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S.V. Bulanov[1], F. Pegoraro and S.G. Shasharina[1]

JET Joint Undertaking, Culham Science Centre, OX14 3DB, Abingdon, UK

[1]*General Physics Institute of Acad. Sci., USSR, Moscow*

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MHD MODES NEAR THE X-LINE OF A MAGNETIC CONFIGURATION

S.V. Bulanov*, F. Pegoraro+, S.G. Shasharina*

*General Physics Institute of Acad. Sci., USSR, Moscow
+JET Joint Undertaking, Abingdon, Oxon. OX14 3EA.

ABSTRACT

The propagation of shear-Alfvén waves in an incompressible plasma is considered in the vicinity of an X-line of the magnetic field. A small dissipation leads to a discrete spectrum of weakly damped MHD modes, with a dissipative layer in the vicinity of the separatrices.

Finite amplitude perturbations imposed from the boundaries and propagating perpendicularly to one of the separatrices are then considered, and the characteristic timescale of their evolution is derived.

1. INTRODUCTION

It is now commonly recognized that the magnetic islands occurring at the resonant surfaces of a toroidal magnetic equilibrium configuration represent a rather typical situation. These islands may occur for different reasons, such as resistive instabilities and perturbations of the plasma boundaries which initiate the breaking of the symmetry of the configuration. Alternatively, structures topologically similar to islands can be produced inside the plasma by an external electric current, as is the case of tokamaks with divertors, and, in JET, of discharges with magnetic nulls.

The properties of the plasma dynamics near an X-line of the magnetic field, the line of intersection of two magnetic separatrix surfaces, are quite distinctive. If the electric current is carried by the plasma, resistivity leads to reconnection of the magnetic field lines in the vicinity of the X-line. When estimated in the framework of the qualitative Parker-Sweet [1] model, which assumed the existence of a thin current sheet directed along the null line of the magnetic field, the characteristic time of this process is of the order of $t_R \epsilon^{1/2}$. Here the small dimensionless parameter

$$\epsilon = t_A/t_R \qquad (1)$$

is the ratio of the Alfvén time $t_A = a/c_A$, with $c_A = B/\sqrt{4\pi\rho}$, and the resistive time $t_R = 4\pi a^2/\eta c^2$, where η is the plasma resistivity and a is the typical size of the inhomogeneity. On the other hand, in connection with the general question of the propagation of MHD waves and of the reconnection of magnetic field lines in a plasma with a non-uniform magnetic field, the analysis of model problems which can be solved analytically may also be of interest.

The aim of the present paper is to analyse the evolution of MHD perturbations in a magnetic configuration with separatrix surfaces. Together with the commonly adopted slab configuration, the linear field that describes the neighbourhood of an X-line is an important model for the investigation of reconnection processes.

In this paper we consider an equilibrium magnetic field of the form

$$\underline{B} = (hx + gy)\underline{e}_x - hy\,\underline{e}_y + B_{\|}\underline{e}_z \; . \qquad (2)$$

In the x-y plane, the magnetic field lines have the structure shown in Fig.1. The z-axis coincides with the line of intersection of the two separatrix surfaces, one in the y = 0 plane and the other at an angle arctg (2h/g). A uniform current density, $\underline{J} = -\frac{4\pi}{c} g\underline{e}_z$, corresponds to the magnetic field (2), where the value of h determines the current free contribution. If h vanishes, Eq.(2) describes the magnetic field configuration considered e.g. in Refs.[2-4], where the y = 0 surface is the resonant surface of perturbations independent of z.

Here we consider two problems. Firstly, we analyse shear Alfvén waves propagating at an arbitrary angle with respect to the separatrices. A small resistivity results in the reflection of the waves from the dissipative regions near the separatrices, which in turn leads to the appearance of a discrete spectrum. An explicit expression for the mode frequency is obtained in the limit of short wavelengths. These modes are weakly damped and the real part of the frequency is much larger than the imaginary part. Secondly we consider finite amplitude, MHD perturbations imposed from the boundary and propagating normally to one of the separatrices. Despite the dependence of the background magnetic field (2) on two coordinates, x and y, exact solutions of the MHD equations are possible, depending only on one coordinate and on time. The amplitude of these perturbations increases as they approach the separatrix. A non vanishing resistivity leads to a saturation of this amplification. The typical time for the redistribution of the electric current density in the plasma is longer than the Alfvén time by a factor $\ell n^3(4/\epsilon)$.

2. MHD EQUATIONS

In contrast to Ref.[5], where the propagation of MHD waves in the vicinity of an X-line of the magnetic field was considered in a zero pressure plasma with $B_{||} = 0$ and $g = 0$, in the present paper we assume that the $B_{||}$ component in Eq.(1) is much greater than $\underline{B}_{\perp}$. This leads to an incompressible plasma flow in the x,y plane. Hence the system of MHD equations

$$\frac{\partial}{\partial t}\underline{v} + (\underline{v}\cdot\nabla)\underline{v} = -\frac{\nabla p}{\rho} + \frac{\underline{J}x\underline{B}}{c\rho} \quad , \tag{3}$$

$$\frac{\partial}{\partial t}\underline{B} = \nabla\ x\ (\underline{v}\ x\ \underline{B}) + \frac{\eta c^2}{4\pi}\nabla^2\underline{B} \quad , \tag{4}$$

can be transformed into

$$(\frac{\partial}{\partial t} + \underline{v}\cdot\nabla)\ \nabla^2\Phi = -\frac{1}{\rho}(\underline{B}\cdot\nabla)\ \nabla^2\Psi \ , \tag{5}$$

$$(\frac{\partial}{\partial t} + \underline{v}\cdot\nabla)\ \Psi = \frac{\eta c^2}{4\pi}\nabla^2\Psi \ . \tag{6}$$

The functions Ψ and Φ are such that the magnetic field in the x-y plane and the plasma velocity can be expressed in the form

$$\underline{B} = (\nabla\ x\ \Psi)\ \underline{e}_z, \qquad \underline{v} = (\nabla\ x\ \Phi)\ \underline{e}_z \ . \tag{7}$$

Linearizing Eqs.(5) and (6) around the equilibrium solution

$$\Phi_o = 0, \quad \Psi_o = hxy + \frac{gy^2}{2} \ , \tag{8}$$

we obtain for the perturbations Φ_1 and Ψ_1

$$\frac{\partial}{\partial t}\Psi_1 + (\underline{B}_o\cdot\nabla)\ \Phi_1 = \epsilon\ \nabla^2\Psi_1 \ , \tag{9}$$

$$\frac{\partial}{\partial t}\nabla^2\Phi_1 + (B_o\cdot\nabla)\ \nabla^2\Psi_1 = 0 \ . \tag{10}$$

Here and below we use dimensionless variables, where t is normalized to the Alfvén time $t_A = (4\pi\rho/h^2)^{1/2}$, lengths are normalized to the characteristic size a and the small dimensionless parameter ϵ defined by (1) becomes

$$\epsilon = \left(\frac{\rho}{4\pi}\right)^{\frac{1}{2}} \frac{\eta c^2}{h a^2} . \tag{11}$$

In the following the subscripts 1 and 0 will be omitted. The operator $(\underline{B}_o \cdot \nabla)$ in Eqs.(9) and (10) has the form

$$(\underline{B}_o \cdot \nabla) = (x + \gamma y) \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} , \tag{12}$$

where $\gamma = g/h$. To simplify the calculations it is convenient to use the Fourier transforms of Eqs.(9) and (10). Thus we obtain

$$\frac{\partial}{\partial t} \hat{\Psi} - [k \frac{\partial}{\partial k} - (q - \gamma k) \frac{\partial}{\partial q}] \hat{\Phi} = - \epsilon (k^2 + q^2) \hat{\Psi} , \tag{13}$$

$$\frac{\partial}{\partial t} \hat{\Phi} = + \frac{1}{k^2 + q^2} [k \frac{\partial}{\partial k} - (q - \gamma k) \frac{\partial}{\partial q}] (k^2 + q^2) \hat{\Psi} . \tag{14}$$

The functions $\hat{\Psi}$ and $\hat{\Phi}$ depend on k and q according to the expressions

$$\hat{\Psi}(k,q,t) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \exp[-i(kx + qy)] \Psi(x,y,t) , \tag{15}$$

$$\hat{\Phi}(k,q,t) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \exp[-i(kx + qy)] \Phi(x,y,t) . \tag{16}$$

In terms of the two new variables ζ and r

$$\zeta = kq - \gamma k^2/2 , \tag{17}$$

$$r = (k/\zeta)(1 + \gamma^2/4)^{1/4} , \tag{18}$$

Eqs.(13) and (14) become

$$\frac{\partial}{\partial t} \hat{\Psi} - r \frac{\partial}{\partial r} \hat{\Phi} = - \epsilon(1 + \frac{\gamma^2}{4})^{1/2} |\zeta| [|\zeta| r^2 + \frac{1}{|\zeta| r^2} + \kappa] \hat{\Psi} , \tag{19}$$

$$\frac{\partial}{\partial t} \hat{\Phi} = [|\zeta| r^2 + \frac{1}{|\zeta| r^2} + \kappa]^{-1} r \frac{\partial}{\partial r} [|\zeta| r^2 + \frac{1}{|\zeta| r^2} + \kappa] \hat{\Psi} , \tag{20}$$

where the constant κ is given by

$$\kappa = [2\gamma/(4 + \gamma^2)^{1/2}] \, \mathrm{sgn}\zeta , \tag{21}$$

and $\mathrm{sgn}x = \pm 1$ for $x \gtrless 0$ respectively. A representation of the (k,q) plane is shown in Fig.(2). We introduce the new variable

$$\ell = \frac{1}{2} \ln |\zeta| r^2 , \tag{22}$$

and define the function

$$S^2(\ell) = |\zeta| r^2 + \frac{1}{|\zeta| r^2} + \kappa = 2 \,\mathrm{ch}(2\ell) + \kappa , \tag{23}$$

which reduces to $S^2(\ell) \sim 2\mathrm{ch}(2\ell)$, for $\gamma \ll 2$, and to $S^2(\ell) \sim 4\mathrm{ch}^2\ell$ or to $S^2(\ell) \sim 4\mathrm{sh}^2\ell$ for $\gamma \gg 2$ and sgn ζ = 1 or -1 respectively. Then, from Eqs.(19) and (20) we obtain

$$\frac{\partial^2}{\partial t^2} \hat{\Psi} - \frac{\partial}{\partial \ell} \frac{1}{S^2(\ell)} \frac{\partial}{\partial \ell} S^2(\ell) \hat{\Psi} = - \epsilon S^2(\ell) \frac{d\hat{\Psi}}{dt} , \tag{24}$$

where

$$\hat{\epsilon} = \epsilon \; |\zeta| (1 + \gamma^2/4)^{\frac{1}{2}} \; . \tag{25}$$

Setting

$$W(\ell) = S(\ell) \; \hat{\Psi} \; , \tag{26}$$

Eq.(24) can be rewritten as

$$\frac{\partial^2}{\partial t^2} W - \frac{\partial^2}{\partial \ell^2} W + S \; (\frac{\partial^2 S^{-1}}{\partial \ell^2}) \; W = - \; \epsilon \; S^2 \; \frac{\partial}{\partial t} W \; . \tag{27}$$

3. IDEAL MHD SOLUTIONS

In this section we investigate the propagation of Alfvén waves in the limit $\epsilon = 0$, i.e. we neglect the effect of dissipation. We consider two limits. The first one corresponds to large values of γ in which case the angle between the two separatrices is small. In this limit, Eq.(27) reduces to

$$\hat{W}'' + (\omega^2 - 1 + \frac{2}{\mathrm{ch}^2 \ell}) \; \hat{W} = 0 \; , \tag{28}$$

and to

$$\hat{W}'' + (\omega^2 - 1 - \frac{2}{\mathrm{sh}^2 \ell}) \; \hat{W} = 0 \; , \tag{29}$$

for sgn $\zeta = 1, -1$ respectively, where $W = \hat{W} \exp(-i\omega t)$ and a prime denotes differentiation with respect to ℓ. These equations can be solved in terms of elementary functions:

$$\hat{W} = C_{\pm} \; \exp(\pm i\alpha\ell)[\mathrm{tgh}\ell \mp i\alpha] \; , \tag{30}$$

and

$$\hat{W} = C_{\pm} \exp(\pm i\alpha\ell)[\mathrm{ctgh}\ell \mp i\alpha] \ , \tag{31}$$

respectively, where

$$\alpha = \sqrt{\omega^2 - 1} \ , \tag{32}$$

and $C_{\pm}$ are integration constants. In the limit where γ is small, Eq.(27) reduces to

$$\hat{W}'' + (\omega^2 - 1 + \frac{3}{\mathrm{ch}^2 2\ell}) \hat{W} = 0 \ , \tag{33}$$

which can be solved in terms of hypergeometric functions [of which (30) and (31) are elementary limits] in the form

$$\hat{W} = C_{\pm}[\mathrm{ch}^2 2\ell]^{\pm\alpha/4} F(-\frac{1}{2} \mp \frac{\alpha}{2}, \frac{3}{2} \mp \frac{\alpha}{2} \ , 1 \mp \frac{\alpha}{2} \ ; \frac{1}{2} (1 - \mathrm{tgh} 2\ell)) \ . \tag{34}$$

For the solutions in x-y space to be regular, $\hat{\Psi}$ must satisfy the appropriate boundary conditions in Fourier space. For $\ell \to \pm \infty$, i.e. for $k \to \infty$ and $k \to 0$ respectively (which correspond to $q \to \gamma k/2$ and to $q \to \infty$, see Fig.2), Eqs.(30), (31) and (34) lead to the asymptotic forms for Ψ

$$\hat{\Psi} \sim \exp[-i(\omega t \pm \alpha\ell) - |\ell|] \ . \tag{35}$$

For real α, i.e. for $\omega^2 > 1$, Eq.(35) corresponds to the superposition of waves propagating in the positive and negative direction along ℓ. In Fourier-space, the amplitude of Ψ decreases as $1/k$ for $k \to \infty$ and as k for $k \to 0$. However, the current density

$$\hat{J} \propto (h^2 + q^2)\hat{\Psi} \sim S^2 \hat{\Psi} \ , \tag{36}$$

increases as k, when $k \to \infty$ and as $1/k$ when $k \to 0$. For $\omega^2 \ll 1$, Eq.(35) has the two asymptotic forms

$$\hat{\Psi} \sim \exp\left[-i\omega t - (|\ell| \pm \ell) \pm \omega^2 \ell/2\right] , \qquad (37)$$

leading again to a current density that diverges as $|\ell| \to \infty$.

These results indicate that the effect of resistivity must be included for $|\ell| \to \infty$. In coordinate space this corresponds to the neighbourhood of the two separatrices. Resistive effects will be considered in the next section.

It is first convenient to express the solutions (30), (31) and (34) in terms of even and odd functions of ℓ. The transformation $\ell \to -\ell$, i.e. $q - \gamma k/2 \leftrightarrow k(1 + \gamma^2/4)^{1/2}$, corresponds in coordinate space to the interchange of the two separatrices, i.e. to $x + \gamma y/2 \leftrightarrow y(1 + \gamma^2/4)^{1/2}$. In the case of Eq.(30), which holds for $\gamma \gg 2$ and sgn $\zeta > 0$, we have simply

$$\hat{W} = C_e[\text{tgh}\ell \sin\alpha\ell - \alpha\cos\alpha\ell] + C_o[\text{tgh}\ell \cos\alpha\ell + \alpha\sin\alpha\ell] , \qquad (38)$$

with C_e and C_o integration constants. The solutions (31) of Eq.(29), which holds for $\gamma \gg 2$ and sgn$\zeta < 0$, behave as $1/\ell$ for $\ell \to 0$. Consequently $\hat{\Psi}$ behaves as $1/\ell^2$. This singularity is the result of the approximation employed for $S(\ell)$. For $\gamma \gg 2$ and $\ell \to 0$, $S(\ell)$ can be approximated as $S(\ell) \propto \ell^2 + 1/\gamma^2$. Then, the correct even and odd solutions are

$$\hat{W} = (\ell^2 + 1/\gamma^2)^{-1/2}[C_e + C_o\ell(\ell^2 + 3/\gamma^2)] , \qquad (39)$$

where we have assumed $\omega^2 \ll \gamma^2$. Matching (39), for $|\ell|\gamma \to \infty$, to (31) for $|\ell| \ll 1$, we find that $C_\pm$ in Eq.(31) take opposite signs for positive and for negative values of ℓ. Rewriting Eq.(31) as a combination of an even and of an odd solution, we obtain, for $|\ell| > 1/\gamma$ and for frequencies such that $|\omega/\gamma| < 1$,

$$\hat{W} = C_e \mathrm{sgn}\ell[\mathrm{ctgh}\ell \cos\alpha\ell + \alpha\sin\alpha\ell] + C_o \mathrm{sgn}\ell[\mathrm{ctgh}\ell\sin\alpha\ell - \alpha\cos\alpha\ell] \ . \quad (40)$$

In the case of Eq.(32), valid for $\gamma << 1$, the following relationship between C_+ and C_- holds for even (s = 1), and for odd (s = -1) modes:

$$C_{\pm} = \frac{\pm iC_{\mp}\, s\alpha}{2[1 \mp s/\sin(i\pi\alpha/2)]} \ \frac{(1 \pm i\alpha)\ \Gamma^2(\mp \frac{i\alpha}{2})}{(1 \mp i\alpha)\ \Gamma^2(\frac{1}{2} \mp \frac{i\alpha}{2})} \ . \quad (41)$$

4. DISCRETE MHD SPECTRUM

Let us assume that the dimensionless parameter ϵ in Eq.(27) is different from zero, but small ($\epsilon << 1$). In this case, even for arbitrary small ϵ, resistive effects become dominant for $\ell \to \pm \infty$.

First we consider the limit $|\gamma| >> 2$, and, instead of Eqs.(28) and (29), obtain

$$\hat{W}'' + (\omega^2 - 1 + \frac{2}{\mathrm{ch}^2\ell})\ \hat{W} = i4\omega\ \hat{\epsilon}\ \mathrm{ch}^2\ell\ \hat{W} \ , \quad (42)$$

and

$$\hat{W}'' + (\omega^2 - 1 - \frac{2}{\mathrm{sh}^2\ell})\ \hat{W} = i4\omega\ \hat{\epsilon}\ \mathrm{sh}^2\ell\ \hat{W} \ , \quad (43)$$

with

$$\hat{\epsilon} = \epsilon|\zeta|\gamma/2 \ . \quad (44)$$

Expanding for large $|\ell|$, as suggested by the results of the previous section, we obtain in both cases

$$\hat{W}'' + (\omega^2 - 1)\,\hat{W} = -\,i\omega\hat{\epsilon}\,\exp(2|\ell|)\,\hat{W}\,. \tag{45}$$

Equation (45) can be solved in terms of Bessel functions. Imposing that $\hat{W}$ vanishes for $|\ell| \to \infty$, the relevant solution can be written as

$$\hat{W}(\ell) = A_{\pm}\, H^{(1)}_{i\alpha}\,[\sqrt{i\omega\hat{\epsilon}}\,\exp\,(|\ell|)]\;, \tag{46}$$

where $\alpha = \sqrt{\omega^2 - 1}$, $H^{(1)}_{\gamma}(z)$ is a Hankel function, $A_{\pm}$ are integration constants and $\mathrm{Im}\,\sqrt{i\omega} > 0$. For $|\ell| \gg |\ell_t|$, where

$$\ell_t = \pm\,\frac{1}{2}\,\ln\,[-\,i\,\frac{\omega^2 - 1}{4\omega\hat{\epsilon}}]\;, \tag{47}$$

are the two symmetrical turning points of Eqs.(42) and (43), this solution decreases exponentially with k for $k \to \infty$ and with 1/k for $k \to 0$. Both $\hat{\Psi}$ and $\hat{J}$ are now well behaved.

Matching Eq.(46) for $|\ell| \ll |\ell_t|$ to the limiting expression (35) for $|\ell| \gg 1$, and using Eqs.(38) and (40) we obtain the dispersion relation for the frequency ω as a function of ζ and ϵ in the limit $\gamma \gg 2$. The resulting expression is

$$[\frac{i\omega\hat{\epsilon}}{4}]^{i\alpha}\,\exp(\pi\alpha) = \pm\,\frac{(1 - i\alpha)\;\Gamma(1 + i\alpha)}{(1 + i\alpha)\;\Gamma(1 - i\alpha)}\;, \tag{48}$$

where the plus sign holds for even modes with sgn $\zeta > 0$ and for odd modes with sgn $\zeta < 0$ and the minus sign holds for odd modes with sgn $\zeta > 0$ and for even modes with sgn $\zeta < 0$. Even and odd refers to the symmetry of $\hat{\Psi}$.

In the limit $|\gamma| << 2$ an analogous procedure leads again to (45), with

$$\hat{\epsilon} = \epsilon|\zeta| \ , \tag{49}$$

and, after matching with the limiting expression of Eq.(35) using (41), to the dispersion relation

$$\left[\frac{i\omega\hat{\epsilon}}{16}\right]^{i\alpha} \exp(\pi\alpha) = \pm i \left[\frac{\sin(i\pi\alpha/2) \pm 1}{\sin(i\pi\alpha/2) \mp 1}\right]^{1/2} \left[\frac{1 + i\alpha}{1 - i\alpha}\right] \frac{\Gamma^2(\frac{1}{2} + \frac{i\alpha}{2})}{\Gamma^2(\frac{1}{2} - \frac{i\alpha}{2})} \ , \tag{50}$$

where the plus and minus sign refer to the even and odd modes respectively.

Eqs.(48) and (50) take a simple form in the high frequency limit $\alpha \sim \omega >> 1$. In this limit from Eq.(48) we obtain

$$\omega_R \ \ell n|\omega_R/\epsilon\zeta\gamma| = (2n \pm 1/2)\pi \ , \tag{51}$$

and

$$\omega_I \ \ell n^2|\omega_R/\epsilon\zeta\gamma| = - \ |2n \pm 1/2|\pi^2/2 \ , \tag{52}$$

where $\omega = \omega_R + i\omega_I$ and n is an integer. Similarly from Eq.(50) we obtain

$$\omega_R \ \ell n|\omega_R/\epsilon\zeta| = m\pi \ , \tag{53}$$

and

$$\omega_I \ \ell n^2|\omega_R/\epsilon\zeta| = - \ |m| \ \frac{\pi^2}{2} \ , \tag{54}$$

where m = 2n + 1 for even modes and m = 2n for odd modes.

Equations.(48) and (50) describe modes with a discrete spectrum for fixed ζ. At large frequencies they are equally spaced, the imaginary part of the frequency is much smaller than the real part, and their characteristic width in the k space is given by

$$\Delta k \simeq \left[\frac{\pi n}{\hat{\epsilon}} \ln^{-1}\left(\frac{\pi n}{\hat{\epsilon}}\right)\right]^{1/2} , \qquad (55)$$

corresponding to a width of the dissipative regions in the neighbourhood of the separatrices of order

$$\Delta x \simeq \left[\frac{\hat{\epsilon}}{\pi n} \ln\left(\frac{\pi n}{\hat{\epsilon}}\right)\right]^{1/2} . \qquad (56)$$

5. FINITE AMPLITUDE PERTURBATIONS

The system of equations (3) and (4) admits solutions of the form

$$\Psi = hxy + A(y,t) \quad , \qquad (57)$$

$$\underline{v} = v(y,t)\, \underline{e}_x \quad . \qquad (58)$$

These solutions describe finite amplitude, incompressible perturbations such that their electric current density and plasma velocity depend only on one coordinate and on time. In terms of the small amplitude perturbations considered in the previous two sections, these solutions correspond to the special case $\zeta \equiv 0$, i.e. to perturbations that propagate in the direction perpendicular to one of the two separatrices. The y-component of Eq. (3) results in the following expression for the pressure

$$p = -\frac{1}{4\pi}\left[hx\frac{\partial A}{\partial y} + \frac{1}{2}\left(\frac{\partial A}{\partial y}\right)^2\right] + P(x,t) \ , \tag{59}$$

where $P(x,t)$ is independent of y. Inserting (59) into the x-component of Eq. (3) we obtain

$$\frac{\partial}{\partial t} v = \frac{h}{4\pi\rho}\left(\frac{\partial A}{\partial y} - y\frac{\partial^2 A}{\partial y^2}\right) - \frac{1}{\rho}\frac{\partial P(x,t)}{\partial x} \ . \tag{60}$$

When combined to (58), it implies

$$\frac{1}{\rho}\frac{\partial P(x,t)}{\partial x} = -a(t) \ , \tag{61}$$

with $a(t)$ a spatially uniform acceleration. From Eq.(4) it follows that

$$\frac{\partial A}{\partial t} = -hvy + \frac{\eta c^2}{4\pi}\frac{\partial^2 A}{\partial y^2} \ . \tag{62}$$

Combining Eqs. (60) and (62), and considering perturbations such that $a(t) = 0$, as is e.g. the case when v is an odd function of y, we find that the x-component of the perturbed magnetic field

$$b(y,t) \equiv \frac{\partial A}{\partial y} \ , \tag{63}$$

obeys the linear equation describing finite amplitude shear-Alfvén waves

$$\frac{\partial^2}{\partial t^2} b - y^2\frac{\partial^2}{\partial y^2} b - y\frac{\partial}{\partial y} b + b = \epsilon\frac{\partial}{\partial t}\frac{\partial^2}{\partial y^2} b \ , \tag{64}$$

where we have reintroduced dimensionless variables. Applying the Laplace transform with respect to time

$$\hat{b}(y,s) = \int_0^\infty \exp(-st)\, b(y,t)\, dt \ , \tag{65}$$

we obtain

$$(\epsilon s + y^2)\, \hat{b}'' + y\hat{b}' - (s^2 + 1)\, \hat{b} = 0 \ , \tag{66}$$

where $b(y, 0) = \partial b(y, 0)/\partial t = 0$ have been assumed for $|y| < 1$ and a prime denotes differentiation with respect to y. In terms of the variable

$$\xi \equiv \ln\, (y + \sqrt{s\epsilon + y^2}) \ . \tag{67}$$

Eq.(66) becomes

$$\frac{d^2\hat{b}}{d\xi^2} = (s^2 + 1)\, \hat{b} \ , \tag{68}$$

and its solutions can be written as

$$\hat{b}(y,\ s) = C_1(\sqrt{s\epsilon + y^2} + y)^{\sqrt{1+s^2}} + C_2(\sqrt{s\epsilon + y^2} - y)^{\sqrt{1+s^2}} \ , \tag{69}$$

with $C_1(s)$ and $C_2(s)$ integration constants. The characteristic scale length across the separatrix is of order $\sqrt{s\epsilon}$. For $y^2 \gg s\epsilon$, the two solutions behave as $y^{\pm\sqrt{1+s^2}}$ respectively.

We consider boundary conditions such that the perturbations of the magnetic field equals $b_o\theta(t)$ at $y = 1$ and $- b_o\theta(t)$ at $y = -1$, with $\theta(t)$ the step function ($\theta(t) = 1$ for $t > 0$ and $\theta(t) = 0$ for $t < 0$). The Laplace transform of the boundary conditions gives $\hat{b}(1,s) = b_o/s$ and $\hat{b}(-1,s) = - b_o/s$. Then (69) can be rewritten as

$$\hat{b}(s,y) = \frac{b_o}{s} \frac{\mathrm{sh}[\sqrt{1+s^2}\ \ell n\ (\frac{\sqrt{\epsilon s + y^2} + y}{\sqrt{\epsilon s}})]}{\mathrm{sh}[\sqrt{1+s^2}\ \ell n\ (\frac{\sqrt{\epsilon s + 1} + 1}{\sqrt{\epsilon s}})]} . \tag{70}$$

The time dependence of the perturbed magnetic field is obtained from the inverse Laplace transform

$$b(y,t) = \frac{1}{2\pi i} \int_C \hat{b}(s,y)\ \exp(st)ds \tag{71}$$

of (70), where C is the Bromwich contour. The singular points of the integrand in (71) are given by $s = 0$ and by

$$\sqrt{1+s^2}\ \ell n[(\sqrt{\epsilon s + 1} + 1)/\sqrt{\epsilon s}] = i\pi n , \tag{72}$$

with n an integer number. The contribution of the pole $s = 0$ leads to the term $b(y,t) = b_o y$, which corresponds to the time independent solution of Eq.(63) for the given boundary conditions. The long time behaviour of (71) is then determined by the solutions of Eq.(72) for which the real part of s is largest [6], corresponding to $n = \pm 1$. The contribution of the branch cuts of (70) along the negative s axis can be shown to be unimportant. Then, combining the contributions at the poles $s = 0$ and

$$s = \pm i[1 + 2\pi^2/\ell n^2(4/\epsilon)] - 2\pi^3/\ell n^3(4/\epsilon) . \tag{73}$$

We obtain, for $|y| \ll \sqrt{\epsilon}$,

$$b(y,t) \simeq b_o y\ [\ 1 + \frac{4\ \cos(t+\pi/4)}{\sqrt{\epsilon}\ \ell n^2(4/\epsilon)} \exp(-\ 2\pi^3 t/\ell n^3(4/\epsilon))] . \tag{74}$$

This shows that the typical time scale for the evolution of one-dimensional shear-Alfvén waves in the vicinity of the separatrix is of order $t_A \ell n^3(4/\epsilon)/\pi^3$, with t_A the Aflvén time.

6. CONCLUSIONS

The results presented in this paper show that characteristic frequencies can be associated with the plasma dynamics around the X-lines of a magnetic field configuration. The presence of the separatrices results in a propagation of the MHD waves towards the dissipative regions in the vicinity of the separatrices that is faster than that found around the resonant surfaces in a slab configuration [4]. In the short wavelength limit, dissipation leads to the reflection of the waves and to the appearance of a discrete spectrum of two-dimensional, localised weakly damped MHD modes. In the one-dimensional approximation, i.e. when the scale length of the perturbations imposed from the boundaries along one of the separatrices is much larger than that along the other, non-linear shear Alfvén waves have been found to decay on a times scale that is considerably longer than the Alfvén time.

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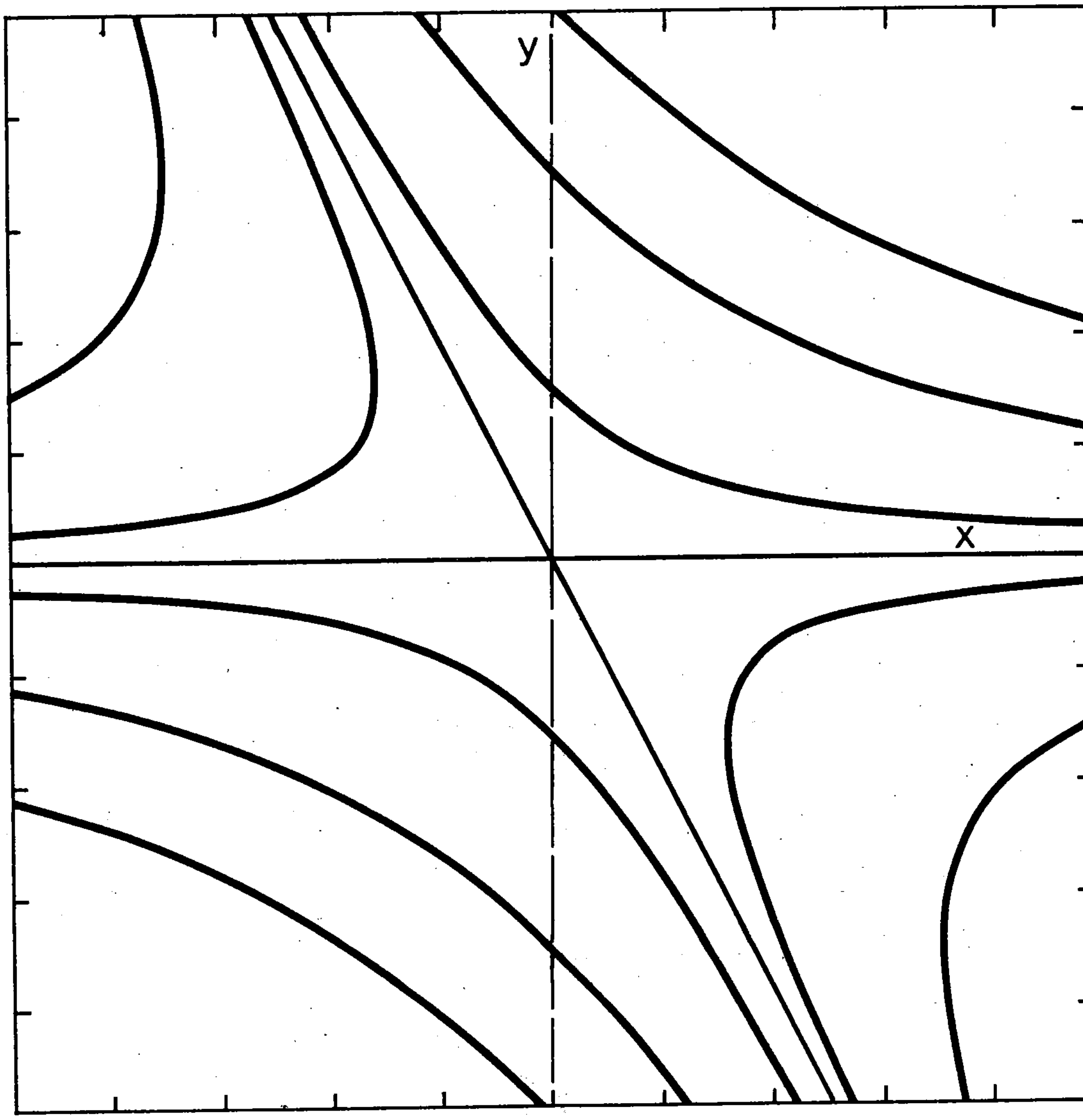

Fig.1 Magnetic field structure near the separatrix surfaces.

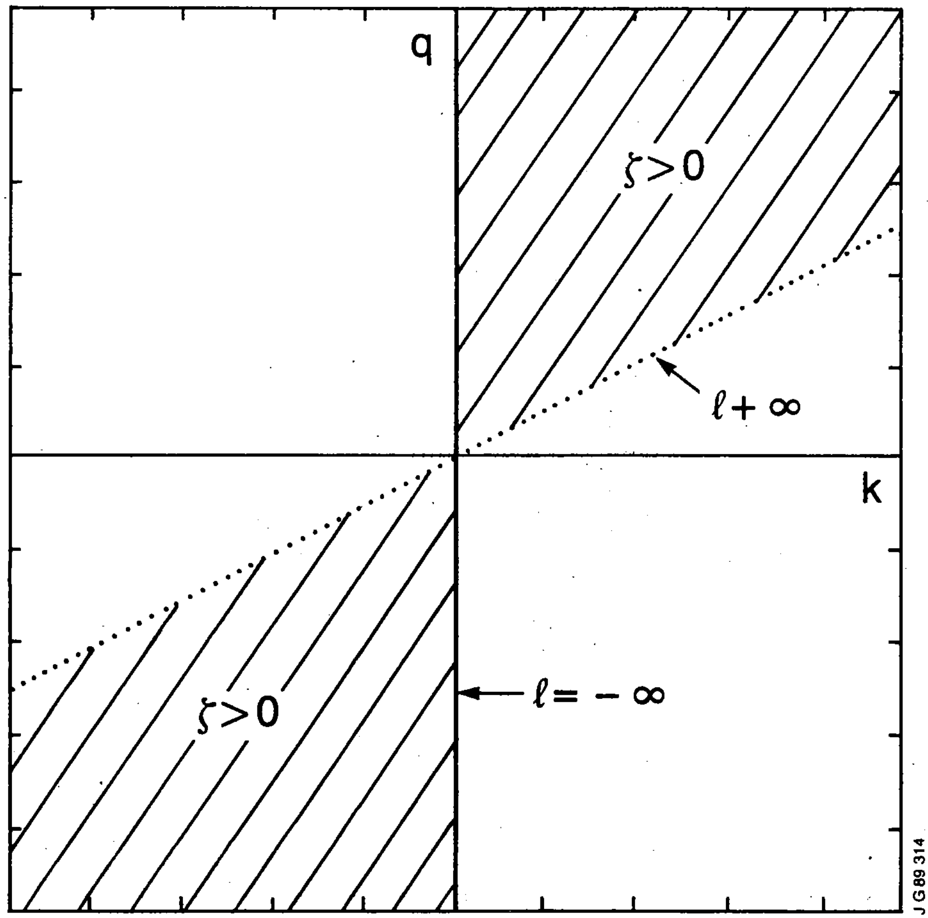

Fig.2 Wave-vector plane. The variable ζ is positive inside the shaded area bounded by the k = 0 axis ($\ell = -\infty$, solid line) and by $q = \gamma k/2$ ($\ell = \infty$, dotted line).