of the electric field of the mode $(\mathbf{k}, \omega)$:

$$\hat{\mathbf{e}} = \hat{\mathbf{g}}_\nu \; , \quad \left\{\mathbf{e}^{-1}\right\}_i = g_{\nu i}^{-1} \quad \text{when} \quad \lambda_\nu = 0 \quad . \tag{17}$$

For the subsequent analysis it is convenient to express $\mathbf{k}$ in polar coordinates:

$$\mathbf{k} = k\hat{\mathbf{k}} \tag{18}$$

where

$$k = \sqrt{\mathbf{k}\cdot\mathbf{k}} \quad ; \qquad \begin{cases} k_{\mathrm{Re}} \geq 0 & \text{for} \quad \mathrm{Re}\{\mathbf{k}\cdot\hat{\mathbf{n}}\} \geq 0 \\ k_{\mathrm{Re}} < 0 & \text{for} \quad \mathrm{Re}\{\mathbf{k}\cdot\hat{\mathbf{n}}\} < 0 \end{cases} \quad . \tag{19}$$

$\hat{\mathbf{n}}$ is for the moment an arbitrary unit vector. Note that if the complex conjugate of $\mathbf{k}$ had been used in the definition of k then $\mathbf{E}$ would not have been an analytic function of k and $\hat{\mathbf{k}}$. With the definition given in (19) $\mathbf{E}$ is an analytic function of $(k, \hat{\mathbf{k}})$, which is important for the subsequent analysis.

The surfaces in $(\mathbf{k}, \omega)$ space, on which $\mathbf{E}(\mathbf{k}, \omega)$ is singular, are the surfaces where the dispersion relation,

$$\mathrm{Det}\{\mathbf{\Lambda}(k_m, \hat{\mathbf{k}}, \omega)\} = 0 \quad , \tag{20}$$

is satisfied. Solving for k_m in (20),

$$k_m = k_m(\hat{\mathbf{k}}, \omega) \quad , \tag{21}$$

gives the location of the pole in the complex k plane as a function of $(\hat{\mathbf{k}}, \omega)$. (21) is of course simply a parametric representation of the singular surfaces, with $(\hat{\mathbf{k}}, \omega)$ as the free parameters.

To define the inverse Fourier transform of (15) it is necessary to specify the contours for the ω and $\mathbf{k}$ integrations. $\mathbf{E}$ is the response of a causal system to the driving force $\mathbf{j}$. From the condition that the system response be causal, it follows (see Appendix A) that the path of integration in the ω plane must pass above all singularities in the finite ω plane. For sufficiently large values of the imaginary part, ω_{Im}, of ω the integration over $\mathbf{k}$ space can be confined to the three-dimensional real space, R^3. Let C_ω be a contour for the ω integration on which the values of ω satisfy this criterion. The inverse transform of $\mathbf{E}(\mathbf{k}, \omega)$ can then be written

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{(2\pi)^4} \int_{C_\omega} \int_{2\pi} \int_{C_k} \mathbf{E}(k\hat{\mathbf{k}}, \omega) e^{i(k\hat{\mathbf{k}}\cdot\mathbf{r} - \omega t)} k^2 \, dk \, d\hat{\mathbf{k}} \, d\omega \quad . \tag{22}$$

Here $C_k = R$ and $\hat{\mathbf{k}}$ is integrated over the half sphere for which $\hat{\mathbf{k}} \cdot \hat{\mathbf{n}} \geq 0$.

Since we are dealing with a boundary value problem, as opposed to an initial value problem, it is preferable to integrate over real values of ω. For this to be possible the medium must be assumed to have no absolute instabilities (see Bers, 1963 and 1972). Furthermore, when the imaginary part of ω is reduced to zero it may happen that one or more of the poles in the k plane cross the real axis. The contour of integration in k, C_k, must then be deformed to keep the poles on the same side of the contour. In order to identify the far field in the direction $\hat{\mathbf{n}}$ from the sources, the contour for the k integration is lifted a distance $i\beta$ up from the real axis. The expression for the field then takes the form

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{(2\pi)^3}\int\limits_R\int\limits_{2\pi}\sum_m \mathbf{E}^p_m(\hat{\mathbf{k}},\omega)e^{i(k_m\hat{\mathbf{k}}\cdot\mathbf{r}-\omega t)}\left|\frac{\partial(k_m\hat{\mathbf{k}})}{\partial(\hat{\mathbf{k}})}\right| d\hat{\mathbf{k}}\,d\omega \tag{23}$$
$$+ \frac{1}{(2\pi)^4}e^{-\beta\hat{\mathbf{k}}\cdot\mathbf{r}}\int\limits_R\int\limits_{2\pi}\int\limits_R \mathbf{E}(\omega,k+i\beta)e^{i(k\hat{\mathbf{k}}\cdot\mathbf{r}-\omega t)}(k+i\beta)^2\,dk\,d\hat{\mathbf{k}}\,d\omega$$

where the summation includes the poles which are situated above the contour C_k and below $i\beta$. $\mathbf{E}^p_m(\hat{\mathbf{k}},\omega)$ is defined as

$$\mathbf{E}^p(\hat{\mathbf{k}},\omega) = ik_m^2\left|\frac{\partial(k_m\hat{\mathbf{k}})}{\partial(\hat{\mathbf{k}})}\right|^{-1}\mathrm{Res}\{\mathbf{E}(\mathbf{k},\omega)dk;k_m\} \tag{24}$$

in which $\mathrm{Res}\{\mathbf{E}(\mathbf{k},\omega)dk;k_m\}$ is the residue of $\mathbf{E}(\mathbf{k},\omega)$ at a singular point $k = k_m$ when the integration is with respect to k:

$$\mathrm{Res}\{\mathbf{E}(\mathbf{k},\omega)dk;k_m\} = \frac{1}{2\pi i}\oint\limits_{k_m}\mathbf{E}(k\hat{\mathbf{k}},\omega)dk \quad . \tag{25}$$

The Jacobian, $|\partial(k_m\hat{\mathbf{k}})/\partial(\hat{\mathbf{k}})|$, of the transformation $k_m\hat{\mathbf{k}} \mapsto \hat{\mathbf{k}}$, gives the ratio between a differential surface element, $\delta(k_m\hat{\mathbf{k}})$, on the singular surface and its corresponding element, $\delta\hat{\mathbf{k}}$, on the unit sphere. While the Jacobian is defined for complex values of k_m we will generally only be interested in situations where $|\mathrm{Im}\{k_m\}| \ll |\mathrm{Re}\{k_m\}|$ and $\mathrm{Im}\{k_m\}$ is a slowly varying function of $(\hat{\mathbf{k}},\omega)$, in which case the physical interpretation of the Jacobian is straightforward. The Jacobian is introduced at this stage to give a definition of $\mathbf{E}^p$ which facilitates an intuitive understanding of the physics contained in subsequent expressions. The expression for the Jacobian is

$$\left|\frac{\partial(k_m\hat{\mathbf{k}})}{\partial(\hat{\mathbf{k}})}\right| = \frac{k_m^2}{\hat{\mathbf{k}}\cdot\hat{\mathbf{v}}_g} \quad , \tag{26}$$

where $\hat{\mathbf{v}}_g$ is a unit vector normal to the singular surface in $\mathbf{k}$ space. Let $\Lambda(\mathbf{k},\omega) = \mathrm{Det}\{\boldsymbol{\Lambda}(\mathbf{k},\omega)\}$. On the singular surface, where $\Lambda = 0$,

$$\delta\Lambda = \frac{\partial\Lambda}{\partial\mathbf{k}}\cdot\delta\mathbf{k} + \frac{\partial\Lambda}{\partial\omega}\delta\omega = 0 \quad . \tag{27}$$

For $\delta\omega = 0$,

$$\frac{\partial \Lambda}{\partial \mathbf{k}} \cdot \delta \mathbf{k} = 0 \quad . \tag{28}$$

It follows that the vector $\partial\Lambda/\partial\mathbf{k}$ is normal to the singular surface in complex $\mathbf{k}$ space. Dividing by the scalar $-\partial\Lambda/\partial\omega$ we find that $\partial\omega/\partial\mathbf{k}$ is normal to the surface:

$$\frac{\partial \Lambda}{\partial \mathbf{k}} \parallel \frac{-\partial\Lambda/\partial\mathbf{k}}{\partial\Lambda/\partial\omega} = \frac{\partial\omega}{\partial\mathbf{k}} \tag{29}$$

and hence that

$$\hat{\mathbf{v}}_g \parallel \frac{\partial\omega}{\partial\mathbf{k}} \quad . \tag{30}$$

With the assumptions made about $\mathbf{k}$ we will generally have that $(\partial\omega/\partial\mathbf{k})_{\mathrm{Im}} : (\partial\omega/\partial\mathbf{k})_{\mathrm{Re}}$ and $\delta\mathbf{k}_{\mathrm{Im}} : \delta\mathbf{k}_{\mathrm{Re}}$ are of the same order as $k_{\mathrm{Im}} : k_{\mathrm{Re}}$. Thus, ignoring the term $\mathrm{Im}\left\{\frac{\partial\omega}{\partial\mathbf{k}}\right\} \cdot \delta\mathbf{k}_{\mathrm{Im}}$, which is of second order in k_{Im}, the real part of (28) reads

$$\left(\frac{\partial\omega}{\partial\mathbf{k}}\right)_{\mathrm{Re}} \cdot \delta\mathbf{k}_{\mathrm{Re}} = 0 \quad . \tag{31}$$

$(\hat{\mathbf{v}}_g)_{\mathrm{Re}}\,(\gg (\hat{\mathbf{v}}_g)_{\mathrm{Im}})$ is therefore parallel to the group velocity, $(\partial\omega/\partial\mathbf{k})_{\mathrm{Re}}$, hence the notation.

At large distances from the source the second integral in (23) vanishes, the field there being due only to the poles which represent the propagating modes. In the direction $\hat{\mathbf{n}}$ from the source the expression for the far field thus takes the form

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{(2\pi)^3} \int_R \int_{2\pi} \sum_m \mathbf{E}^p_m(\hat{\mathbf{k}},\omega) e^{i(k_m \hat{\mathbf{k}}\cdot\mathbf{r} - \omega t)} \frac{k_m^2}{\hat{\mathbf{k}}\cdot\hat{\mathbf{v}}_g}\, d\hat{\mathbf{k}}\, d\omega \quad , \tag{32}$$

where it must be remembered that k_m is a function of $(\hat{\mathbf{k}},\omega)$.

To express $\mathbf{E}^p_m$ in terms of the source currents and the dielectric properties of the medium, (15) is inserted in (24):

$$E^p_{m_i}(\hat{\mathbf{k}},\omega) = \mathrm{Res}\left\{\frac{1}{\lambda_\nu}\, dk\, ;\, k_m\right\} \left. \frac{\left\{g_{i\nu}\, \hat{\mathbf{k}}\cdot\hat{\mathbf{v}}_g\, g^{-1}_{\nu j}\, j_j\right\}}{\omega\varepsilon_0} \right|_{k=k_m} \quad . \tag{33}$$

In (33) there is implicit summation over the index j but not over ν.

Assuming that $1/\lambda_\nu$ has a simple pole at $k = k_m$,

$$\operatorname{Res}\left\{\frac{1}{\lambda_\nu}\, dk\, ;\; k_m\right\} = \left(\left.\frac{\partial \lambda_\nu}{\partial k}\right|_{k=k_m}\right)^{-1} \quad . \tag{34}$$

Solving for λ in (12):

$$\lambda_\nu = g_{\nu i}^{-1}\, \Lambda_{ij}\, g_{j\nu} \quad . \tag{35}$$

Differentiating with respect to k and making use of the fact that $\Lambda_{ij}\, g_{j\nu}|_{k=k_m} = 0$ and $g_{\nu i}^{-1}\, \Lambda_{ij}\Big|_{k=k_m} = 0$ (the latter is easily seen by substituting in (13) for Λ_{ij}), we find

$$\frac{\partial \lambda_\nu}{\partial k} = g_{\nu i}^{-1}\, \frac{\partial \Lambda_{ij}}{\partial k}\, g_{j\nu} \quad . \tag{36}$$

Summation is implicit, except over ν. From (10),

$$\frac{\partial \mathbf{\Lambda}}{\partial k} = \frac{2\mu^2}{k}\left\{\hat{\mathbf{k}} \times \hat{\mathbf{k}} \times\right\} + \frac{\partial \mathbf{K}}{\partial k} \quad . \tag{37}$$

With (34), (36) and (37) and making use of (16) and (17), the expression for $\mathbf{E}_m$ takes the form

$$\mathbf{E}_m^p = \hat{\mathbf{e}}\, \frac{\hat{\mathbf{k}} \cdot \hat{\mathbf{v}}_g\, (\mathbf{e}^{-1} \cdot \mathbf{j})}{\omega \varepsilon_0\, \mathbf{e}^{-1} \cdot \left(\dfrac{-2\mathbf{K}}{k} + \dfrac{\partial \mathbf{K}}{\partial k}\right) \cdot \hat{\mathbf{e}}} \quad , \tag{38}$$

where $k = k_m$. In the subsequent development the index m will be dropped.

The energy flux associated with the field $\mathbf{E}^p(\hat{\mathbf{k}}, \omega)$ is analysed in Appendix B for modes where $|k_{\rm Im}| \ll |k_{\rm Re}|$. A finite observation time T is assumed. It is shown that the power, $\partial^3 P/\partial\hat{\mathbf{k}}\partial\omega$, carried by a mode $(\hat{\mathbf{k}}, \omega)$ per unit angular frequency and per unit solid angle of $\hat{\mathbf{k}}$, is given by

$$\frac{\partial^3 P(\hat{\mathbf{k}},\omega)}{\partial\hat{\mathbf{k}}\,\partial\omega} = \frac{2}{(2\pi)^3}\frac{k_{\mathrm{Re}}^2}{|\hat{\mathbf{k}}\cdot\hat{\mathbf{v}}_g|}\,\varepsilon_0 c\mathcal{F}\,\frac{1}{T}\Big[E^{p*}E^p\Big]_T \quad , \tag{39}$$

$$\mathcal{F} = \frac{1}{\varepsilon_0 c}\left|\left(\mathbf{S}_{ij} - \left(\frac{\omega\varepsilon_0}{2}\right)\left\{\frac{\partial K_{ij}}{\partial\mathbf{k}}\right\}^h\right) e_i^* e_j\right| \quad , \tag{40}$$

where ω takes only positive values. $\mathcal{F}$ is the energy flux, associated with a field in the plasma, normalized by the flux of a vacuum field with identical amplitude. $\mathbf{S}_{ij}$, referred to as the *Poynting tensor* (see equation B.16 in Appendix B), is a rank three tensor which is Hermitian in the indices ij and has the property that its bilinear product with the electric field vector is equal to the real part of the Poynting vector:

$$\mathbf{S}_{ij}E_i^*E_j = \mathrm{Re}\{\mathbf{E}\times\mathbf{H}\}$$

Superscript h indicates the Hermitian part of the tensor concerned. $[E^{p*}E^p]_T/T$ is the power spectrum of E^p in the observation period which has a duration T. It should be noted that

$$\left\{\frac{\partial\mathbf{K}}{\partial\mathbf{k}}\right\}^h = \frac{\partial\mathbf{K}^h}{\partial\mathbf{k}_{\mathrm{Re}}} \quad .$$

With expression (38) for E^p the power spectrum takes the form

$$\frac{1}{T}\Big[E^{p*}E^p\Big]_T = \frac{|\hat{\mathbf{k}}\cdot\hat{\mathbf{v}}_g|^2\,\frac{1}{T}\Big[\left|\mathbf{e}^{-1}\cdot\mathbf{j}\right|^2\Big]_T}{\omega^2\varepsilon_0^2\left|\mathbf{e}^{-1}\cdot\left(\frac{-2\mathbf{K}}{k}+\frac{\partial\mathbf{K}}{\partial k}\right)\cdot\hat{\mathbf{e}}\right|^2} \quad . \tag{41}$$

Inserting (41) in (39) we obtain

$$\frac{\partial^3 P(\hat{\mathbf{k}},\omega)}{\partial\hat{\mathbf{k}}\,\partial\omega} = \frac{2|\mu|^2|\hat{\mathbf{k}}\cdot\hat{\mathbf{v}}_g|\,\mathcal{F}\,\frac{1}{T}\Big[\left|\mathbf{e}^{-1}\cdot\mathbf{j}\right|^2\Big]_T}{(2\pi)^3\varepsilon_0 c\left|\mathbf{e}^{-1}\cdot\left(\frac{-2\mathbf{K}}{k}+\frac{\partial\mathbf{K}}{\partial k}\right)\cdot\hat{\mathbf{e}}\right|^2} \quad . \tag{42}$$

To reiterate: this is the energy flux per unit solid angle of wave vector and unit angular frequency of the far field resulting from a set of source currents, $\mathbf{j}$, embedded in a spatially dispersive medium.

2.2 Inhomogeneous magnetized plasma

In this section a stationary inhomogeneous magnetized plasma is investigated. It is assumed that the sources are in a homogeneous region of the plasma which is sufficiently large that the results of the previous section can be used in that region. In the inhomogeneous region gradients are assumed to be small enough for the WKB approximation to be a valid description of the propagation of electromagnetic waves. k will be assumed to be real. It should be noted that assuming that k and ω are real does not imply that the the dielectric tensor is Hermitian. The main aim of this section is to derive an expression for the power received at a detector outside the medium in which the source currents are embedded. To this end it is necessary to introduce a number of new quantities.

Let I be energy flowing across a surface, A, per unit area. It should be noted that I is an integral over directions of propagation. The energy flow, in a beam emanating from an antenna with an aperture which is large relative to the wave length, can have a very small spread in directions of propagation. I, with A perpendicular to the beam, is a convenient way of describing the energy flow in such a beam. We will refer to I as the intensity in accordance with the definition given by Born and Wolf (1987). Let $\mathcal{I}$, the *normalized intensity*, be the intensity of a beam divided by the total power, P, of that beam:

$$\mathcal{I} = I/P \quad . \tag{43}$$

When describing radiation emanating from sources in the plasma it is convenient to differentiate with respect to direction of propagation. We will write the intensity per unit solid angle of wave vector as $\partial^2 I/\partial\hat{\mathbf{k}}$. In an anisotropic medium the ray direction, identical to the direction of the group velocity, is in general not parallel to the direction of the wave vector. We shall therefore also define an intensity per solid angle of ray direction: $\partial^2 I/\partial\hat{\mathbf{v}}_g$. Bekefi (1966), amongst others, has shown that

$$\frac{\partial^2 I/\partial\hat{\mathbf{v}}_g}{\mu_{\text{ray}}^2} = \text{constant} \tag{44}$$

along a ray trajectory. μ_{ray} is the ray refractive index which, for (44) to hold, takes the form

$$\mu_{\mathrm{ray}}^2 = v_g \mu^2 \left|\frac{\partial \hat{\mathbf{k}}}{\partial \hat{\mathbf{v}}_g}\right| \mid \frac{\partial k}{\partial \omega}\bigg|_{\hat{\mathbf{k}}} \mid \quad . \tag{45}$$

Noting that

$$\frac{\partial k}{\partial \omega}\bigg|_{\hat{\mathbf{k}}} = \frac{1}{(\partial \omega / \partial \mathbf{k}) \cdot \hat{\mathbf{k}}} = \frac{1}{\mathbf{v}_g \cdot \hat{\mathbf{k}}} \quad ,$$

(45) may be written as

$$\left|\frac{\partial \hat{\mathbf{k}}}{\partial \hat{\mathbf{v}}_g}\right| = \frac{|\hat{\mathbf{k}} \cdot \hat{\mathbf{v}}_g| \mu_{\mathrm{ray}}^2}{\mu^2} \quad . \tag{46}$$

Given the power radiated per unit solid angle of wave vector (equation 42) the power radiated per unit solid angle of ray direction is found by simple multiplication with the Jacobian, $|\partial \hat{\mathbf{k}} / \partial \hat{\mathbf{v}}_g|$.

At a detector located outside the medium the power, P^{s}, received from a point source embedded in the anisotropic medium is given (see Appendix C) by

$$\frac{\partial P^{\mathrm{s}}}{\partial \omega} = \frac{\lambda_0^{\mathrm{s}\,2} \mathcal{I}^{\mathrm{s}}}{\mu_{\mathrm{ray}}^2} \frac{\partial^3 P(\hat{\mathbf{k}}, \omega)}{\partial \hat{\mathbf{v}}_g \, \partial \omega} \tag{47}$$

$$= \lambda_0^{\mathrm{s}\,2} \mathcal{I}^{\mathrm{s}} \frac{\hat{\mathbf{k}}^{\mathrm{s}} \cdot \hat{\mathbf{v}}_g^{\mathrm{s}}}{\mu^{\mathrm{s}2}} \frac{\partial^3 P(\hat{\mathbf{k}}, \omega)}{\partial \hat{\mathbf{k}} \, \partial \omega} \tag{48}$$

where λ_0^{s} is the vacuum wave length of the received radiation and $\mathcal{I}^{\mathrm{s}}$ is the normalized intensity of the beam pattern of the receiving antenna at the position of the source. Note that in vacuum $\hat{\mathbf{k}}^{\mathrm{s}} \cdot \hat{\mathbf{v}}_g^{\mathrm{s}} / \mu^{\mathrm{s}2} = 1$, reducing (48) to the well known vacuum relation.

For coherent detection of the field resulting from an extended source the fields at the detector resulting from each point in the source must be added up to give the total field, from which the total power follows. The field at the detector, $E_{\mathrm{det}}^{\mathrm{s}}$ resulting from a point source

$$\mathbf{j}(\mathbf{r},t) = \hat{\mathbf{j}}\delta(\mathbf{r}-\mathbf{r}_p, t-t_p) \Leftrightarrow \mathbf{j}(\mathbf{k},\omega) = \hat{\mathbf{j}}e^{-i(\mathbf{k}\cdot\mathbf{r}_p-\omega t_p)}$$

is

$$\{E^{\rm s}_{\rm det}\}_p = C\sqrt{\mathcal{I}^{\rm s}_p}\mathbf{e}^{-1}\cdot\hat{\mathbf{j}}e^{-i(\mathbf{k}\cdot\mathbf{r}_p-\omega t_p)} \quad , \tag{49}$$

where C is a constant of proportionality, which can be inferred from (48) and (42). C is independent of $(\mathbf{r}_p, t_p)$ except for a possible phase factor which would appear if neighbouring ray paths had different optical lengths (i.e. if waves emitted from the detector antenna had buckled phase fronts). Assuming that the refractive index of the plasma varies slowly across the detector beam, it follows that the optical lengths of adjacent rays will be equal at planes perpendicular to the rays and no phase factor enters the constant of proportionality. In this case the field at the detector resulting from a distributed source is

$$E^{\rm s}_{\rm det} = CJ \tag{50}$$

where

$$J(\mathbf{k},\omega) = \int\sqrt{\mathcal{I}^{\rm s}(\mathbf{r}_p,t_p)}\mathbf{e}^{-1}\cdot\mathbf{j}(\mathbf{r}_p,t_p)e^{-i(\mathbf{k}\cdot\mathbf{r}_p-\omega t_p)}\,d\mathbf{r}_p\,dt_p \quad . \tag{51}$$

We let $\mathcal{I}$ depend on t_p to account for the finite extent, T, of the observation period. In the observation period, $\mathcal{I}$ is defined in (43), outside it is identically zero. From equations 42, 48 and 50 it follows that the total power at the detector is

$$\frac{\partial P^{\rm s}}{\partial\omega} = \frac{2\lambda_0^{{\rm s}2}|\hat{\mathbf{k}}\cdot\hat{\mathbf{v}}_g|^2\,\mathcal{F}\,\dfrac{|J|^2}{T}}{(2\pi)^3\varepsilon_0 c\left|\mathbf{e}^{-1}\cdot\left(\dfrac{-2\mathbf{K}}{k}+\dfrac{\partial\mathbf{K}}{\partial k}\right)\cdot\hat{\mathbf{e}}\right|^2} \quad ; \qquad \{k\in R\} \quad . \tag{52}$$

3 Scattering by density fluctuations

In this section we will derive an expression for the power scattered from an incident field by fluctuations in the conductivity tensor caused by fluctuations in the electron density. (Scattering by fluctuations in other parameters have been considered amongst others by Akhiezer *et al.*, 1967 and Aamodt and Russell, 1990.) Let $\mathbf{E}^{\mathrm{i}}$ be the incident field, $\mathbf{E}^{\mathrm{s}}$ the scattered field and in general let upper index i or s signify that the quantity refers to the incident or scattered field respectively.

The total field, $\mathbf{E} = \mathbf{E}^{\mathrm{i}} + \mathbf{E}^{\mathrm{s}}$, satisfies the relation

$$\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\frac{1}{\varepsilon_0 c^2}\frac{\partial}{\partial t}\left\{\int \boldsymbol{\sigma}'(\mathbf{r}, t, \mathbf{r}', t')\mathbf{E}(\mathbf{r}', t')\, d\mathbf{r}'\, dt'\right\} \tag{53}$$

Assume that the medium is stationary and homogeneous, apart from the fluctuating electron density. Furthermore, assume that the characteristic times and lengths of the density perturbations are large relative to the response time and lengths of the medium. (The response time and lengths are the maximum values of $|t - t'|$ and $|\mathbf{r} - \mathbf{r}'|$ for which $\boldsymbol{\sigma}$ remains significantly different from zero.) Finally assume that $\boldsymbol{\sigma}$ is linear in the electron density, n_e. The conductivity tensor, $\boldsymbol{\sigma}'$, then takes the form

$$\boldsymbol{\sigma}'(\mathbf{r}, t, \mathbf{r}', t') = \left(1 + \frac{\delta n_e(\mathbf{r}, t) + \delta n_e(\mathbf{r}', t')}{2n_e}\right)\boldsymbol{\sigma}(\mathbf{r} - \mathbf{r}', t - t') \tag{54}$$

The expansion used in (54) preserves the symmetry of $\boldsymbol{\sigma}'$ with respect to $(\mathbf{r}, t)$ and $(\mathbf{r}', t')$. This symmetry follows from the symmetry under time reversal of the underlying equations of motion. A discussion of this point is given in Appendix D. Inserting (54) in (53) we obtain

$$\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} =$$

$$-\frac{1}{\varepsilon_0 c^2}\frac{\partial}{\partial t}\left\{\int \left(1 + \frac{\delta n_e(\mathbf{r}, t) + \delta n_e(\mathbf{r}', t')}{2n_e}\right)\boldsymbol{\sigma}(\mathbf{r} - \mathbf{r}', t - t')\mathbf{E}(\mathbf{r}', t')\, d\mathbf{r}'\, dt'\right\} \quad . \tag{55}$$

In the absence of fluctuations, the total field would consist of the incident field only:

$$\nabla \times \nabla \times \mathbf{E}^{\mathrm{i}} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}^{\mathrm{i}}}{\partial t^2} = -\frac{1}{\varepsilon_0 c^2}\frac{\partial}{\partial t}\left\{\int \boldsymbol{\sigma}(\mathbf{r} - \mathbf{r}', t - t')\mathbf{E}^{\mathrm{i}}(\mathbf{r}', t')\, d\mathbf{r}'\, dt'\right\} \quad . \tag{56}$$

Subtracting (56) from (55) we find, in the Born approximation (neglecting multiple scattering)

$$\nabla \times \nabla \times \mathbf{E}^{s} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}^{s}}{\partial t^2} + \frac{1}{\varepsilon_0 c^2}\frac{\partial}{\partial t}\left\{\int \boldsymbol{\sigma}(\mathbf{r}-\mathbf{r}'', t-t'')\mathbf{E}^{s}(\mathbf{r}'', t'')\, d\mathbf{r}''\, dt''\right\} =$$

$$-\frac{1}{\varepsilon_0 c^2}\frac{\partial}{\partial t}\left\{\int \left(\frac{\delta n_e(\mathbf{r}, t) + \delta n_e(\mathbf{r}', t')}{2n_e}\right) \boldsymbol{\sigma}(\mathbf{r}-\mathbf{r}', t-t')\mathbf{E}^{i}(\mathbf{r}', t')\, d\mathbf{r}'\, dt'\right\} \quad . \tag{57}$$

Hence, the currents, associated with the incident field acting on the perturbation of the conductivity tensor, are the source currents for the scattered field:

$$\mathbf{j}(\mathbf{r}, t) = \int \left(\frac{\delta n_e(\mathbf{r}, t) + \delta n_e(\mathbf{r}', t')}{2n_e}\right) \boldsymbol{\sigma}(\mathbf{r}-\mathbf{r}', t-t')\mathbf{E}^{i}(\mathbf{r}', t')\, d\mathbf{r}'\, dt' \quad . \tag{58}$$

Let the incident field be a monochromatic beam:

$$\mathbf{E}^{i}(\mathbf{r}, t) = \hat{\mathbf{e}}^{i}\frac{\sqrt{\mathcal{I}^{i}(\mathbf{r}, t)}}{2}\left(\mathcal{E}^{i} e^{i(\mathbf{k}^{i}\cdot\mathbf{r}-\omega^{i}t)} + \mathcal{E}^{i*} e^{-i(\mathbf{k}^{i}\cdot\mathbf{r}-\omega^{i}t)}\right) \tag{59}$$

The total power P^{i} in this beam is

$$P^{i} = \frac{\varepsilon_0 c}{2}\mathcal{F}^{i}|\mathcal{E}^{i}|^2 \tag{60}$$

Inserting (59) in (58) and the resulting equation for $\mathbf{j}$ into (51) yields

$$\begin{aligned} J(\mathbf{k}^{s}, \omega^{s}) = & \frac{1}{2}\int \sqrt{\mathcal{I}^{s}(\mathbf{r}, t)\mathcal{I}^{i}(\mathbf{r}', t')}\left(\frac{\delta n_e(\mathbf{r}, t) + \delta n_e(\mathbf{r}', t')}{2n_e}\right) \\ & \left(\mathbf{e}^{-1}\right)^{s}\cdot\boldsymbol{\sigma}(\mathbf{r}-\mathbf{r}', t-t')\cdot\hat{\mathbf{e}}^{i} \\ & e^{-i(\mathbf{k}^{s}\cdot\mathbf{r}-\omega^{s}t)}\left(\mathcal{E}^{i} e^{i(\mathbf{k}^{i}\cdot\mathbf{r}'-\omega^{i}t')} + \mathcal{E}^{i*} e^{-i(\mathbf{k}^{i}\cdot\mathbf{r}'-\omega^{i}t')}\right) d\mathbf{r}'\, dt'\, d\mathbf{r}\, dt \quad . \end{aligned} \tag{61}$$

Assuming that the response length and response time of the plasma are short relative to the characteristic lengths and times for $\mathcal{I}^{i}$ and $\mathcal{I}^{s}$, (61) can be simplified to

$$
\begin{aligned}
J(\mathbf{k}^s,\omega^s) &= \frac{1}{2}\int\left(\frac{\widehat{\delta n_e}(\mathbf{r},t)+\widehat{\delta n_e}(\mathbf{r}',t')}{2n_e}\right) \\
&\quad \left(\mathbf{e}^{-1}\right)^s\cdot\boldsymbol{\sigma}(\mathbf{r}-\mathbf{r}',t-t')\cdot\hat{\mathbf{e}}^i \\
&\quad e^{-i(\mathbf{k}^s\cdot\mathbf{r}-\omega^s t)}\left(\mathcal{E}^i e^{i(\mathbf{k}^i\cdot\mathbf{r}'-\omega^i t')}+\mathcal{E}^{i*}e^{-i(\mathbf{k}^i\cdot\mathbf{r}'-\omega^i t')}\right)d\mathbf{r}'\,dt'\,d\mathbf{r}\,dt \quad .
\end{aligned} \tag{62}
$$

where

$$
\widehat{\delta n_e}(\mathbf{r},t)=\sqrt{\mathcal{I}^s(\mathbf{r},t)\mathcal{I}^i(\mathbf{r},t)}\,\delta n_e(\mathbf{r},t) \quad . \tag{63}
$$

The square root of the normalized intensity, $\sqrt{\mathcal{I}}$, plays the same role here as the "weight function", w, in the paper by Holzhauer and Massig (1978). Carrying out the integrations in (62) gives

$$
\begin{aligned}
J(\mathbf{k}^s,\omega^s) &= \frac{1}{2}\mathcal{E}^i\frac{\widehat{\delta n_e}(\mathbf{k}^s-\mathbf{k}^i,\omega^s-\omega^i)}{n_e}\left(\mathbf{e}^{-1}\right)^s\cdot\left(\frac{\boldsymbol{\sigma}(\mathbf{k}^i,\omega^i)+\boldsymbol{\sigma}(\mathbf{k}^s,\omega^s)}{2}\right)\cdot\hat{\mathbf{e}}^i \\
&+ \frac{1}{2}\mathcal{E}^{i*}\frac{\widehat{\delta n_e}(\mathbf{k}^s+\mathbf{k}^i,\omega^s+\omega^i)}{n_e}\left(\mathbf{e}^{-1}\right)^s\cdot\left(\frac{\boldsymbol{\sigma}^*(\mathbf{k}^i,\omega^i)+\boldsymbol{\sigma}^*(\mathbf{k}^s,\omega^s)}{2}\right)\cdot\hat{\mathbf{e}}^i \quad .
\end{aligned} \tag{64}
$$

The two last terms represent the effect of density fluctuations at approximately twice the probing frequency and generally propagating faster than the phase velocity of the probing light. Such density fluctuations are unlikely to exist as noted by Hutchinson (1987) in his treatment of scattering based on vacuum propagation. Neglecting the two last terms in (64) we obtain

$$
|J|^2=\frac{1}{4n_e}\,|\mathcal{E}^i|^2\,O_b\,T\,S(\mathbf{k},\omega)\left|\left(\mathbf{e}^{-1}\right)^s\cdot\left(\frac{\boldsymbol{\sigma}(\mathbf{k}^i,\omega^i)+\boldsymbol{\sigma}(\mathbf{k}^s,\omega^s)}{2}\right)\cdot\hat{\mathbf{e}}^i\right|^2 \tag{65}
$$

where

$$
\mathbf{k}=\mathbf{k}^s-\mathbf{k}^i \quad , \qquad \omega=\omega^s-\omega^i \quad .
$$

O_b is the *beam overlap* (Bindslev, 1989):

$$
O_b=\int\mathcal{I}^i(\mathbf{r},t)\mathcal{I}^s(\mathbf{r},t)\,d\mathbf{r} \tag{66}
$$

and $S(\mathbf{k},\omega)$ is the spectral density function:

$$S(\mathbf{k},\omega) = \frac{|\widehat{\delta n_e}(\mathbf{k},\omega)|^2}{n_e O_b T} \quad . \tag{67}$$

Inserting (65) into (52) and making use of (60) to express $|\mathcal{E}^{\mathrm{i}}|^2$ in terms of the incident power, we find

$$\frac{\partial P^{\mathrm{s}}}{\partial \omega} = \frac{P^{\mathrm{i}}\, O_b\, S(\mathbf{k},\omega)\, \lambda_0^{\mathrm{s}\,2} |\hat{\mathbf{k}}^{\mathrm{s}} \cdot \hat{\mathbf{v}}_g^{\mathrm{s}}|^2\, \mathcal{F}^{\mathrm{s}} \left| \left(\mathbf{e}^{-1}\right)^{\mathrm{s}} \cdot \left(\dfrac{\boldsymbol{\sigma}^{\mathrm{i}} + \boldsymbol{\sigma}^{\mathrm{s}}}{2}\right) \cdot \hat{\mathbf{e}}^{\mathrm{i}} \right|^2}{(2\pi)^3\, n_e\, (\varepsilon_0 c)^2\, \mathcal{F}^{\mathrm{i}} \left| \mathbf{e}^{-1} \cdot \left(\dfrac{-2\mathbf{K}}{k} + \dfrac{\partial \mathbf{K}}{\partial k}\right) \cdot \hat{\mathbf{e}} \right|^2} \quad . \tag{68}$$

With the classical electron radius, r_e, and the plasma frequency, ω_p,

$$r_e = \frac{q_e^2}{4\pi\varepsilon_0 m_e c^2} \quad ; \quad \frac{r_e^2}{\omega_p^4} = \frac{1}{(4\pi)^2 c^4 n_e^2} \tag{69}$$

(68) can be written

$$\frac{\partial P^{\mathrm{s}}}{\partial \omega} = P^{\mathrm{i}}\, O_b\, \lambda_0^{\mathrm{i}} \lambda_0^{\mathrm{s}} r_e^2\, n_e\, \frac{S(\mathbf{k},\omega)}{2\pi}\, G \quad , \tag{70}$$

where the geometrical factor takes the form

$$G = \frac{|\mu^{\mathrm{s}}|^2 |\hat{\mathbf{k}}^{\mathrm{s}} \cdot \hat{\mathbf{v}}_g^{\mathrm{s}}|^2\, \mathcal{F}^{\mathrm{s}}\, \dfrac{\omega^{\mathrm{i}}\omega^{\mathrm{s}}}{\omega_p^2} \left| \left(\mathbf{e}^{-1}\right)^{\mathrm{s}} \cdot \left(\dfrac{\omega^{\mathrm{i}}\mathbf{Q}^{\mathrm{i}} + \omega^{\mathrm{s}}\mathbf{Q}^{\mathrm{s}}}{2\omega_p}\right) \cdot \hat{\mathbf{e}}^{\mathrm{i}} \right|^2}{\mathcal{F}^{\mathrm{i}} \left| \left\{ \mathbf{e}^{-1} \cdot \left(\mathbf{K} - \dfrac{k}{2}\dfrac{\partial \mathbf{K}}{\partial k}\right) \cdot \hat{\mathbf{e}} \right\}^{\mathrm{s}} \right|^2} \quad . \tag{71}$$

Equation (70) gives the power received in a scattering system, given the incident power, the beam patterns and the nature of the density fluctuations. If from a statistical knowledge of the density fluctuations we wanted a prediction of the received power the best estimate would be obtained by using the ensemble average for the spectral density function,

$$S(\mathbf{k},\omega) = \frac{< |\widehat{\delta n_e}(\mathbf{k},\omega)|^2 >}{n_e O_b T} \quad . \tag{72}$$

This definition of the spectral density function is identical to the definition given by Sheffield (1975) when the beams have rectangular cross sections and uniform intensities.

If the anti–Hermitian part of the dielectric tensor can be ignored, i.e.

$$\mathbf{K} = \mathbf{K}^h \quad ,$$

it follows that

$$\mathbf{e}^{-1} = \hat{\mathbf{e}}^*$$

and we have (see Appendix B):

$$\hat{\mathbf{e}}^* \cdot \left(\mathbf{K} - \frac{k}{2} \frac{\partial \mathbf{K}}{\partial k} \right) \cdot \hat{\mathbf{e}} = \mu \hat{\mathbf{k}} \cdot \hat{\mathbf{v}}_g \mathcal{F} \quad . \tag{73}$$

With this relation between the terms stemming from the residue and the flux, (71) can be written in the compact and symmetrical form

$$G = \frac{\mathcal{C}}{\mathcal{F}^{\mathrm{i}} \mathcal{F}^{\mathrm{s}}} \tag{74}$$

where the *coupling term*, $\mathcal{C}$, is given by

$$\mathcal{C} = \frac{\omega^{\mathrm{i}} \omega^{\mathrm{s}}}{\omega_p^2} \left| (\hat{\mathbf{e}}^*)^{\mathrm{s}} \cdot \left(\frac{\omega^{\mathrm{i}} \mathbf{Q}^{\mathrm{i}} + \omega^{\mathrm{s}} \mathbf{Q}^{\mathrm{s}}}{2\omega_p} \right) \cdot \hat{\mathbf{e}}^{\mathrm{i}} \right|^2 \tag{75}$$

and the $\mathcal{F}$'s are the normalized fluxes given in equation (40).

In the limit of no spatial dispersion, the expression for the geometrical factor, G, reduces to that given by Hughes and Smith (1989), apart from a minor point about symmetry, which is discussed in Appendix D. In the low density limit the term $\mathcal{C}$ tends towards $|(\hat{\mathbf{e}}^*)^{\mathrm{s}} \cdot \hat{\mathbf{e}}^{\mathrm{i}}|^2$, while $\mathcal{F}$ tends towards unity and consequently G tends towards $|(\hat{\mathbf{e}}^*)^{\mathrm{s}} \cdot \hat{\mathbf{e}}^{\mathrm{i}}|^2$.

The differential scattering cross section, $\partial^3\Sigma/\partial\hat{\mathbf{k}}\,\partial\omega$, is often given as an intermediate result. From (47) it is readily seen that

$$\frac{\partial P^{s}}{\partial \omega}=\frac{\lambda_0^{s\,2} P^{i} O_b n_e}{\mu_{ray}^2}\frac{\partial^3 \Sigma}{\partial \hat{\mathbf{v}}_g\, \partial \omega} \tag{76}$$

and hence, with use of (46)

$$\frac{\partial^3 \Sigma}{\partial \hat{\mathbf{k}}\, \partial \omega}=\frac{\lambda_0^{i}}{\lambda_0^{s}} r_e^2 \frac{S(\mathbf{k},\omega)}{2\pi}\frac{\mu^{s2}}{|\hat{\mathbf{k}}^{s}\cdot \hat{\mathbf{v}}_g^{s}|} G \quad . \tag{77}$$

In the limit of no spatial dispersion (77) is identical to the expression given by Hughes and Smith (1989) after symmetrization, as discussed in Appendix D.

4 Numerical results

Equation (70) is the equation describing the power transfer in a scattering system. The quantities which depend on the properties of the plasma are the beam overlap, O_b, the spectral density function, $S(\mathbf{k},\omega)$ and the geometrical factor, G. The beam overlap was investigated by Bindslev (1989) while the spectral density function has been the subject of many investigations, e.g. Hughes and Smith (1988). In this paper the main new result is the generalized expression for the geometrical factor which allows for spatial dispersion. This means that the dielectric tensor and derivatives thereof, which enter into the expression for G, can be evaluated not only on the basis of the cold plasma model but also using hot or relativistic models.

Computer codes have been developed, as part of this work, to evaluate G with dielectric tensors and their derivatives derived from four magnetized plasma models: (a) cold, (b) hot equilibrium, (c) weakly relativistic equilibrium based on Shkarofsky (1986) and (d) weakly relativistic equilibrium based on Yoon and Krauss–Varban (1990). With a number of corrections to Yoon and Kraus–Varban's work (see Appendix F), the two relativistic codes give identical results.

In the low temperature limit the relativistic, the hot and the cold codes all give identical results. For propagation close to perpendicular to the magnetic field, both the real and the imaginary part of the refractive index found with codes c and d agree accurately with curves given by Batchelor, Goldfinger and Weitzner (1984). The location of cutoffs are shifted by relativistic effects. The density at which the O–mode is cutoff, is given by

$$\left(\frac{\omega_p^2}{\omega^2}\right)_{\text{O-cutoff}} = \frac{3K_2(\rho)}{\rho^2 \int_0^\infty (p^4/\gamma^2) e^{-\rho\gamma}\, dp} \tag{78}$$

while the density of the R–cutoff is given by

$$\left(\frac{\omega_p^2}{\omega^2}\right)_{\text{R-cutoff}} = \frac{3K_2(\rho)}{\rho^2 \int_0^\infty \left(1+\frac{\Omega}{\gamma}\right) \frac{p^4 e^{-\rho\gamma}}{p^2+1-\Omega^2}\, dp} \tag{79}$$

where

$$\Omega = \frac{\omega_{ce}}{\omega} \quad , \qquad \rho = \frac{m_e c^2}{T_e} \quad , \qquad \gamma = \sqrt{1+p^2} \quad ,$$

and K_2 is the modified Bessel function of second order. Equation (78) is identical to equation (43) in Batchelor, Goldfinger and Weitzner (1984) (subsequently referred to as B.G.W.) while (79) is derived from B.G.W.'s equations 39 to 41. (Please note that there is a miss print in B.G.W.'s equations 38 and 39, X–mode. They should read $n_\perp^2 - (\epsilon_{xx}\epsilon_{yy} + \epsilon_{xy}^2)/\epsilon_{xx} = 0$ and $\epsilon_{xx}\epsilon_{yy} + \epsilon_{xy}^2 = 0$.) The locations of the O–mode cutoff and the R–cutoff found with codes c and d (see figures 1 (d), 3 and 4) agree accurately with those found with equations (78) and (79).

In the investigated range of electron temperatures, $T_e = 0$ to 18 keV, G as well as the terms $\frac{\omega^i\omega^s}{\omega_p^2}\left|(\mathbf{e}^{-1})^s \cdot \left(\frac{\omega^i\mathbf{Q}^i+\omega^s\mathbf{Q}^s}{2\omega_p}\right)\cdot \hat{\mathbf{e}}^i\right|^2$, $\mathcal{F}$, and $\left(\mathbf{e}^{-1}\cdot\left(\mathbf{K} - \frac{k}{2}\frac{\partial \mathbf{K}}{\partial k}\right)\cdot\hat{\mathbf{e}}\right)$ appear to depend on the norm of the anti–Hermitian part of the dielectric tensor, $||\mathbf{K}^a|| = \sqrt{\sum_{ij}|K_{ij}|^2}$, only to second order (the author has been able to show this result analytically for $\left(\mathbf{e}^{-1}\cdot\left(\mathbf{K} - \frac{k}{2}\frac{\partial \mathbf{K}}{\partial k}\right)\cdot\hat{\mathbf{e}}\right)$ but not for the other terms) and the effect of $\mathbf{K}^a$ is negligible except in the vicinity of resonances. We will therefore discuss numerical results in terms of equation (74).

Parameter space is clearly very large and a comprehensive survey of it is outside the scope of this paper. Here we will only present some illustrative examples with parameters relevant to the planned scattering experiments at JET (Costley *et al.*, 1988, 1989 a, b) and TFTR (Woskov *et al.*, 1988).

To describe the geometry of the scattering let

$$\cos\psi_{i/s} = \hat{\mathbf{k}}_{i/s}\cdot\hat{\mathbf{B}} \quad , \qquad \cos\chi = \frac{(\mathbf{k}_i\times\mathbf{B})\cdot(\mathbf{k}_s\times\mathbf{B})}{|\mathbf{k}_i\times\mathbf{B}|\,|\mathbf{k}_s\times\mathbf{B}|}$$

and

$$\cos\theta = \hat{\mathbf{k}}_i \cdot \hat{\mathbf{k}}_s \quad , \qquad \cos\phi = \hat{\mathbf{k}} \cdot \hat{\mathbf{B}} \quad .$$

The angles ψ_i, ψ_s and χ are the angles used by Bretz (1987) and Hughes and Smith (1989) to describe the scattering geometry. We shall only present data resulting from geometries where $\psi_s = 180° - \psi_i$. With this symmetry the scattering geometry is fully described by θ and ϕ.

The plots given in figure 1 are calculated with parameters relevant for scattering in JET (parameters are given in the figure caption). The curves show the geometrical factor, the coupling term (equation 75), the normalized flux (equation 40) and the refractive index as functions of electron density. $\mathcal{F}$ and μ are identical for incident and scattered fields due to the symmetry in the direction of propagation relative to the magnetic field. Both incident and scattered fields are in the extraordinary mode (X to X scattering). The frequency of the radiation is higher than the cyclotron frequency. This implies that the R–cutoff determines the maximum density to which the radiation can propagate. The effect of the R–cutoff is clearly visible in the plots. While the hot plasma predictions tend toward the cold plasma predictions at the R–cutoff, the relativistic plasma model produces a shift in the R–cutoff toward higher densities. This shift can be attributed to the relativistic mass increase of the electrons. Although the hot plasma model does produce a change in predictions relative to the cold model, in this regime much more substantial effects are found with the relativistic model. It is noteworthy that a nonvanishing imaginary part to the refractive index is found with the relativistic model and not with the hot model. This absorption is attributable to the relativistic smearing of the cyclotron absorption.

In figure 2, cold and relativistic versions of the geometrical factor are plotted against ω^s for a range of electron densities around $n_e = 6.5 \cdot 10^{19}\text{m}^{-3}$. It is evident that as the R–cutoff is approached the shape of the curves become increasingly sensitive to the electron density.

The reliability of the analysis of light scattered for diagnostic purposes depends, among many factors, on the accuracy of the model. Figures 1 and 2 clearly illustrate the need for a relativistic model. Another factor of importance to the reliability of the analysis is the sensitivity of the spectrum of scattered light to various plasma parameters. Sensitivity to quantities which the diagnostic seeks to measure is beneficial while sensitivity to other quantities like the electron density reduces the reliability of the analysis. As the R–cutoff is approached the geometrical factor and hence the spectrum of scattered light for X to X scattering becomes increasingly sensitive to the electron density and other parameters, making reliable analysis impossible in the vicinity of the R–cutoff. The practical consequence of the relativistic shift of the R–cutoff is therefore to increase the upper limit on the density range in which reliable measurements can be made with X to X scattering.

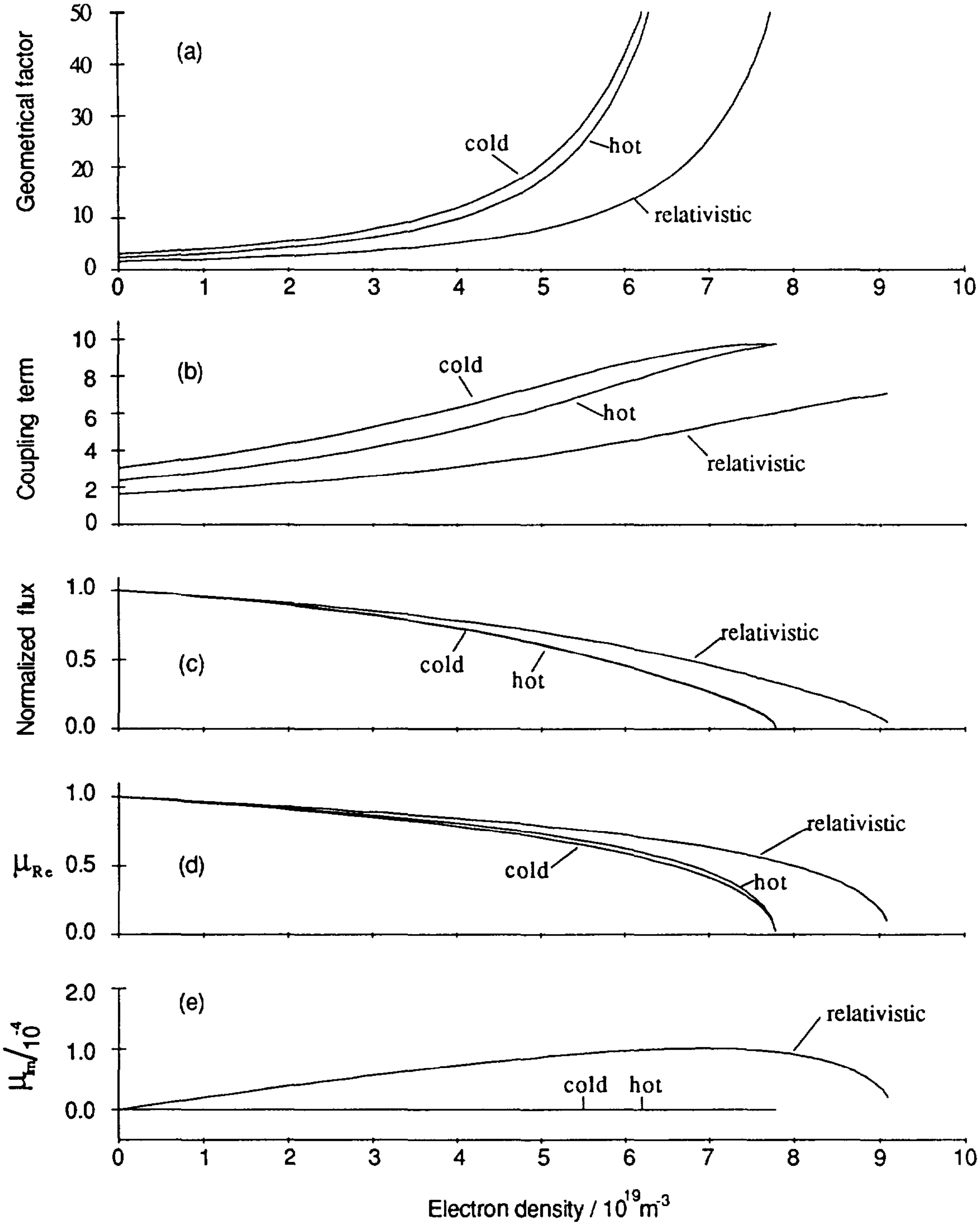

Figure 1: (a) Geometrical factor, (b) coupling term, (c) flux term, (d) and (e) real and imaginary part of refractive index, as functions of n_e. Parameters: X to X scattering, $\omega^i = \omega^s = 2\pi \cdot 140$ GHz, $\theta = 30°$, $\phi = 30°$ ($\psi^i \approx 103°$, $\psi^s \approx 77°$, $\chi \approx 15°$), $B = 3.4$ T, $T_e = 12$ keV.

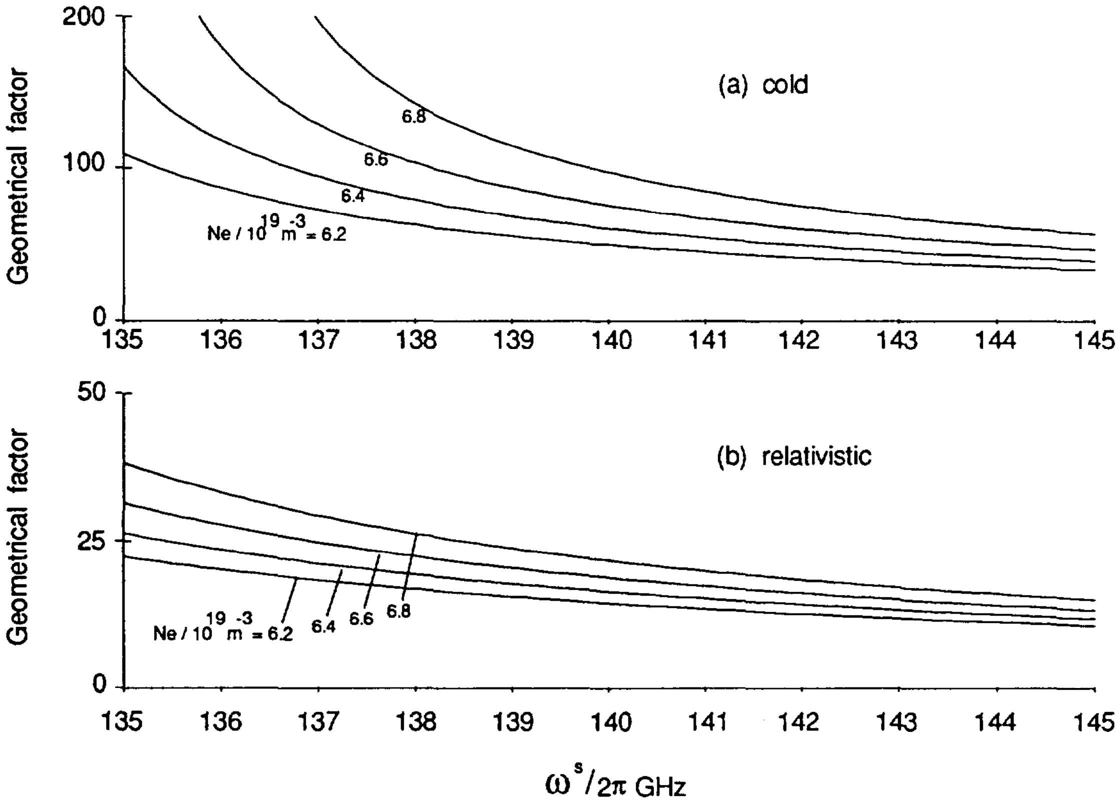

Figure 2: Geometrical factor, (a) cold, (b) relativistic, as functions of ω^s. Parameters: as in figure 1 except that n_e = 6.2, 6.4, 6.6, 6.8 $\cdot$ $10^{19}\mathrm{m}^{-3}$.

At any given electron density, the reduced sensitivity found with the relativistic model, which is illustrated in figure 2, implies that the reliability of the analysis will be better than expected from the cold plasma predictions.

In figure 3 the geometrical factor and the real part of the refractive index are plotted against electron density for a range of electron temperatures. The rest of the parameters are the same as in figure 1. Only relativistic curves are plotted, the $T_e = 50$ eV curve being indistinguishable from the equivalent cold plasma curve. The dependence of the R–cutoff on temperature is evident.

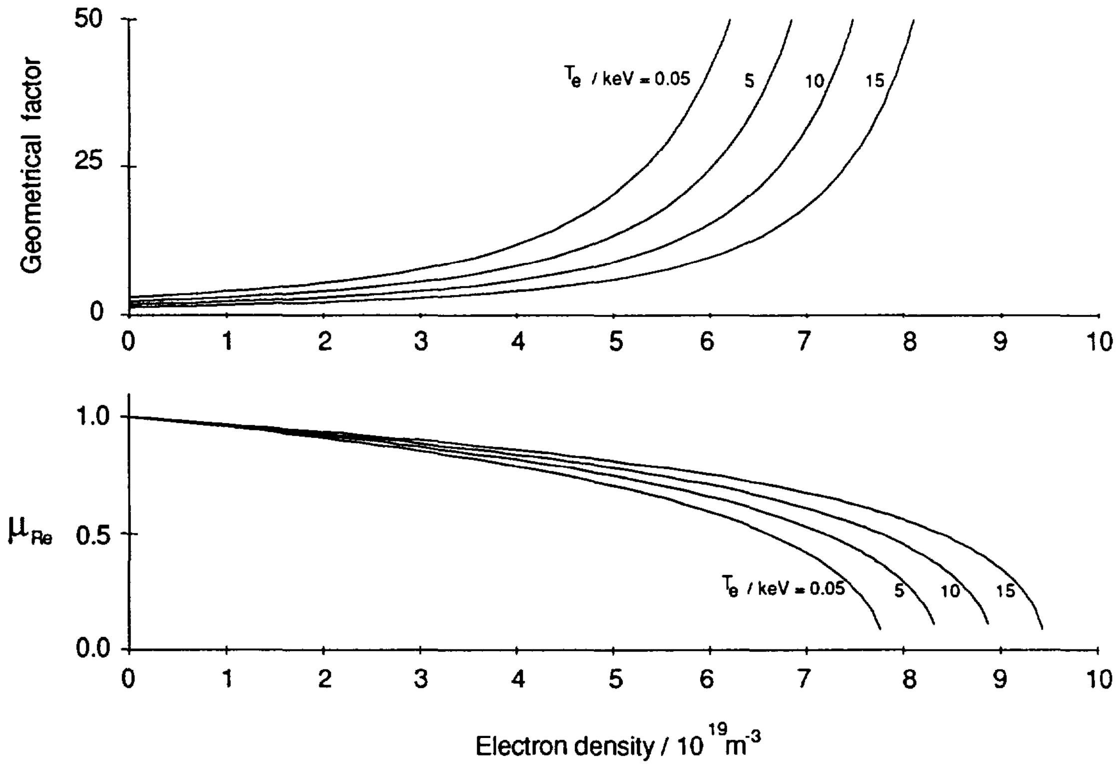

Figure 3: Relativistic geometrical factor and real part of refractive index as functions of n_e.
Parameters: as in figure 1 except that $T_e = 0.05$, 5, 10, 15 keV.

For O to X and X to O scattering the difference between cold and relativistic predictions are similarly dominated by the shift in the R–cutoff.

Figure 4 shows plots for O to O scattering with the other parameters identical to those for figure 3. Radiation in the ordinary mode is cut off at the plasma frequency. Again a relativistic shift in the cutoff frequency is found. Attention is drawn to the different scale for the density. At densities found in JET the difference between the cold and relativistic predictions are of little practical importance.

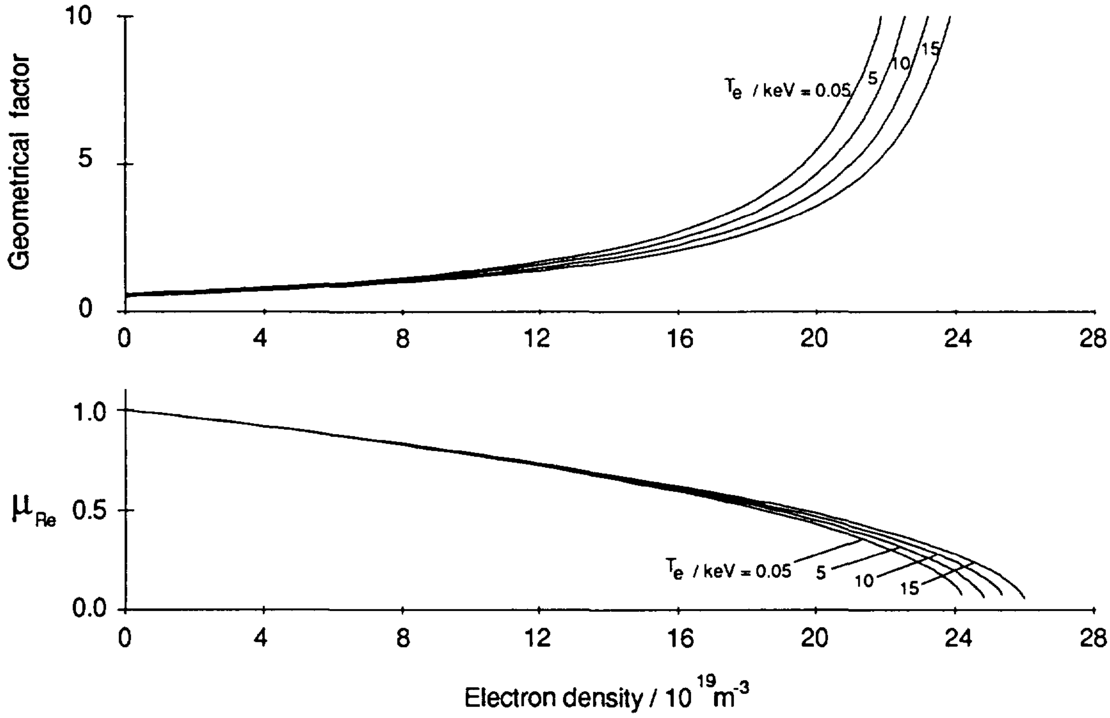

Figure 4: Relativistic geometrical factor and real part of refractive index as functions of n_e.
Parameters: O to O scattering, otherwise as in figure 3

Figure 5 shows plots similar to figure 3 with parameters relevant for the scattering experiment planned at TFTR (Woskov *et al.*, 1988). Here the frequency of the probing radiation is below the cyclotron frequency. This implies that the radiation, which is in the extraordinary mode, is cut off at the L–cutoff.

Though a small relativistic shift of the L–cutoff is observed, relativistic effects in the geometrical factor appear to be negligible for the TFTR parameters.

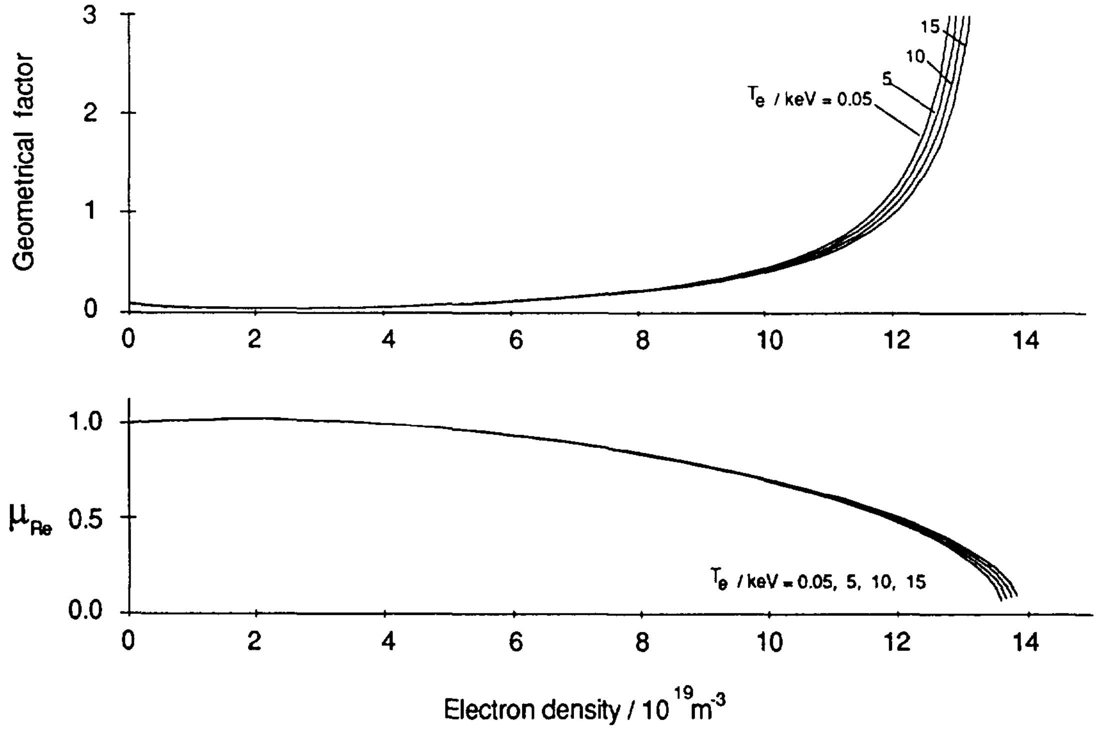

Figure 5: Relativistic geometrical factor and real part of refractive index as functions of n_e.
Parameters: X to X scattering, $\omega^i = \omega^s = 2\pi \cdot 56$ GHz, $\theta = 30°$, $\phi = 80°$, ($\psi^i \approx 92.6°$, $\psi^s \approx 87.4°$, $\chi \approx 29.6°$), $B = 5.0$ T, $T_e = 0.05, 5, 10, 15$ keV.

5 Conclusions

A theory of scattering has been developed which takes the dielectric effects of the plasma into account and, as a new element, allows for spatial dispersion. Thermal motion results in spatial dispersion. This new expression is therefore required when hot or relativistic effects are included in the dielectric properties of the plasma.

Symmetry with respect to incident and scattered fields has been demonstrated in the limit where the anti–Hermitian part of the dielectric tensor can be neglected and shown to be in agreement with the reciprocity relation. The source of asymmetry in earlier results has been identified.

Earlier results are confirmed in the cold plasma limit.

Significant relativistic effects, of practical importance for the planned collective scattering diagnostic at JET, have been found for the advantageous X to X scattering. Due to the relativistic shift of the R–cutoff to higher densities, reliable analysis of radiation scattered from X mode to X mode appears feasible in an important density range which would not have been considered possible on the basis of the cold plasma predictions.

For O to O scattering in JET the relativistic effects appear to be of no importance to the signal level in O–mode. However, it is important, even for O to O scattering, to stay clear of the R–cutoff, preferably at densities above it, in order to minimize or eliminate spurious signals from X to X scattering. This requirement accentuates the importance of an accurate knowledge of the location of the R–cutoff.

For the collective scattering diagnostic under consideration at TFTR (Woskov, 1988) no relativistic effects of any importance are predicted.

Finally it should be noted that relativistic effects are likely to be important for reflectometry which relies on reflection of radiation by the R–cutoff or O–mode cutoff layers.

6 Acknowledgements

The author would like to thank T.P. Hughes and S.R.P. Smith for many very fruitful discussions. Thanks are also extended to I.H. Hutchinson for a useful discussion and helpful criticism of a draft version of this article.

Appendix A Fourier–Laplace transformation

In this paper a spatial Fourier transform and temporal Laplace transform are used:

$$A(\mathbf{r},t) = \frac{1}{(2\pi)^4}\int_{-\infty+i\gamma}^{\infty+i\gamma}\int_{R^3} A(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,d\mathbf{k}\,d\omega \qquad \text{(A.1)}$$

$$A(\mathbf{k},\omega) = \int_0^{\infty}\int_{R^3} A(\mathbf{r},t)e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,d\mathbf{r}\,dt \quad . \qquad \text{(A.2)}$$

The Fourier–Laplace transform is defined for values of γ large enough for the integral (A.2) to exist. In the application, use will be made of the analytic continuation of the transform. It is assumed that in the time domain only the asymptotic response is of interest. Implicit in this is that the response to any initial conditions can be neglected.

Appendix B Energy flux

The energy flux associated with propagating modes in a plasma will be derived here. It is well known that in a spatially dispersive medium the energy flux is the sum of the electromagnetic flux, accounted for in the conventional Poynting vector, and a kinetic flux due to correlated movement of particles with the wave. The expression for the kinetic flux associated with a quasi–monochromatic mode has been derived by a number of authors, e.g. Bers (1963 and 1972) and Landau, Lifshitz and Pitaevskii (1984). Here, however, a broadband field, propagating in all directions, is investigated. To ensure that integrations over frequency and direction of propagation are taken correctly, the expression for the energy flux in a spatially dispersive medium is developed from Poynting's theorem. The Poynting vector is defined as

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} \qquad \text{(B.1)}$$

From Maxwell's equations (1) and (2) and the constitutive equations (3) and (5) in section 2, Poynting's theorem is readily derived:

$$\begin{aligned}\nabla\cdot\mathbf{S} &= \mathbf{H}\cdot\nabla\times\mathbf{E}-\mathbf{E}\cdot\nabla\times\mathbf{H}\\ &= -\frac{\mu_0}{2}\frac{\partial H^2}{\partial t}-\frac{\varepsilon_0}{2}\frac{\partial E^2}{\partial t}-\mathbf{E}\cdot\mathbf{j}_p-\mathbf{E}\cdot\mathbf{j} \quad . \end{aligned} \tag{B.2}$$

$\mathbf{j}_p$ is the plasma current induced by $\mathbf{E}$:

$$\mathbf{j}_p(\mathbf{r},t)=\int \boldsymbol{\sigma}(\mathbf{r}-\mathbf{r}',t-t')\mathbf{E}(\mathbf{r}',t')\,d\mathbf{r}'\,dt' \quad , \tag{B.3}$$

while $\mathbf{j}$ represents externally induced currents. For most of the terms in (B.2) the interpretation is straightforward: the term on the left hand side is the divergence of the electromagnetic flux, the first two on the right account for the change in electrostatic and magnetostatic energy while the last on the right accounts for the exchange of energy with external sources. The term $\mathbf{E}\cdot\mathbf{j}_p$ is the exchange of energy between the field and the plasma particles. However, $\mathbf{j}_p$ is a function of $\mathbf{E}$ which complicates the interpretation of this term. In the limit where $\mu_{\rm Im}\ll\mu_{\rm Re}$, $\mathbf{E}\cdot\mathbf{j}_p$ can be approximated by three terms which account for dissipation, change in energy and divergence of flux associated with correlated particle movement. It is this latter term, the *kinetic flux*, which is of particular interest here.

For a field of the type given in equation (32) we have

$$\begin{aligned}\mathbf{E}(\mathbf{r},t)\cdot\mathbf{j}_p(\mathbf{r},t) &= \frac{1}{(2\pi)^6}\int E_i^{p*}(\hat{\mathbf{k}}_1,\omega_1)\cdot\boldsymbol{\sigma}_{ij}(\hat{\mathbf{k}}_2,\omega_2)\cdot\mathbf{E}_j^{p}(\hat{\mathbf{k}}_2,\omega_2)\\ &\quad e^{i\left((\mathbf{k}_2-\mathbf{k}_1^*)\cdot\mathbf{r}-(\omega_2-\omega_1)t\right)}\left(\frac{k_1{}^2}{\hat{\mathbf{k}}_1\cdot\hat{\mathbf{v}}_{g1}}\right)^*\left(\frac{k_2{}^2}{\hat{\mathbf{k}}_2\cdot\hat{\mathbf{v}}_{g2}}\right)d\hat{\mathbf{k}}_1\,d\omega_1\,d\hat{\mathbf{k}}_2\,d\omega_2\end{aligned} \tag{B.4}$$

where

$$\mathbf{k}=k\hat{\mathbf{k}} \quad , \qquad k=k(\hat{\mathbf{k}}) \quad .$$

The summation over modes is dropped, implicitly assuming that the flux does not contain cross–mode terms. Let

$$\mathbf{k}_0=\frac{\mathbf{k}_2+\mathbf{k}_1^*}{2} \quad , \qquad \omega_0=\frac{\omega_2+\omega_1}{2} \quad ,$$

Integrating (B.4) over time and space (e.g. to find the time–averaged power crossing a surface) the integrand will, due to the harmonic term, vanish except where

$(\mathbf{k}_2, \omega_2) \approx (\mathbf{k}_1, \omega_1)$. In this case expanding $\boldsymbol{\sigma}$ around $(\mathbf{k}_0, \omega_0)$, retaining only zero and first order terms, is a valid approximation:

$$
\begin{aligned}
\boldsymbol{\sigma}(\mathbf{k}_2, \omega_2) \quad = \quad \boldsymbol{\sigma}(\mathbf{k}_0, \omega_0) \; &+ \; \frac{\partial \boldsymbol{\sigma}(\mathbf{k}_0, \omega_0)}{\partial \mathbf{k}} \cdot (\mathbf{k}_2 - \mathbf{k}_1^*)/2 \\
&+ \; \frac{\partial \boldsymbol{\sigma}(\mathbf{k}_0, \omega_0)}{\partial \omega} (\omega_2 - \omega_1)/2 \\
&+ \; O(\mathbf{k}_2 - \mathbf{k}_1^*, \omega_2 - \omega_1) \quad .
\end{aligned} \tag{B.5}
$$

Substituting (B.5) into (B.4) and adding the complex conjugate expression, making use of the fact that the left hand side is real, we find

$$
\begin{aligned}
&\int_V \int_T \mathbf{E}(\mathbf{r}, t) \cdot \mathbf{j}_p(\mathbf{r}, t) \, d\mathbf{r} \, dt \\
&\quad = \int_V \int_T \mathcal{L} \left\{ E_i^{p*}(\hat{\mathbf{k}}_1, \omega_1) \cdot \boldsymbol{\sigma}_{ij}^h \cdot \mathbf{E}_i^p(\hat{\mathbf{k}}_2, \omega_2) \right\} \, d\mathbf{r} \, dt \\
&\quad + \int_V \int_T \nabla \cdot \mathcal{L} \left\{ \mathbf{E}_i^{p*}(\hat{\mathbf{k}}_1, \omega_1) \cdot \left(\frac{1}{2} \right) \left(\frac{\partial \boldsymbol{\sigma}_{ij}}{\partial \mathbf{k}} \right)^a \cdot \mathbf{E}_j^p(\hat{\mathbf{k}}_2, \omega_2) \right\} \, d\mathbf{r} \, dt \\
&\quad + \int_V \int_T \frac{\partial}{\partial t} \mathcal{L} \left\{ \mathbf{E}_i^{p*}(\hat{\mathbf{k}}_1, \omega_1) \cdot \left(\frac{-1}{2} \right) \left(\frac{\partial \boldsymbol{\sigma}_{ij}}{\partial \omega} \right)^a \cdot \mathbf{E}_j^p(\hat{\mathbf{k}}_2, \omega_2) \right\} \, d\mathbf{r} \, dt \quad ,
\end{aligned} \tag{B.6}
$$

where $\mathcal{L} \{ \cdots \}$ is the integral operator:

$$
\mathcal{L} \{ \cdots \} = \frac{1}{(2\pi)^6} \int \cdots e^{i\left((\mathbf{k}_2 - \mathbf{k}_1^*) \cdot \mathbf{r} - (\omega_2 - \omega_1) t\right)} \left(\frac{k_1{}^2}{\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{v}}_{g1}} \right)^* \left(\frac{k_2{}^2}{\hat{\mathbf{k}}_2 \cdot \hat{\mathbf{v}}_{g2}} \right) \, d\hat{\mathbf{k}}_1 \, d\omega_1 \, d\hat{\mathbf{k}}_2 \, d\omega_2
$$

The second term on the right hand side of (B.6) is clearly the kinetic flux leaving the volume V.

For convenience we define a *Poynting tensor*

$$
\{\mathbf{S}_{ij}\}_l = \frac{1}{2\mu_0} \left(-\frac{\{\mathbf{k}_2\}_i \, \delta_{jl}}{\omega_2} - \frac{\{\mathbf{k}_1^*\}_j \, \delta_{il}}{\omega_1} + \frac{\{\mathbf{k}_1^*\}_l \, \delta_{ij}}{\omega_1} + \frac{\{\mathbf{k}_2\}_l \, \delta_{ij}}{\omega_2} \right) \quad , \tag{B.7}
$$

which has the property

$$\begin{aligned}
&\mathbf{S}_{ij}E_i^*(\mathbf{k}_1,\omega_1)E_j(\mathbf{k}_2,\omega_2) \\
&= \frac{\mathbf{E}(\mathbf{k}_2,\omega_2)\times(\mathbf{k}_1^*\times\mathbf{E}^*(\mathbf{k}_1,\omega_1))}{2\mu_0\omega_1} + \frac{\mathbf{E}^*(\mathbf{k}_1,\omega_1)\times(\mathbf{k}_2\times\mathbf{E}(\mathbf{k}_2,\omega_2))}{2\mu_0\omega_2} \qquad \text{(B.8)} \\
&= \frac{\mathbf{E}(\mathbf{k}_2,\omega_2)\times\mathbf{H}^*(\mathbf{k}_1,\omega_1)+\mathbf{E}^*(\mathbf{k}_1,\omega_1)\times\mathbf{H}(\mathbf{k}_2,\omega_2)}{2} \quad .
\end{aligned}$$

From (B.2) and (B.6) it follows that the average power, P, carried across a surface, A, over a period, T, is given by

$$P = \int_A \frac{1}{T}\int_{\tau-\frac{T}{2}}^{\tau+\frac{T}{2}} \mathcal{L}\left\{ \left(\mathbf{S}_{ij} + \left(\frac{1}{2}\right)\left\{\frac{\partial\sigma_{ij}}{\partial\mathbf{k}}\right\}^a\right)\cdot\hat{\mathbf{n}}E_i^{p*}(\mathbf{k}_1,\omega_1)E_j^p(\mathbf{k}_2,\omega_2)\right\} d\mathbf{r}\, dt \quad , \qquad \text{(B.9)}$$

where $\hat{\mathbf{n}}$ is a unit vector normal to the surface A. Integrating over A and T,

$$\begin{aligned}
P &= \frac{1}{(2\pi)^3}\frac{1}{T}\int\left(\mathbf{S}_{ij} + \left(\frac{1}{2}\right)\left\{\frac{\partial\sigma_{ij}}{\partial\mathbf{k}}\right\}^a\right)\cdot\hat{\mathbf{n}} \qquad \text{(B.10)} \\
&\quad E_i^{p*}(\hat{\mathbf{k}}_1,\omega_1)E_j^p(\hat{\mathbf{k}}_2,\omega_2)\left(\frac{{k_1}^2}{\hat{\mathbf{k}}_1\cdot\mathbf{v}_{g1}}\right)^*\left(\frac{{k_2}^2}{\hat{\mathbf{k}}_2\cdot\mathbf{v}_{g2}}\right)e^{i((\mathbf{k}_2-\mathbf{k}_1^*)\cdot\rho_a-(\omega_2-\omega_1)\tau)} \\
&\quad \delta_A^2(\hat{\mathbf{n}}\times(\mathbf{k}_2-\mathbf{k}_1^*))\,\delta_T(\omega_2-\omega_1)\,d\hat{\mathbf{k}}_1\,d\omega_1\,d\hat{\mathbf{k}}_2\,d\omega_2 \quad .
\end{aligned}$$

In the integration over $\mathbf{r}$, it was assumed that $k_{\text{Im}} \ll (L|\hat{\mathbf{k}}\times\hat{\mathbf{n}}|)^{-1}$ where L is the dimension of A in the direction of $(\hat{\mathbf{k}}\times\hat{\mathbf{n}})$. ρ_a is the position vector of the centre of the surface A and τ the midpoint in the time interval. $\delta_T(\omega)$ is a peaked function with a maximum of $T/2\pi$, a width of the order $2\pi/T$ and $\int_{-\infty}^{\infty}\delta_T(\omega)\,d\omega = 1$. δ_A^2 is defined similarly, only in two dimensions.

Carrying out one of the $\hat{\mathbf{k}}$ and one of the ω integrations,

$$P = \frac{1}{(2\pi)^3}\frac{1}{T}\int_{-\infty}^{\infty}\int_{2\pi} e^{-2k_{\mathrm{Im}}\hat{\mathbf{k}}\cdot\rho_a}\frac{|k|^4}{|\hat{\mathbf{k}}\cdot\hat{\mathbf{v}}_g|^2}\left|\left|\frac{\partial(\mathbf{k}_{\mathrm{Re}}\times\hat{\mathbf{n}})}{\partial(\hat{\mathbf{k}})}\right|\right|^{-1} \tag{B.11}$$

$$\left(\mathbf{S}_{ij} + \left(\frac{1}{2}\right)\left\{\frac{\partial\sigma_{ij}}{\partial\mathbf{k}}\right\}^a\right)\left[E_i^{p*}E_j^p\right]_{A,T}(\rho_a,\tau)\cdot\hat{\mathbf{n}}\,d\hat{\mathbf{k}}\,d\omega \quad .$$

The Jacobian, $\left|\left|\frac{\partial(\mathbf{k}_{\mathrm{Re}}\times\hat{\mathbf{n}})}{\partial(\hat{\mathbf{k}})}\right|\right|$, stems from the integration of δ_A^2 over $\hat{\mathbf{k}}$:

$$\begin{aligned}\left|\left|\frac{\partial(\mathbf{k}_{\mathrm{Re}}\times\hat{\mathbf{n}})}{\partial(\hat{\mathbf{k}})}\right|\right| &= \left|\left|\frac{\partial(\mathbf{k}_{\mathrm{Re}}\times\hat{\mathbf{n}})}{\partial(\mathbf{k}_{\mathrm{Re}}|_\omega)}\right|\left|\frac{\partial(\mathbf{k}_{\mathrm{Re}}|_\omega)}{\partial(\hat{\mathbf{k}})}\right|\right| \\ &= |\hat{\mathbf{n}}\cdot\hat{\mathbf{v}}_g|\,\frac{k_{\mathrm{Re}}^2}{|\hat{\mathbf{k}}\cdot\hat{\mathbf{v}}_g|} \quad ,\end{aligned} \tag{B.12}$$

where $|_\omega$ means for fixed ω.

$\left[E_i^{p*}E_j^p\right]_{A,T}$ is the power spectrum of E^p around the location ρ and time τ. Subscripts A, T indicate that the power spectrum is obtained from knowledge of $E(\mathbf{r},t)$ on the surface A and time period T. It is therefore also indicated that the resolution in the power spectrum is approximately $2\pi/T$ in ω while the resolution of $\mathbf{k}\times\hat{\mathbf{n}}$ is $(2\pi)^2/A$. The resulting resolution in $\hat{\mathbf{k}}$ is

$$\begin{aligned}\Delta\hat{\mathbf{k}} &= \left|\left|\frac{\partial(\hat{\mathbf{k}})}{\partial(\mathbf{k}_{\mathrm{Re}}\times\hat{\mathbf{n}})}\right|\right|\Delta\{\mathbf{k}_{\mathrm{Re}}\times\hat{\mathbf{n}}\} \\ &= \frac{(2\pi)^2}{k_{\mathrm{Re}}^2 A}\frac{|\hat{\mathbf{k}}\cdot\hat{\mathbf{v}}_g|}{|\hat{\mathbf{n}}\cdot\hat{\mathbf{v}}_g|} \quad .\end{aligned} \tag{B.13}$$

Inserting (B.12) in (B.11), integrating only over positive frequencies and making use of the fact that the direction of the power flow is parallel to the group velocity (see for instance Bers, 1963), we find that

$$P = \frac{1}{(2\pi)^3}\frac{1}{T}\int_{-\infty}^{\infty}\int_{2\pi} e^{-2k_{\rm Im}\hat{\mathbf{k}}\cdot\rho_a}\frac{k_{\rm Re}^2}{|\hat{\mathbf{k}}\cdot\hat{\mathbf{v}}_g|} \left|\left(\mathbf{S}_{ij} - \left(\frac{\omega\varepsilon_0}{2}\right)\left\{\frac{\partial\mathbf{K}_{ij}}{\partial\mathbf{k}}\right\}^h\right)\left[E_i^{p*}E_j^p\right]_{A,T}(\rho_a,\tau)\right| d\hat{\mathbf{k}}\, d\omega \quad . \tag{B.14}$$

Note that the difference between $|k|^2$ and ${k_{\rm Re}}^2$ is of second order in $k_{\rm Im}$ and hence not included here. The dielectric tensor was substituted for the conductivity tensor in (B.14). Letting A extend to infinity in both directions and neglecting damping, we find that the energy flux associated with a mode is given by

$$\frac{\partial^3 P_m(\hat{\mathbf{k}},\omega,\tau)}{\partial\hat{\mathbf{k}}\,\partial\omega} = \frac{2}{(2\pi)^3}\frac{k_{\rm Re}^2}{|\hat{\mathbf{k}}\cdot\hat{\mathbf{v}}_g|}\left|\left(\mathbf{S}_{ij} - \left(\frac{\omega\varepsilon_0}{2}\right)\left\{\frac{\partial\mathbf{K}_{ij}}{\partial\mathbf{k}}\right\}^h\right)\frac{1}{T}\left[E^{p*}{}_i E_j\right]_T(\tau)\right| \quad . \tag{B.15}$$

where the Poynting tensor now takes the simpler form

$$\{\mathbf{S}_{ij}\}_l = \frac{-\{\mathbf{k}\}_i\,\delta_{jl} - \{\mathbf{k}^*\}_j\,\delta_{il} + \{\mathbf{k}^*\}_l\,\delta_{ij} + \{\mathbf{k}\}_l\,\delta_{ij}}{2\mu_0\omega} \quad . \tag{B.16}$$

When $\mathbf{K} = \mathbf{K}^h$ we find the following useful relations:

$$\begin{aligned}\hat{\mathbf{k}}\cdot(\hat{\mathbf{e}}^*\cdot\mathbf{S}\cdot\hat{\mathbf{e}}) &= \varepsilon_0 c\mu\,\hat{\mathbf{k}}\cdot\hat{\mathbf{e}}^*\times(\hat{\mathbf{k}}\times\hat{\mathbf{e}}) \\ &= -\varepsilon_0 c\mu\,\hat{\mathbf{e}}^*\cdot\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\hat{\mathbf{e}}) \\ &= \frac{\varepsilon_0 c}{\mu}\,\hat{\mathbf{e}}^*\cdot\mathbf{K}\cdot\hat{\mathbf{e}} \quad ,\end{aligned} \tag{B.17}$$

$$\hat{\mathbf{k}}\cdot\frac{\omega\varepsilon_0}{2}\frac{\partial\mathbf{K}}{\partial\mathbf{k}} = \frac{\varepsilon_0 c}{\mu}\,\frac{k}{2}\frac{\partial\mathbf{K}}{\partial k} \quad . \tag{B.18}$$

From (B.17) and (B.18) we find that the normalized flux

$$\mathcal{F} = \frac{1}{\varepsilon_0 c}\left|\left\{\mathbf{S}_{ij} - \left(\frac{\omega\varepsilon_0}{2}\right)\left\{\frac{\partial K_{ij}}{\partial\mathbf{k}}\right\}^h\right\} e_i{}^* e_j\right|$$

satisfies the following relation:

$$\begin{aligned}\hat{\mathbf{k}} \cdot \mathcal{F} &= \frac{1}{\mu}\hat{\mathbf{e}}^* \cdot \left(\mathbf{K} - \frac{k}{2}\frac{\partial \mathbf{K}}{\partial k}\right) \cdot \hat{\mathbf{e}} \\ &= \frac{-k}{2\mu}\hat{\mathbf{e}}^* \cdot \frac{\partial \mathbf{\Lambda}}{\partial k} \cdot \hat{\mathbf{e}} \quad .\end{aligned} \tag{B.19}$$

Appendix C Étendue

To obtain equation 48 in section 2.2 we first find the radiation at a point v in vacuum resulting from a source at a point p in the plasma and then, at the point v, we use the well–known expression for the étendue of a receiving antenna. Let the power emitted from the source be represented by

$$\frac{\partial^3 P(\hat{\mathbf{k}},\omega)}{\partial \hat{\mathbf{v}}_g\, \partial \omega} = \left.\frac{\partial^3 I}{\partial \hat{\mathbf{v}}_g\, \partial \omega}\right|_p \delta a_p \tag{C.1}$$

where δa_p is the cross sectional area of the source perpendicular to the ray direction. Along any ray, (44) gives the following relation between the vacuum intensity and the intensity in the plasma:

$$\left.\frac{\partial^3 I}{\partial \hat{\mathbf{k}}\, \partial \omega}\right|_v = \left.\frac{\partial^3 I}{\partial \hat{\mathbf{v}}_g\, \partial \omega}\right|_v = \frac{1}{\mu_{\rm ray}^2} \left.\frac{\partial^3 I}{\partial \hat{\mathbf{v}}_g\, \partial \omega}\right|_p \tag{C.2}$$

The ray tube with cross sectional area δa_p at the source will, at a point v in vacuum have the cross sectional area δa_v. For the normalized beam intensity we have

$$\mathcal{I}_p \delta a_p = \mathcal{I}_v \delta a_v \tag{C.3}$$

The receiver would see no difference between a source with intensity I_p and cross sectional area δa_p located at the point p and one with intensity I_v and area δa_v located at v.

The power, P^s, received from a source, which radiates uniformly across the beam pattern of the receiver, can be related to the intensity, I, of the emission from the source through

$$\frac{\partial P^{s}}{\partial \omega} = \Omega \frac{\partial^3 I}{\partial \hat{\mathbf{k}}\, \partial \omega} A \quad , \tag{C.4}$$

where A is the cross sectional area of the beam pattern and Ω is a constant of proportionality with dimension of solid angle. Often Ω is visualized as the cone of radiation from the source which is accepted by the receiving antenna. In vacuum the product ΩA, which is called the étendue, is equal to the square of the vacuum wave length, λ_0 (Siegman, 1966):

$$\Omega = \lambda_0^2 \frac{1}{A} \quad . \tag{C.5}$$

If the cross sectional area of the source is small relative to the cross section of the beam then the above result is not applicable. Based on results given in chapter 4 of Collin and Zucker (1969), an expression similar to (C.4) and (C.5), but applicable for small sources is readily derived (Bindslev, 1989, Appendix A). The power received form a small source with cross sectional area δa perpendicular to the beam is given by

$$\frac{\partial P^{s}}{\partial \omega} = \Omega \frac{\partial^3 I}{\partial \hat{\mathbf{k}}\, \partial \omega} \delta a \tag{C.6}$$

$$\Omega = \lambda_0^2 \mathcal{I} \quad , \tag{C.7}$$

where $\mathcal{I}$ is the normalized beam intensity introduced in equation (43). Thus we find that the power received from the point v is given by

$$\frac{\partial P^{s}}{\partial \omega} = \lambda_0^{s\,2} \mathcal{I}_v \left. \frac{\partial^3 I}{\partial \hat{\mathbf{k}}\, \partial \omega} \right|_v \delta a_v \tag{C.8}$$

Using (C.2) and (C.3), (C.8) takes the form

$$\frac{\partial P^{s}}{\partial \omega} = \frac{\lambda_0^{s\,2} \mathcal{I}_p}{\mu_{\mathrm{ray}}^2} \left. \frac{\partial^3 I}{\partial \hat{\mathbf{k}}\, \partial \omega} \right|_p \delta a_p \quad , \tag{C.9}$$

which, with (C.1), gives equation (48).

Appendix D Symmetry

In this appendix the reciprocity relation for electromagnetic fields is extended to a scattering system and it is found that the equation of transfer for a scattering system should be symmetrical in incident and scattered fields. This provides a convenient check on derivations of the equation of transfer. Earlier expressions for the equation of transfer (e.g. Hughes and Smith, 1989) were slightly asymmetrical. The step in these derivations which introduced the asymmetry is identified and compared with the approach adopted here. It is further noted that assumptions are made in the derivation given by Hughes and Smith (1989) and in the derivation given here, which limit the range of validity of the expressions to situations where $\omega \ll \omega^{\mathrm{i}}$ and $k \ll k^{\mathrm{i}}$. In this regime the quantitative difference between the asymmetrical result given by Hughes and Smith and cold plasma limit of the symmetrical result given here is negligible.

The reciprocity relation has been shown to hold in a generalized form in a magnetized plasma with spatial dispersion (see e.g. discussions by Ginzburg, 1970 and Budden, 1985). To extend the reciprocity relation to a scattering system consider the following fields and currents:

$\mathbf{j}^1$, $\mathbf{j}^2$: the source currents in antennas 1 and 2 respectively.

$\mathbf{E}^1$, $\mathbf{E}^2$: the fields resulting from currents $\mathbf{j}^1$ and $\mathbf{j}^2$ respectively.

$\mathbf{j}^a$, $\mathbf{j}^b$: the currents resulting from the interaction of fields $\mathbf{E}^1$, $\mathbf{E}^2$ with the density fluctuations.

$\mathbf{E}^a$, $\mathbf{E}^b$: the fields resulting from currents $\mathbf{j}^a$ and $\mathbf{j}^b$ respectively.

The generalized form of the reciprocity relation states that

$$\int \mathbf{E}^b(\mathbf{r},-t)\cdot\mathbf{j}^1(\mathbf{r},t)\,d\mathbf{r}\,dt = \int \mathbf{E}^1(\mathbf{r},t)\cdot\mathbf{j}^b(\mathbf{r},-t)\,d\mathbf{r}\,dt \qquad \text{(D.1)}$$

$$\int \mathbf{E}^a(\mathbf{r},t)\cdot\mathbf{j}^2(\mathbf{r},-t)\,d\mathbf{r}\,dt = \int \mathbf{E}^2(\mathbf{r},-t)\cdot\mathbf{j}^a(\mathbf{r},t)\,d\mathbf{r}\,dt \quad , \qquad \text{(D.2)}$$

where the quantities with superscript 2 or b refer to a plasma where the external magnetic field is reversed. The fluctuation derived currents are related to the incident fields through

$$\mathbf{j}^a(\mathbf{r},t) = \int \delta\sigma(\mathbf{r},t,\mathbf{r}',t')\cdot\mathbf{E}^1(\mathbf{r}',t')\,d\mathbf{r}'\,dt' \quad ,$$

$$\mathbf{j}^b(\mathbf{r},t) \;=\; \int \delta\boldsymbol{\sigma}^{\dagger}(\mathbf{r},t,\mathbf{r}',t')\cdot\mathbf{E}^2(\mathbf{r}',t')\,d\mathbf{r}'\,dt' \quad ,$$

where $\delta\boldsymbol{\sigma} = \boldsymbol{\sigma}' - \boldsymbol{\sigma}$ is the plasma response associated with the density fluctuations. It is readily seen that the right hand sides of (D.1) and (D.2) are identical if

$$\delta\boldsymbol{\sigma}^{\dagger}(\mathbf{r}',-t',\mathbf{r},-t) = \delta\boldsymbol{\sigma}(\mathbf{r},t,\mathbf{r}',t') \quad . \tag{D.3}$$

Equation (D.3) is satisfied if $\delta\boldsymbol{\sigma}^{\dagger}$ is the response due to fluctuations in a plasma where the direction of time is reversed. Physically this means that the direction of the external magnetic field is reversed and that the Fourier components of the fluctuations propagate in the opposite direction. That this is the case is seen from the following considerations. $\delta\boldsymbol{\sigma}(\mathbf{r}_b,t_b,\mathbf{r}_a,t_a)$ can be regarded as the kinetic coefficient which relates the electric field at the space–time point $(\mathbf{r}_a,t_a)$ to the resulting current at the point $(\mathbf{r}_b,t_b)$. The equation of motion from which $\delta\boldsymbol{\sigma}$ is derived is the linearized Vlasov equation with the velocity distribution associated with the fluctuations, which cause the scatter, as the background velocity distribution. This equation is symmetrical under time reversal when time reversal includes reversal of the external magnetic field and reversal of the temporal evolution of the fluctuations. The latter implies reversal of the direction of propagation of a Fourier component of the fluctuations. We shall refer to the plasma (or system) with the external magnetic field and fluctuations reversed as the *conjugate* plasma (or system). Since the equation underlying $\delta\boldsymbol{\sigma}$ is symmetrical under time reversal it follows, by Onsager's principle (see e.g. Landau & Lifshitz, 1986, §120), that

$$\delta\boldsymbol{\sigma}^{\dagger}(\mathbf{r}_a,\tau_a,\mathbf{r}_b,\tau_b) = \delta\boldsymbol{\sigma}(\mathbf{r}_b,t_b,\mathbf{r}_a,t_a) \quad , \tag{D.4}$$

where $\delta\boldsymbol{\sigma}^{\dagger}$ is the fluctuation derived plasma response in the conjugate plasma. τ_a and τ_b correspond to the time points t_a and t_b respectively, but since the direction of time is reversed in the conjugate system we have that $\tau_a - \tau_b = t_b - t_a$. It is worth noting that in the conjugate system the causality requirement is $\delta\boldsymbol{\sigma}^{\dagger} = 0$ for $\tau_a - \tau_b < 0$ and hence if $\delta\boldsymbol{\sigma}$ describes a causal response then $\delta\boldsymbol{\sigma}^{\dagger}$, as given by (D.4), automatically satisfies the causality requirement in the conjugate system.

Having shown that (D.3) holds when $\delta\boldsymbol{\sigma}^{\dagger}$ refers to the conjugate plasma, the plasma to which also superscripts 2 and b refer, we find that

$$\int \mathbf{E}^b(\mathbf{r},-t)\cdot\mathbf{j}^1(\mathbf{r},t)\,d\mathbf{r}\,dt = \int \mathbf{E}^a(\mathbf{r},t)\cdot\mathbf{j}^2(\mathbf{r},-t)\,d\mathbf{r}\,dt \tag{D.5}$$

where the field $\mathbf{E}^a$ is the scattered field at antenna 2 resulting from currents $\mathbf{j}^1$

in antenna 1 while the field $\mathbf{E}^b$ is the scattered field at antenna 1 resulting from the currents $\mathbf{j}^2$ in antenna 2 when the external magnetic field is reversed and the Fourier components of the density fluctuations propagate backwards. This is the reciprocity relation for a scattering system. It clearly has the same form as the original generalized reciprocity relation. By the methods outlined in Collin and Zucker (1969) the equality of the fractions of power transfer for forward and reverse transmission is readily shown to follow from (D.5). From this it follows that the equation of transfer, (70), must be symmetrical in the incident and scattered fields.

The equations of transfer given in previous work, e.g. Hughes & Smith (1989) (referred to below as H & S), were not entirely symmetrical. It is shown here that in the approximation used by H & S and others, it is not necessary to break the symmetry. H & S implicitly expand the conductivity tensor in the density fluctuations at the response point, $(\mathbf{r}, t)$:

$$\boldsymbol{\sigma}'(\mathbf{r}, t, t') = \left(1 + \frac{\delta n_e(\mathbf{r}, t)}{n_e}\right) \boldsymbol{\sigma}(t - t') \quad . \tag{D.6}$$

We find that the equation of transfer becomes symmetrical if the conductivity tensor is expanded in the mean of the density fluctuations at the input point, $(\mathbf{r}', t')$, and the response point, $(\mathbf{r}, t)$:

$$\boldsymbol{\sigma}'(\mathbf{r}, t, \mathbf{r}', t') = \left(1 + \frac{\delta n_e(\mathbf{r}, t) + \delta n_e(\mathbf{r}', t')}{2n_e}\right) \boldsymbol{\sigma}(\mathbf{r} - \mathbf{r}', t - t') \quad . \tag{D.7}$$

Even to first order in the density fluctuations, this is only an approximation: $\boldsymbol{\sigma}$ does not depend only on the values of the electron density at the end points, $(\mathbf{r}, t)$ and $(\mathbf{r}', t')$.

Having assumed that the response length and time are small relative to the characteristic length and time of the density perturbations (section 3) it clearly follows that (D.6) and (D.7) are approximations of the same order. The relative merit of (D.7) is that with this definition $\boldsymbol{\sigma}'$ satisfies Onsager's relations and a symmetrical equation of transfer results. The two are of course linked. The choice of expansion manifests itself in the coupling term. Expansion (D.6) leads to

$$\mathcal{C} = \frac{\omega^{\mathrm{i}}\omega^{\mathrm{s}}}{\omega_p^2} \left| (\hat{\mathbf{e}}^*)^{\mathrm{s}} \cdot \left(\frac{\omega^{\mathrm{i}}\mathbf{Q}^{\mathrm{i}}}{\omega_p}\right) \cdot \hat{\mathbf{e}}^{\mathrm{i}} \right|^2 \quad . \tag{D.8}$$

while (D.7) gives

$$\mathcal{C} = \frac{\omega^{\mathrm{i}}\omega^{\mathrm{s}}}{\omega_p^2} \left| (\hat{\mathbf{e}}^*)^{\mathrm{s}} \cdot \left(\frac{\omega^{\mathrm{i}}\mathbf{Q}^{\mathrm{i}} + \omega^{\mathrm{s}}\mathbf{Q}^{\mathrm{s}}}{2\omega_p} \right) \cdot \hat{\mathbf{e}}^{\mathrm{i}} \right|^2 \quad . \tag{D.9}$$

(D.9) is symmetrical in incident and scattered fields while (D.8) is not. Since the rest of the equation of transfer for a scattering system (70) is symmetrical in incident and scattered field it follows that, with expansion (D.7), the equation of transfer becomes symmetrical in incident and scattered fields.

An estimate of the response length and response time is given by $\frac{\|\partial\boldsymbol{\sigma}/\partial\mathbf{k}\|}{\|\boldsymbol{\sigma}\|}$ and $\frac{\|\partial\boldsymbol{\sigma}/\partial\omega\|}{\|\boldsymbol{\sigma}\|}$. These have to be small relative to $1/k$ and $1/\omega$ for the above expansions to be valid. From this it follows that the difference between $\mathbf{Q}^{\mathrm{i}}$ and $\mathbf{Q}^{\mathrm{s}}$, and hence between (D.8) and (D.9), is small when the expansion is valid.

The decision to expand the conductivity, $\boldsymbol{\sigma}$, rather than for instance the susceptibility, $\mathbf{Q}$, appears arbitrary yet the result depends on it. Again the expressions converge to those given above in the limit where $k \ll k^{\mathrm{i}}$ and $\omega \ll \omega^{\mathrm{i}}$.

To find a theory of scattering in plasmas which does not require that $k \ll k^{\mathrm{i}}$ and $\omega \ll \omega^{\mathrm{i}}$ it appears necessary to break with the conventional approach of treating the fluctuations and the scattering separately. Treating the problem as a two wave coupling may be the way forward.

Appendix E Symbol list

$\mathbf{A}^a$	Anti–Hermitian part of the tensor $\mathbf{A}$.	
$\mathbf{A}^h$	Hermitian part of the tensor $\mathbf{A}$.	
A^{i}	A, referring to the *incident* field.	
A_{Im}	Imaginary part of A.	
A_{Re}	Real part of A.	
A^{s}	A, referring to the *scattered* field.	
$\hat{\mathbf{A}}$	Unit vector.	
A^*	Complex conjugate of A.	
c	Vacuum speed of light.	(6)
$\hat{\mathbf{e}}$	Unit electric field vector.	(16)
$\hat{\mathbf{g}}$	Unit eigenvector of $\mathbf{A}$.	(12)
$\mathbf{j}$	Current.	(2)
$\mathbf{k}$	Wave vector.	(A.1)
k	Complex norm of the wave vector.	(19)
$\hat{\mathbf{k}}$	Unit wave vector.	(18)

m_e	Electron mass.	(69)
n_e	Electron density.	(54)
r_e	Classical electron radius.	(54)
$\hat{\mathbf{v}}_g$	Unit group velocity vector.	(30)
q_e	Elementary charge.	(54)
$\mathbf{B}$	Magnetic flux density.	(1)
$\mathbf{D}$	Electric displacement.	(2)
$\mathbf{E}$	Electric field strength.	(1)
$\mathbf{E}^p$	Electric field strength in polar Fourier space.	(15)
G	Geometrical factor.	(71)
$\mathbf{H}$	Magnetic field strength.	(2)
$\mathbf{J}$	Fourier component of weighted current distribution.	(51)
$\mathbf{K}$	Dielectric tensor	(8)
O_b	Beam overlap.	(66)
P	Power.	(39)
$\mathbf{Q}$	Suseptibility tensor.	(9)
$\mathbf{S}$	Poynting vector.	(B.1)
$\mathbf{S}_{ij}$	Poynting tensor.	(B.16)
$S(\mathbf{k},\omega)$	Spectral density function.	(72)
T	Observation periode.	(B.9)
$\mathcal{C}$	Coupling term.	(75)
$\mathcal{E}$	Amplitude of monochromatic electric field.	(59)
$\mathcal{F}$	Normalized energy flux.	(40)
$\mathcal{I}$	Normalized beam intensity.	(43)
ε_0	Vacuum permitivity.	(4)
λ	Eigenvalue of $\mathbf{\Lambda}$.	(12)
λ_0	Vacuum wave length.	(47)
μ	Refractive index.	(7)
μ_0	Vacuum permeability.	(3)
μ_{ray}	Ray refractive index.	(45)
$\boldsymbol{\sigma}'(\mathbf{r},t,\mathbf{r}',t)$	Kernel of conductivity operator.	(4)
$\boldsymbol{\sigma}(\mathbf{r}-\mathbf{r}',t-t')$	Kernel of conductivity operator.	(5)
$\boldsymbol{\sigma}(\mathbf{k},\omega)$	Conductivity tensor.	(8)
ω	Angular frequency.	(A.1)
ω_{ce}	Angular electron cyclotron frequency.	
ω_p	Angular plasma frequency.	(69)
Λ	Determinant of the Maxwell tensor.	(27)
$\mathbf{\Lambda}$	Maxwell tensor.	(10)
Σ	Differential scattering cross section.	(77)

Appendix F Notes on Yoon and Krauss–Varban (1990)

Yoon and Krauss–Varban (1990) (referred to below as Y & K) give expressions for the dielectric tensor elements of a weakly relativistic plasma with a loss cone distribution. Setting the "loss–cone index", l, equal to zero reduces the loss–cone to a Maxwellian distribution. Their result with $l = 0$ and the corrections given below formed the basis of one of the relativistic codes used here.

Y & K use the Shkarofsky function, F, (Shkarofsky, 1966) which is calculated by means of the relation between F and the plasma dispersion function (Fried and Conte, 1961) given by Krivenski and Orefice (1983). With ψ and ϕ as defined by Krivenski and Orefice and h and z as defined by Y & K, the expression given by Y & K assumes that $\psi = h$ and $\phi = -i\sqrt{z-h}$ where the branch cut for the argument of the square root is along the negative real axis. The correct relations are $\psi = \mu_{\|}\sqrt{m_e c^2/2T_e} = \sqrt{h}$ and $\phi = \sqrt{h-z}$ for $(h-z)_{\mathrm{Re}} > 0$ and $\phi = -i\sqrt{z-h}$ for $(h-z)_{\mathrm{Re}} < 0$.

In Y & K's expression for M_{yy}^m, $(m-1)$ should be replaced by $(m+1)$ and in their expression for M_{zz}^m, $C_{m-1}^l(m-1)$ should for $m = 1$ be replaced by C_m^l.

With these corrections Y & K's expressions give the same numerical results as the expressions by Shkarofsky (1986).

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APPENDIX 1.

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CR 88.49.1A