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EXTENDED VARIABLE REPRESENTATIONS

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ABSTRACT

The extended-variable ("ballooning") representation, commonly used for the study of short wavelength plasma modes in a toroidal magnetic equilibrium, is generalised to include:

i) the effect of a differential toroidal rotation

ii) non linear terms in the mode amplitude.

These results are obtained by relating this representation to the irreducible representations of the relevant invariance group, which is shown to be a discrete Abelian subgroup of the Heisenberg group.

1. INTRODUCTION

The extended variable representation was introduced in 1977 by different plasma theoreticians, Refs.[1-4], as a convenient mathematical tool in order to describe the linear phase of high-n modes in toroidal, magnetically confined, plasmas.

In these analyses the equilibrium configuration is assumed to be axisymmetric, with the toroidal angle ϕ an ignorable coordinate, and n is the mode toroidal number. The radial shear of the magnetic field lines generates a mismatch between the (changing) helical pitch ι of the field lines winding along the torus, and the pitch of the mode. This mismatch gives a radially dependent contribution to the longitudinal mode number $k_{||} = -i\, \underline{b}\cdot\nabla\, \ell n\tilde{\xi}$. The radial direction is defined as perpendicular to the toroidal magnetic surfaces, $\underline{b}$ is the unit vector along field lines and $\tilde{\xi}$ is a representative perturbed quantity. An additional contribution to $k_{||}$, which is characteristic of toroidal configurations, arises from the non uniformity of the equilibrium along field lines caused by the absence of rotational invariance in the poloidal plane around the plasma magnetic axis.

In a strong magnetic field, modes with finite $k_{||}$ are difficult to excite, as a consequence, for example in the case of magnetic modes, of the tension of the stretched field lines. Thus, if we write $\tilde{\xi}(r,\theta,\phi) = \xi(r,\theta)\exp(in\phi)$, the amplitude $\xi(r,\theta)$ proves to be a non factorisable function of the radial coordinate r and of the poloidal angle θ. In fact, the spectrum of the mode poloidal harmonics, which are coupled by the lack of poloidal rotational invariance, changes with r so as to accommodate the variation of ι and to minimise the longitudinal mode number. The n/m ratio, with m the central value of the poloidal mode number spectrum, can match the pitch ι only on mode-rational surfaces where n/ι is an integer, and a residual mismatch is present outside these surfaces.

For modes with large toroidal mode numbers n, if the dimensionless shear parameter $s = - d\ell n\ \iota/d\ell n\ r$ is not small, a convenient separation occurs between the radial scale length related to the equilibrium variations, which are of the order of the torus minor radius, and the distance between adjacent mode-rational surfaces. This allows us to introduce the auxiliary "fast" radial variable S

$$S = n/\iota - m^o \ , \qquad (1)$$

where $m^o = n/\iota(r_o)$, with r_o a reference mode rational surface. To leading order in the resulting asymptotic expansion, the mode dispersion relation for the amplitude $X(S,\theta,r)$, where

$$\xi(r,\theta) = X(S,\theta,r)\ \exp(-\ im^{o}\theta)\ , \tag{2}$$

characteristically takes the form of a partial differential equation in θ and S. The dependence on r is determined by successive orders in the asymptotic expansion [2]. For the sake of definiteness in this letter we consider the following reference equation

$$(\frac{\partial}{\partial\theta} + iS)(1 - s^2 \frac{\partial^2}{\partial S^2})(\frac{\partial}{\partial\theta} + iS)\ X(S,\theta) + \Gamma(\cos\theta + is\ \sin\theta\ \frac{\partial}{\partial S})\ X(S,\theta)$$

$$+ \Omega^2(1 - s^2 \frac{\partial^2}{\partial S^2})\ X(S,\theta) = 0\ , \tag{3}$$

which was derived [5] in a (simplified) treatment of the ideal magnetohydrodynamic ballooning modes [6]. These modes are the magnetic analogue of the Rayleigh-Taylor instabilities [7]. They are driven by the combined effect, represented by the term proportional to $\Gamma = \Gamma(r)$ in Eq.(3), of the magnetic curvature and of the plasma pressure gradient, which have a relative orientation which depends on the poloidal angle as indicated by the periodic coefficients in θ. In Eq.(3) the only explicit dependence on S occurs through the longitudinal mode number $k_{||}$ represented by the operator $\partial/\partial\theta + iS$, where the linear term in S is due to the shear of the magnetic field lines. The term proportional to $k_{||}^2$, arises from the tension of the perturbed field lines and Ω is the normalised mode frequency. Equation (3) is to be solved subject to the condition that $X(S,\theta)$ be periodic in θ as the mode amplitude must be a single valued function of the poloidal angle.

The extended variable representation reduces Eq.(3) to an ordinary differential equation, preserving the required amplitude periodicity in θ. This is achieved by introducing an extended poloidal variable $\hat{\theta}$ ranging from $-\infty$ to $+\infty$, and an extended amplitude $\hat{X}(\hat{\theta})$ from which the physical amplitude

$X(S,\theta)$ is reconstructed through a summation procedure. A detailed account of this method can be found in [8]. Here we quote two key formulae:

$$X(S,\theta) = \int_{-\pi}^{+\pi} \frac{d\alpha}{2\pi} X_\alpha(S,\theta) \exp(iS\alpha) \ , \tag{4}$$

with α, $-\pi < \alpha \leq \pi$, the effective radial wave number, and

$$X'_\alpha(S,\theta) = \sum_{-\infty}^{+\infty}{}_m \hat{X}_\alpha(\hat{\theta}) \exp(-\ i2\pi mS) \ , \tag{5}$$

with $X'_\alpha(S,\theta) = X_\alpha(S,\theta) \exp(iS\theta)$ and $\hat{\theta} = \theta + 2\pi m$. Using (4) and (5), the dispersion relation (3) reduces to

$$\frac{\partial}{\partial\hat{\theta}} [1 + s^2(\hat{\theta} - \alpha)^2] \frac{\partial}{\partial\hat{\theta}} \hat{X}_\alpha(\hat{\theta}) + \Gamma[\cos\hat{\theta} + s^2(\hat{\theta} - \alpha) \sin\hat{\theta}] \hat{X}_\alpha(\hat{\theta})$$

$$+ \Omega_\alpha^2 [1 + s^2(\hat{\theta} - \alpha)^2] \hat{X}_\alpha(\hat{\theta}) = 0. \tag{6}$$

2. INVARIANCE GROUP

In order to establish the group theoretical basis of Eqs.(4) and (5), we notice that the differential operators $\partial/\partial\theta + iS$ and $\partial/\partial S$ in (3) satisfy the commutation relation

$$[\partial/\partial S, \partial/\partial\theta + iS] = i. \tag{7}$$

We denote by L the three-dimensional Lie-algebra generated by $\partial/\partial S$, $\partial/\partial\theta + iS$ and i. The transformation $X'(S,\theta) = X(S,\theta) \exp(iS\theta)$ leads to an equivalent realisation of this algebra in terms of $\partial/\partial S - i\theta$, $\partial/\partial\theta$ and i. We denote the algebra generated by the latter operators by L', and find that

$$[L,L'^*] = 0 \ , \tag{8}$$

where a star denotes complex conjugation. Then, in order to determine the invariance group of Eq.(3), we consider the Heisenberg group H [9] generated by L'^*, i.e. by $\partial/\partial S + i\theta$, $\partial/\partial\theta$ and $-i$. Its action on the mode amplitude is defined by $X(S,\theta) \rightarrow X(S + \lambda_1,\theta)\ \exp(i\lambda_1\theta)$, $X(S,\theta) \rightarrow X(S,\theta + \lambda_2)$ and $X(S,\theta) \rightarrow X(S,\theta)\ \exp(-i\lambda_3)$ with λ_1, λ_2 and λ_3 the parameters that label the transformations generated by $\partial/\partial S + i\theta$, $\partial/\partial\theta$ and $-i$ respectively.

The θ-dependent terms make Eq.(3) invariant only under discrete poloidal transformations such that $\lambda_2/2\pi$ is an integer. This restriction follows from the absence of poloidal rotational symmetry in the toroidal equilibrium. Furthermore, the mode amplitude remains periodic in θ under the action of transformations along S only for integer values of λ_1 [8]. We conclude that the invariance group of Eq.(3), subject to the condition that its solutions be periodic in θ, is the discrete subgroup G of H defined by

$$X(S,\theta) \rightarrow X(S + 1,\theta)\ \exp(i\theta) \ , \tag{9a}$$

$$X(S,\theta) \rightarrow X(S,\theta + 2\pi) \ . \tag{9b}$$

The group G is Abelian. Thus we can expand the solutions of Eq.(3) on a basis of common eigenfunctions of the transformations (9). Periodicity in θ requires that the eigenvalue of the first transformation be equal to one. The eigenvalue of the second transformation can be written as $\exp(i\alpha)$, with α defined below Eq.(4), which represents the expansion of $X(S,\theta)$ into eigenfunctions of (9a). Then the primed amplitude $X'_\alpha(S,\theta) \equiv X_\alpha(S,\theta)\ \exp(iS\theta)$ is periodic in S with period one, while $X'_\alpha(S,\theta + 2\pi) = X'_\alpha(S,\theta)\ \exp(i2\pi S)$.

Equation (5) follows immediately by expanding $X'_\alpha(S,\theta)$ into a Fourier series in S and by imposing the poloidal periodicity constraint on $X_\alpha(S,\theta)$. This constraint leads to the relationship $\hat{X}_{\alpha,m}(\theta) = \hat{X}_\alpha(\theta + 2\pi m)$ [8] between the Fourier coefficients $X_{\alpha,m}(\theta)$ which are thus expressed in terms of a single function of an extended poloidal variable, $\hat{X}_\alpha(\hat{\theta})$.

Alternatively, Eq.(5) can be derived as a consequence of the restriction from the group H to its subgroup G [10]. The eigenfunctions of the $\partial/\partial S + i\theta$, with eigenvalues differing by 2π times an integer, are coupled when only discrete radial transformations are considered, leading to the summation in (5). Since G is Abelian, these coupled eigenfunctions can be chosen such that they are simultaneous eigenfunctions of (9b). A complementary representation in terms of an extended radial variable [8] is obtained by reversing the order with which this restriction is imposed, i.e. by starting from the periodic eigenfunction of $\partial/\partial\theta$, which are coupled when only discrete poloidal transformations are considered, and by successively expanding them into eigenfunctions of (9a).

3. DIFFERENTIAL TOROIDAL ROTATION

The usefulness of identifying the normal modes of the dispersion equation in terms of eigenfunctions of its invariance group can be convincingly illustrated when the extended variable representation is generalised to perturbations occurring in a plasma with sheared equilibrium mass flows. The evolution of ballooning perturbations in a rotating toroidal configuration has been analysed in Ref.[11] and more recently in [12] and [13]. A difficulty with the standard extended variable representation, as given by Eq.(5), arises when the plasma rotation is not uniform. This introduces an explicit dependence on the fast radial variable S that cannot be removed from the dispersion equation, if perturbations with a factorised time dependence of the form $\exp(-i\Omega t)$ are chosen.

We consider the case of a purely toroidal differential plasma rotation and, in order to simply illustrate the relevant mathematical steps, adopt the following model dispersion equation

$$\left(\frac{\partial}{\partial\theta} + iS\right)\left(1 - s^2 \frac{\partial^2}{\partial S^2}\right)\left(\frac{\partial}{\partial\theta} + iS\right) X(S,\theta,t) + \bar{\Gamma}\left(\theta, \frac{\partial}{\partial S}\right) X(S,\theta,t)$$

$$- \left(\frac{\partial}{\partial t} + i\mu S\right)\left(1 - s^2 \frac{\partial^2}{\partial S^2}\right)\left(\frac{\partial}{\partial t} + i\mu S\right) X(S,\theta,t) = 0 . \qquad (10)$$

Here $\bar{\Gamma}(\theta,\partial/\partial S)$ is periodic in θ and corresponds to the combination $\Gamma(\cos\theta + is \sin\theta\ \partial/\partial S)$ in Eq.(3). The amplitude $X(S,\theta,t)$ depends explicitly on time. The operator $\partial/\partial t + i\mu S$ plays the rôle of the Doppler shifted frequency $\Omega - n\Omega_R(r)$, with $\Omega_R(r)$ the plasma toroidal rotation frequency. The term $i\mu S$ represents the differential part of the Doppler shift,

$$\mu S \equiv n[\Omega_R(r) - \Omega_R(r_o)] , \qquad (11)$$

with $\mu = \mu(r)$, while the constant part $n\Omega_R(r_o)$ is included in $\partial/\partial t$.

In full analogy to the mode poloidal structure described in the previous section, we expect $X(S,\theta,t)$ to be a non factorisable function of t, and the mode frequency spectrum to adjust itself with S so as to minimise the relative change of the phase caused by the S-dependent Doppler shift. This adjustment is not complete for all values of t, and we expect solutions that, apart from a phase factor which defines the mode "frequency" and growth rate, are periodic functions of t with period $|\mu|^{-1}$, corresponding to the phase lag between adjacent mode rational surfaces.

In order to identify the invariance group of Eq.(10) we start from the Lie algebra L_R of the operators $\partial/\partial S$, $\partial/\partial\theta + iS$, $\partial/\partial t + i\mu S$ and i, define L_R' through the transformation $X'(S,\theta,t) = X(S,\theta,t) \exp[iS(\theta + \mu t)]$ and find

$$[L_R,\ L_R'^*] = 0\ . \tag{12}$$

The group H_R generated by $L_R'^*$, is the direct product of the Heisenberg group generated by $\partial/\partial S + i\eta$, $\partial/\partial\eta$, and $-i$, where $\eta = \theta + \mu t$ and we have used η, S and t as coordinates, and of the group generated by $\partial/\partial t$. As in Sec. 2 we must restrict H_R to discrete poloidal, $\theta \to \theta + 2\pi$, and radial, $S \to S + 1$, transformations. Furthermore, to obtain an Abelian subgroup, we restrict H_R to discrete time transformations $t \to t + 2\pi/\mu$. The resulting invariance group G_R is the direct product of the group of G of Sec. 2 with η substituted for θ, and of a group of discrete time translations (at constant η).

The expansion of $X(S,\theta,t)$ into eigenfunctions of G_R provides the natural definition of the normal modes of (10) and identifies two eigenvalues, the radial wave number α and the "frequency" Ω. The latter is in general a complex number with Re Ω restricted to an interval of width $|\mu|$. The amplitude $X'_{\alpha,\Omega}(S,\eta,t) \equiv X_{\alpha,\Omega}(S,\theta,t)\ \exp[iS(\theta + \mu t)]$ is periodic in S and in μt at constant η. It can be expanded into eigenfunctions of the discrete time translations and, in analogy to Eq.(5), expressed in the extended variable $\hat{\eta} = \eta + 2\pi m$ in the form

$$X'_{\alpha,\Omega}(S,\eta,t) = \sum_{k=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \hat{X}_{k,\alpha,\Omega}(\hat{\eta})\ \exp[-\ i(2\pi mS + k\mu t)]\ . \tag{13}$$

A complementary extended variable representation can be derived in terms of an extended radial variable.

Inserting (13) into (10), we obtain an infinite system of coupled ordinary differential equations

$$\frac{\partial}{\partial\hat{\eta}}\ (1+s^2\hat{\eta}^2)\ \frac{\partial}{\partial\hat{\eta}}\ \hat{X}_k(\hat{\eta})\ +\ [\Omega+\mu k+i\mu\ \frac{\partial}{\partial\hat{\eta}}](1+s^2\hat{\eta}^2)[\Omega+\mu k+i\mu\ \frac{\partial}{\partial\hat{\eta}}]\ \hat{X}_k(\hat{\eta})$$

$$+ \frac{\Gamma}{2} [\cos\hat{\eta}+s\hat{\eta}\ \sin\hat{\eta}][\hat{X}_{k-1}(\hat{\eta})+\hat{X}_{k+1}(\hat{\eta})]+ \frac{\Gamma}{2i} [\sin\hat{\eta}-s\hat{\eta}\ \cos\hat{\eta}][\hat{X}_{k-1}(\hat{\eta})-\hat{X}_{k+1}(\hat{\eta})], \quad (14)$$

where, for the sake of illustration, we have taken $\alpha = 0$, dropped the index Ω, and used the form of $\bar{\Gamma}$ given in (3). A convenient approach to solving the system (14) is to use μ as an expansion parameter, since $\mu < 1$ for most cases of interest. Then, to lowest order in μ, and for values of k such that $|\mu k| < \Omega$, $\hat{X}_k$ becomes independent of k and Eq.(14) reduces to (6). For larger k this approximation is not valid and would lead to a singular time behaviour.

4. NON LINEAR AMPLITUDE REPRESENTATION

The extended variable representation can be generalised to products of the mode amplitude by suitably extending the action of the group G [10]. A finer radial scale appears according to the substitution $S \rightarrow hS$, with h the order of the nonlinearity, as consistent with the rule for the toroidal number $n \rightarrow hn$, which follows from Eq.(2). In this letter we restrict our discussion to $h = 2$, but the extension to arbitrary h is straightforward.

We consider the product $Z_{\alpha''}(S,\theta) = X_{\alpha}(S,\theta)\ Y_{\alpha'}(S,\theta)$, with $\alpha'' = \alpha + \alpha'$ and $X_{\alpha}(S,\theta)$ and $Y_{\alpha'}(S,\theta)$ eigenfunctions of G. Then $Z_{\alpha''}(S,\theta)$ is periodic in θ, $Z_{\alpha''}(S + 1,\theta) = Z_{\alpha''}(S,\theta) \exp(- 2i\theta)$ and $Z_{\alpha''}(S,\theta) \exp(iS\alpha'')$ belongs to the irreducible representation labelled by $\langle\alpha''\rangle \equiv \alpha'' - 2\pi\rho$, of the group $G^{(2)}$, with $\rho = 0, \pm 1$, such that $-\pi < \langle\alpha''\rangle \leq \pi$. This group extends the action of G to terms quadratic in the mode amplitude and is defined by the transformations $Z(S,\theta) \rightarrow Z(S,\theta + 2\pi)$ and $Z(S,\theta) \rightarrow Z(S + 1,\theta) \exp(i2\theta)$.

The group $G^{(2)}$ can be seen as a subgroup of G_2. This is isomorphic to G and its action is obtained from that of G by substituting 2S for S. The subgroup $G^{(2)}$ is obtained from G_2 by restricting the transformations $2S \rightarrow 2S + \ell$ to even values of ℓ. The latter group acts on functions $K(2S,\theta)$, periodic in θ, which can be expanded into irreducible representations following a procedure analogous to that developed in Sec. 2. The generalisation of the extended variable representation is then obtained by relating the eigenfunctions of $G^{(2)}$ to those of G_2. Since $G^{(2)}$ is a subgroup, this relationship involves the coupling of two eigenfunctions belonging to different representations of G_2. We find

$$Z_{\alpha''}(S,\theta)\ \exp(iS\alpha'') = K_\beta(2S,\theta)\ \exp(i2S\beta) + K_{\beta\pm\pi}(2S,\theta)\ \exp[i2S(\beta\pm\pi)] \ , \qquad (15)$$

where $K_\beta(2S,\theta + 2\pi) = K_\beta(2S,\theta)$, $K_\beta(2S + 1,\theta) = K_\beta(2S,\theta)\ \exp(-\ i\theta)$, $\beta = \alpha''/2$, $-\ \pi < \beta \leq \pi$ and the sign in $\beta \pm \pi$ is chosen such that $-\ \pi < \beta \pm \pi \leq +\ \pi$. Using the extended poloidal variable representation on the left and on the right hand side of Eq.(15) we obtain, for $Z'_{\alpha''}(S,\theta) \equiv X'_\alpha(S,\theta)\ Y'_{\alpha'}(S,\theta)$ with $Z'_{\alpha''}(S,\theta) = Z_{\alpha''}(S,\theta)\ \exp(i2S\theta)$,

$$Z'_{\alpha''}(S,\theta) = \sum_{-\infty}^{+\infty}{}_m\ [\hat{K}_\beta(\hat{\theta}) + \hat{K}_{\beta\pm\pi}(\hat{\theta})\ \exp(\pm\ i2\pi S)]\ \exp(-\ i2\pi m2S) \ , \qquad (16)$$

which is the counterpart of Eq.(5). Representation (16) involves two functions (in the general case h functions) of a single extended variable. These are given by the convolution products

$$\hat{K}_\beta(\hat{\theta}) = \sum_{-\infty}^{+\infty}{}_p\ \hat{X}_\alpha(\hat{\theta}^+)\ \hat{Y}_{\alpha'}(\hat{\theta}^-) \ , \qquad (17a)$$

$$\hat{K}_{\beta\pm\pi}(\hat{\theta}) = \sum_{-\infty}^{+\infty}{}_p\ \hat{X}_\alpha(\hat{\theta}^+ \mp 2\pi)\ \hat{Y}_{\alpha'}(\hat{\theta}^-) \ , \qquad (17b)$$

where $\hat{\theta} = \theta + 2\pi m$, $\hat{\theta}^\pm = \hat{\theta} \pm 2\pi p = \theta + 2\pi(m \pm p)$. A complementary representation in terms of an extended radial variable is described in [10] where the inversion formulae of Eqs.(15) and (16) are also given.

5. CONCLUSIONS

In this letter the relationship between the so called extended variable ("ballooning") representation and the invariance group of the mode dispersion relation has been identified. As a result, the extension of this representation to differentially rotating plasma equilibria has been derived. Central to this extension is the choice of the eigenfunctions of the invariance group as the normal modes of the dispersion relation. In the presence of differential rotation, this leads to modes with a non-factorisable time dependence. Nevertheless, a mode frequency can be defined. The mode amplitude is periodic in time with a period given by the time lag between adjacent mode rational surfaces.

The same group-theoretical formalism makes it possible to derive the proper convolution product that expresses the extended variable representation of a non linear term in the mode amplitude in terms of the representations of the individual amplitudes. This method applies to modes with equal (or multiple) toroidal mode numbers n and provides a convenient "Fourier" basis for studying the non linear evolution of high n-modes in a toroidal axisymmetric configuration.

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