JET–P(89)41

F. Pegoraro
and JET Team

Plasma Modes, Periodicity and Symplectic Structure

Plasma Modes, Periodicity and Symplectic Structure

F. Pegoraro[1]
and JET Team*

JET-Joint Undertaking, Culham Science Centre, OX14 3DB, Abingdon, UK

Scuola Normale Superiore, 56100 Pisa, Italy
** See Appendix 1*

Preprint of Paper to be submitted for publication in
Festschrift Volurre

JET-P(89)41

PLASMA MODES, PERIODICITY AND SYMPLECTIC STRUCTURE

F. Pegoraro*

JET Joint Undertaking, Abingdon, Oxon, OX14 3EA, U.K.

*Permanent address: Scuola Normale Superiore, 56100 Pisa, Italy.

Preprint of a paper to be submitted for publication in Festschrift Volume in Honour of prof. L.A. Radicati, Scuola Normale Superiore, 1989

May 1989

PLASMA MODES, PERIODICITY AND SYMPLECTIC STRUCTURE

F. Pegoraro

Scuola Normale Superiore, 56100 Pisa, Italy.

JET Joint Undertaking, Abingdon, Oxon. OX14 3EA, UK

ABSTRACT

A group theoretical formulation of the so-called "ballooning" or "extended variable" representation, commonly adopted for the study of the linear eigenmodes of short wavelength excitations in a toroidal, axisymmetric electromagnetic plasma, is presented. With the help of this novel formulation, this representation is generalised so as to include the non-linear evolution of the excitations.

1. INTRODUCTION

The behaviour of a high temperature plasma in a strong magnetic field depends on the global geometrical and topological features of its configuration. Its properties are anisotropic because charged particles move essentially freely along field lines, while they are effectively tied to them in the perpendicular directions. These features are reflected in the behaviour and spatial structure of the plasma collective excitations. These generally exceed the thermal fluctuation level as plasmas are in most cases far from thermodynamic equilibrium. In particular, short wavelength collective excitations (micro-instabilities) are held responsible for the observed enhanced particle and energy diffusion across the field lines in inhomogeneous magnetically confined plasmas of the type that are of interest for controlled thermonuclear fusion experiments.

It has long since been recognised that the most dangerous excitations are those with an amplitude which is approximately constant along the equilibrium magnetic field of the configuration. The reason for this can be easily shown, e.g. in the case of magnetic plasma excitations, as a strong restoring force would be produced by the tension of the field lines which are stretched when the perturbed magnetic field varies along the equilibrium field. Whether, or to what degree, this condition can be satisfied, depends on the global structure of the magnetic configuration.

In many experiments, the largely unrestrained motion of the plasma particles along the field lines has led to the adoption of closed confinement configurations. The resulting topology is characteristically that of a (full) torus with helically shaped magnetic field lines winding along toroidal surfaces (magnetic surfaces) as sketched in Fig.1a. Generally the pitch of these helices is not constant across the toroidal surfaces (Fig.1b), i.e. the magnetic field is sheared, and non-rational surfaces are covered ergodically by the field lines. Sheared magnetic configurations of this type are advantageous for plasma stability.

To illustrate this point we refer to a ubiquitous form of instability in magnetically confined plasmas. Charged particles, as they move along curved magnetic field lines, e.g. in the configuration depicted in Fig.1a, experience a centrifugal acceleration which is equivalent to an effective gravity. If this acceleration points in the direction opposite to the gradient of the plasma pressure, the plasma can decrease its energy simply by exchanging regions of high and low pressure. This instability is similar to the Rayleigh-Taylor instability [1] which occurs in inhomogeneous or stratified fluids. However, the motion of a highly conductive plasma is constrained by the (approximate) conservation of the magnetic flux. A convenient way of accounting for this constraint is by imagining the plasma to be made of tubes of constant magnetic flux that maintain their identity as they move. If the magnetic field is sheared, these tubes can be thought of as forming sheaths of skewed elastic strings. This net of strings cannot be opened to allow the exchange of different plasma regions without the individual strings being stretched.

In a toroidally confined plasma the direction of the centrifugal acceleration relative to that of the pressure gradient changes from the inside to the outside of the torus as indicated in Fig.2, so that the energy of the configuration is most efficiently reduced by displacing only that portion of the magnetic tube where these two directions are opposite. Such a deformation involves wave lengths that are short on the scale of the torus minor radius and stretches the magnetic tube leading to a restoring force.

These qualitative arguments are taken here as a justification for writing the linearised, local dispersion relation for magnetic plasma excitations (plasma modes) of the type described above in the heuristic form

$$\omega^2 = k_{||}^2(r,\theta)\ c_A^2 - g(r,\theta)/r_p. \tag{1}$$

Here ω is the mode frequency, $k_{||}(r,\theta)$ is the local value of the component of the mode wave vector along the equilibrium magnetic field, r and θ are coordinates that label the magnetic surfaces and the poloidal angle respectively (see Fig.2), c_A is the plasma Alfvén velocity (i.e. the propagation velocity of magnetic perturbations in a plasma), $-g$ is the effective gravity projected onto the direction of the pressure gradient and r_p is the characteristic scale length of the pressure gradient. The dispersion relation (1) applies to small amplitude excitations, and, if they are growing in time, is restricted to their linear phase. Being local (its r.h.s. is a function of r and θ), Eq.(1) cannot be used to determine the mode frequency, which can only be obtained by solving the relevant dispersion equation for the full mode spatial eigenfunction, but can be interpreted as relating $k_{||}$ to ω, for example in an eikonal (WKBJ) approximation scheme [2]. The term quadratic in $k_{||}$ arises from the restoring force due to the tension of the magnetic field lines and depends on r because the direction of the magnetic field changes with r (magnetic shear). It also depends on θ since the relative orientation of the centrifugal acceleration and the pressure gradient is a function of θ so that energy is to be gained if the amplitude of the mode is localised on the outside (θ = 0). If, for modes so localised, the term $-g/r_p$ prevails in the r.h.s. of (1), ω^2 is negative and a purely growing instability occurs, which is called the "ballooning" instability [3] in the plasma physics literature in view of its spatial appearance.

2. SMALL AMPLITUDE EIGENFUNCTIONS

The onset of plasma excitations can be described in terms of a dispersion equation for the mode eigenfunction, obtained by linearising the appropriate set of dynamical equations around a prescribed equilibrium configuration. In the case of the magnetic modes introduced above, the simplest significant set of dynamical equations is provided by the so-called "ideal magnetohydrodynamic" approximation. This portrays the plasma as a perfectly conducting fluid. This description has a rather limited validity, but will suffice for the scope of this presentation which simply aims to elucidate the general mathematical properties and spatial structure of the excitations.

The plasma confinement experiments that are considered in the present analysis consist of toroidal magnetic equilibrium configurations (see Fig.1) which are approximately axisymmetric. We may thus consider excitations characterised, during their linear phase, by a well defined toroidal (the long way around the torus) mode number n. We thus write

$$\xi(t,\phi,\theta,r) = \hat{\xi}(\theta,r) \exp(in\phi - i\omega t), \tag{2}$$

where ϕ is the toroidal angle and we have chosen a scalar quantity ξ to represent the excitation. For instance ξ may be defined as the radial (i.e. normal to the magnetic surfaces) component of the plasma displacement vector associated with the perturbed magnetic field. On the contrary the excitation amplitude $\hat{\xi}(\theta,r)$ contains several poloidal (the short way around the torus) mode numbers, as the different poloidal harmonics are coupled by the lack of rotational symmetry in the poloidal plane. For the mode amplitude to be poloidally localised, the expansion of $\hat{\xi}(\theta,r)$ in poloidal harmonics must extend to large poloidal mode numbers m. This condition can be consistent with the requirement that the excitation be approximately constant along the equilibrium field lines only if the toroidal mode number n is also large (short wavelength excitations). More precisely the n/m ratio, with m the central value of the poloidal mode number spectrum, must match the pitch ι of the magnetic field lines. This is defined as the ratio on each toroidal surface between the number of turns a field line must follow along the short and the long way around the torus respectively before closing onto itself. Thus we expect $\hat{\xi}(r,\theta)$ to be a non-factorisable function of r and θ and the spectrum of poloidal mode numbers to change with r so as to accommodate the variation of ι across magnetic surfaces. A residual mismatch between the pitch of the mode and that of the field lines must remain on magnetic surfaces which are not "mode rational" i.e. for which $n/\iota(r)$ is not an integer.

Since n is large, and provided the dimensionless shear parameter s

$$s \equiv -d \ln \iota / d \ln r \tag{3}$$

is not small, the distance between adjacent mode rational surfaces is much smaller than the characteristic scale length of variation of the equilibrium configuration (such as r_p in Eq.(1)). The latter is generally of the order of the minor radius of the torus. Thus, when solving the mode dispersion equation for $\hat{\xi}(r,\theta)$ we will adopt a two scale approach [2] corresponding to an asymptotic expansion for large n. The additional radial coordinate is defined as

$$S = n/\iota - m^o, \tag{4}$$

with $m^o = n/\iota(r_o)$ and r_o a reference mode rational surface. The distance between two adjacent mode rational surfaces corresponds to $\Delta S = 1$. Following this procedure we consider S and r to be independent variables and write

$$\hat{\xi}(r,\theta) = X(r,S,\theta) \exp(- i\ m^o\ \theta), \tag{5}$$

where, for the sake of convenience, part of the mode poloidal dependence has been factorised. The dependence of the new amplitude X on r and S must be determined separately by solving the mode dispersion equation in successive orders of approximation for large n. The dependence on S describes the response of the mode to the shear of the magnetic field lines, whereas the dependence on r is due to the radial change of the equilibrium. To leading order in the asymptotic expansion, only the S dependence comes into play.

The detailed derivation of the dispersion equation requires algebraic steps that we think convenient to bypass by referring to a model equation (see e.g. ref.[4]) which retains the features essential for the present analysis. Then, to leading order in 1/n, we write the mode dispersion equation as

$$(\frac{\partial}{\partial\theta} + iS)\ (1 - s^2 \frac{\partial^2}{\partial S^2})\ (\frac{\partial}{\partial\theta} + iS)\ X(S,\theta) + \Gamma[\cos\theta + is\ \sin\theta\ \frac{\partial}{\partial S}]\ X(S,\theta)$$

$$+ \Omega^2[1 - s^2 \frac{\partial^2}{\partial S^2}]\ X(S,\theta) = 0. \qquad (6)$$

The function $\Gamma = \Gamma(r)$, which is a constant on the S scale, is proportional to the radial gradient of the plasma pressure. The poloidal variation of the effective gravity is represented by the trigonometric functions of θ, while Ω is a properly normalised frequency. The operator $(\partial/\partial\theta + iS)$ plays the role of the mode wave number along the equilibrium field and the linear term in S is due to the shear of the magnetic field. If, formally, we substitute $-ik_{||}$ for $(\partial/\partial\theta + iS)$ and disregard the S derivatives, we recover the heuristic dispersion relation (1) with $\Gamma\ \cos\theta \rightarrow g$.

3. PERIODICITY AND EXTENDED POLOIDAL VARIABLE

It would appear that Eq.(6) can be reduced to an ordinary differential equation in θ by means of the transformation

$$X'(S,\theta) = X(S,\theta)\ \exp(iS\theta), \qquad (7)$$

and by subsequently expanding $X'(S,\theta)$ into plane-wave solutions in S. However $X(S,\theta)$ is a single valued function of the poloidal angle θ. This periodicity reintroduces an S-dependence through the condition

$$X'(S,\theta + 2\pi) = X'(S,\theta)\ \exp(i2\pi S). \qquad (8)$$

A solution to this problem was found in almost the same months by different researchers, including the present author in collaboration with Dr. T.J. Schep (see refs.[5-8]). Setting aside differences in their formulation, all the solutions that were presented relied on introducing an "extended" poloidal variable $\hat{\theta}$ ranging from $-\infty$ to $+\infty$, and an extended mode amplitude $\hat{X}(\hat{\theta})$ from which the physical amplitude $X(S,\theta)$ can be reconstructed through a summation procedure.

The approach of ref.[5], which is described in detail in ref.[9], is based on the observation that the dependence on S introduced by the exponential factor in (8) is periodic with period one. Then, solutions for $X'(S,\theta)$ can be sought that, aside for a phase factor, are periodic in S with period one.

These are obtained by writing

$$X'(S,\theta) = \int_{-\pi}^{+\pi} \frac{d\alpha}{2\pi} X'_\alpha(S,\theta) \exp(iS\alpha), \tag{9}$$

with

$$X'_\alpha(S,\theta) \exp(iS\alpha) = \sum_{-\infty}^{+\infty}{}_p X'(S + p,\theta) \exp(-ip\alpha), \tag{10}$$

where

$$X'_\alpha(S + 1,\theta) = X'_\alpha(S,\theta) \tag{11}$$

and α, $-\pi < \alpha \leq \pi$, plays the role of a radial wave number. The S-periodic amplitude X'_α is subsequently expanded in a Fourier series

$$X'_\alpha(S,\theta) = \sum_{-\infty}^{+\infty}{}_m X_{\alpha,m}(\theta) \exp(-i2\pi mS). \tag{12}$$

When (8) is imposed, the following relationship between the Fourier coefficients $X_{\alpha,m}(\theta)$ is found

$$X_{\alpha,m}(\theta) = X_{\alpha,o}(\theta + 2\pi m) \equiv \hat{X}_\alpha(\hat{\theta}), \tag{13}$$

finally leading to the "extended poloidal variable" representation

$$X'_\alpha(S,\theta) = \sum_{-\infty}^{+\infty}{}_m \hat{X}_\alpha(\hat{\theta}) \exp(-i2\pi mS), \tag{14}$$

with $\hat{\theta} = \theta + 2\pi m$ the extended poloidal variable.

Equation (14) can be inverted in the form

$$\hat{X}_\alpha(\hat{\theta}) = \int_{-1/2}^{1/2} dS\, X'_\alpha(S,\theta) \exp(i2\pi mS). \tag{15}$$

Combining (15) and (10) with (7), and (14) and (9) with (7) we have

$$\hat{X}_\alpha(\hat{\theta}) = \int_{-\infty}^{+\infty} dS\, X(S,\theta) \exp[iS(\hat{\theta} - \alpha)] \tag{16}$$

and

$$X(S,\theta) = \sum_{-\infty}^{+\infty}{}_m \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} \hat{X}_\alpha(\hat{\theta}) \exp[-iS(\hat{\theta} - \alpha)]. \tag{17}$$

A corresponding representation can be obtained in terms of an extended S-variable by expanding the θ-periodic amplitude $X_\alpha(S,\theta) \equiv X'_\alpha(S,\theta) \exp(-iS\theta)$ in poloidal harmonics and subsequently using Eq.(11) to find a relationship analogous to (13) between the poloidal Fourier coefficients. The resulting expressions are

$$X_\alpha(S,\theta) = \sum_{-\infty}^{+\infty}{}_\ell \bar{X}_\alpha(\bar{S}) \exp(i\ell\theta), \tag{18}$$

with $\bar{S} = S + \ell$ the extended S variable, and

$$\bar{X}_\alpha(\bar{S}) = \int_{-\pi}^{+\pi} \frac{d\theta}{2\pi} X_\alpha(S,\theta) \exp(-i\ell\theta). \tag{19}$$

It is easy to verify that the representation in the extended poloidal amplitude and that in the extended S-variable are related by the Fourier transformation

$$\bar{X}_\alpha(\bar{S}) = \int_{-\infty}^{+\infty} \frac{d\hat{\theta}}{2\pi} \hat{X}_\alpha(\hat{\theta}) \exp(-i\bar{S}\hat{\theta}), \tag{20}$$

and

$$\hat{X}_\alpha(\hat{\theta}) = \int_{-\infty}^{+\infty} d\bar{S}\, \bar{X}_\alpha(\bar{S}) \exp(i\bar{S}\hat{\theta}). \tag{21}$$

Furthermore the norm is preserved as

$$\int_{-1/2}^{+1/2} dS \int_{-\pi}^{+\pi} \frac{d\theta}{2\pi} |X_\alpha(S,\theta)|^2 = \int_{-\infty}^{+\infty} d\bar{S}\, |\bar{X}_\alpha(\bar{S})|^2 = \int_{-\infty}^{+\infty} \frac{d\hat{\theta}}{2\pi} |\hat{X}_\alpha(\hat{\theta})|^2. \tag{22}$$

Representations (14) and (18) associate a function of a single variable on the interval $-\infty$, $+\infty$ to the functions $X'_\alpha(S,\theta)$ and $X_\alpha(S,\theta)$. When applied to the dispersion equation (6) they reduce it into an ordinary differential equation which includes the periodicity conditions.

The presence of the trigonometric terms in θ in Eq.(6) makes the use of the extended poloidal representation more suitable. Then, for each α-component, we obtain

$$\frac{\partial}{\partial\hat{\theta}} [1 + s^2(\hat{\theta}-\alpha)^2] \frac{\partial}{\partial\hat{\theta}} \hat{X}_\alpha(\hat{\theta}) + \Gamma[\cos\hat{\theta} + s^2(\hat{\theta}-\alpha) \sin\hat{\theta}] X_\alpha(\hat{\theta})$$

$$+ \Omega_\alpha^2[1 + s^2(\hat{\theta}-\alpha)^2] \hat{X}_\alpha(\hat{\theta}) = 0. \qquad (23)$$

Equation (23) is to be solved for the eigenvalue Ω_α^2 with the condition that for $|\hat{\theta}| \to \infty$, $\hat{X}_\alpha(\hat{\theta})$ vanishes sufficiently rapidly* for the inverse transformation (15) to converge for all S.

The linear stability condition against these "ballooning" excitations is obtained by requiring that an initial perturbation does not grow (exponentially) i.e. by imposing that the eigenvalue Ω_α^2 is positive for all values of α. This leads to an instability threshold of the form $\Gamma = \Gamma_{th}(s)$, which expresses the balance between the combined destabilising effect of the pressure gradient and of the magnetic curvature, Γ, and the opposing effect of the magnetic shear s.

4. EXTENDED VARIABLE REPRESENTATIONS

The extended variable representations can be conveniently reinterpreted by adopting a group theoretical formalism. First we notice that the differential operators $\partial/\partial\theta + iS$ and $\partial/\partial S$ in (6) satisfy the commutation relation of the Heisenberg algebra [11]

$$[\frac{\partial}{\partial S}, \frac{\partial}{\partial\theta} + iS] = i. \qquad (24)$$

This is the same symplectic condition [12] that is satisfied by a coordinate and its conjugate momentum in phase space, which hints directly at an

*Weaker conditions can be adopted when this method is extended [10] to the description of excitations characterised by the presence of a boundary layer.

interpretation for the Fourier relationships (19) and (20). We denote the three-dimensional Lie algebra generated by the operators $\partial/\partial S$, $\partial/\partial\theta + iS$ and i by L.

Transformation (7), leads to an equivalent realisation of this algebra in terms of the operators $\partial/\partial S - i\theta$, $\partial/\partial\theta$ and i. We denote the algebra generated by the latter operators by L'. Then we find

$$[L, L'^*] = [L', L^*] = 0, \tag{25}$$

where a star denotes complex conjugation. We are thus led to consider the invariance properties of the dispersion equation (23) under the action of the group H generated by L'^*, i.e. by the operators $\partial/\partial S + i\theta$, $\partial/\partial\theta$ and $-i$. Its action on the mode amplitude $X(S,\theta)$ is defined by

$$\Lambda_1 X(S,\theta) = X(S + \lambda_1, \theta) \exp(i\lambda_1\theta) \tag{26a}$$

$$\Lambda_2 X(S,\theta) = X(S, \theta + \lambda_2) \tag{26b}$$

$$\Lambda_3 X(S,\theta) = X(S,\theta) \exp(-i\lambda_3) \tag{26c}$$

with Λ_1, Λ_2 and Λ_3 elements of the one dimensional subgroups generated by $\partial/\partial S + i\theta$, $\partial/\partial\theta$ and $-i$ respectively, and λ_1, λ_2, λ_3 the parameters that label the transformations.

Due to its θ-dependent terms, the dispersion equation (23) is invariant only under the subgroup of H defined by the condition that $\lambda_2/2\pi$ be an integer. In addition, for the mode amplitude to remain periodic in θ, we must further reduce the invariance group to integer values of λ_1. We are thus left with the discrete subgroup G of H generated by the transformations corresponding to $\theta \to \theta + 2\pi$ and to $S \to S + 1$.* Since G is Abelian, we can expand the mode amplitude $X(S,\theta)$ into common eigenfunctions of the transformations $X(S,\theta) \to X(S,\theta + 2\pi)$ and $X(S,\theta) \to X(S + 1,\theta) \exp(i\theta)$. Periodicity in θ requires that the eigenvalue under the first transformation be equal to one. The eigenvalue of the second transformation can be written as $\exp(i\alpha)$, where α coincides with the "radial" mode number in Eq.(9) and labels the irreducible unitary representations** [11] of G on the space of the functions $X_\alpha(S,\theta) \exp(iS\alpha)$. The transformation (7) leads to an equivalent representation of G on the functions $X'_\alpha(S,\theta) \exp(iS\alpha)$.

The extended variable representations (14) and (18) are thus simply a consequence of the restriction of the invariance group from H to G and arise from the decomposition of irreducible representations of H into irreducible representations of G. The possibility of employing two different extended variable representations stems from the fact that the group H is not Abelian. In the case of Eq.(18) the starting point is the set of the (periodic) eigenfunctions of the transformations (26b) generated by the operator $\partial/\partial\theta$. The restriction of the invariance group couples these eigenfunctions, leading to the summation over ℓ, but, since the subgroup G is Abelian, allows for the simultaneous implementation of the discrete invariance $S \to S + 1$, which introduces a relationship between the Fourier coefficients. In the case of Eq.(14) (rewritten in terms of the amplitude $X_\alpha(S,\theta)$ instead of $X'_\alpha(S,\theta)$) the starting point is the set of the eigenfunctions of the transformations (26a) generated by the operator $\partial/\partial S + i\theta$. The restricted invariance results now in the coupling between eigenfunctions with eigenvalues differing by 2π times an integer number and in the relationship (13) between their coefficients.

*Since these transformations commute we do not need the constant phase transformations of H.

**An ambiguity in denominations can arise here. We recall that the representation of the group action on a linear space, and the representation of a function of two variables in terms of a function of a single extended variable are different mathematical objects.

5. REPRESENTATION PRODUCT

The extended variable representation can be generalised to the product of the amplitudes of the excitations. Let $X_\alpha(S,\theta)\exp(iS\alpha)$ and $Y_{\alpha'}(S,\theta)\exp(iS\alpha')$ belong to the α and to the α' representations of G respectively. The product $Z_{\alpha''}(S,\theta) = X_\alpha(S,\theta)\ Y_{\alpha'}(S,\theta)$ satisfies the periodicity conditions

$$Z_{\alpha''}(S,\theta + 2\pi) = Z_{\alpha''}(S,\theta) \ , \tag{27}$$

and

$$Z_{\alpha''}(S + 1,\theta) = Z_{\alpha''}(S,\theta)\exp(-i2\theta) \ , \tag{28}$$

where $\alpha'' = \alpha + \alpha'$. Thus $Z_{\alpha''}(S,\theta)\exp(iS\alpha'') = X_\alpha(S,\theta)\ Y_{\alpha'}(S,\theta)\exp[iS(\alpha+\alpha')]$ belongs to the irreducible representation labelled by $\langle\alpha''\rangle = \alpha'' - 2\pi k$ (with $k = 0,\pm1$ such that $-\pi < \langle\alpha''\rangle \leq \pi$) of the group $G^{(2)}$ generated by the transformations $Z(S,\theta) \rightarrow Z(S,\theta + 2\pi)$ and $Z(S,\theta) \rightarrow Z(S + 1,\theta)\exp(i2\theta)$.

The group $G^{(2)}$ can be seen as a subgroup of the group G_2. This is isomorphic to G and its action is obtained from that of G by substituting 2S for S. The group $G^{(2)}$ is obtained from G_2 by restricting the transformations $2S \rightarrow 2S + p$ to even values of p.

The group G_2 acts on functions $K(2S,\theta)$ periodic in θ. These can be expanded into components belonging to irreducible representations of G_2 which can be expressed as functions of a single extended variable following a procedure analogous to that developed in the previous two sections. The new radial scale length is half that of the group G and corresponds to the distance between the mode rational surfaces of perturbations with toroidal number 2n, as consistent with the multiplication of the mode amplitudes in Eqs.(2) and (5).

The generalisation of the extended variable representation to quadratic terms in the excitation amplitude is then obtained by finding the relationship between the eigenfunctions of $G^{(2)}$ and those of G_2. Since $G^{(2)}$ is a subgroup of G_2, this relationship must involve the coupling of (two) eigenfunctions of G_2 belonging to different representations. It is easily seen that

$$Z_{\alpha''}(S,\theta)\ \exp(iS\alpha'') = K_\beta(2S,\theta)\ \exp(i2S\beta) + K_{\beta\pm\pi}(2S,\theta)\ \exp[i2S(\beta\pm\pi)], \quad (29)$$

where $K_\beta(2S,\theta + 2\pi) = K_\beta(2S,\theta)$, $K_\beta(2S + 1,\theta) = K_\beta(2S,\theta)\ \exp(-i\theta)$, $\beta = \alpha''/2 = (\alpha + \alpha')/2$, $-\pi < \beta \leq \pi$ and the sign in $(\beta \pm \pi)$ must be chosen such that $-\pi < \beta \pm \pi \leq + \pi$. Equation (29) is inverted in the form

$$K_\beta(2S,\theta) = \frac{1}{2}\ [Z_{\alpha''}(S,\theta) + Z_{\alpha''}(S + ½,\theta)\ \exp(i\theta)]\ , \quad (30)$$

and

$$K_{\beta\pm\pi}(2S,\theta) = \frac{1}{2}\ [Z_{\alpha''}(S,\theta) - Z_{\alpha''}(S + ½,\theta)\ \exp(i\theta)]\ \exp(\mp i2\pi S). \quad (31)$$

The explicit form of the generalised extended variable representations can be derived by inserting the extended variable representations of $X(S,\theta)$, of $Y(S,\theta)$ and of $K(2S,\theta)$ into Eq.(9).

In the following part of this section we list a few groups of relevant formulae:

i) Addition rule for the radial mode number

If $Z'(S,\theta) \equiv X'(S,\theta)\ Y'(S,\theta) = X(S,\theta)\ Y(S,\theta)\ \exp(i2S\theta)$, then

$$Z'(S,\theta) = \int_{-\pi}^{+\pi} \frac{d\gamma}{2\pi}\ K'_\gamma(2S,\theta)\ \exp(i2S\gamma)\ , \quad (32)$$

where $K'_\gamma(2S,\theta) = K_\gamma(2S,\theta)\ \exp(i2S\theta)$, $K'_\gamma(2S+1,\theta) = K'_\gamma(2S,\theta)$ and

$$K'_\gamma(2S,\theta) = \int_{-\pi}^{+\pi} \frac{d\alpha}{2\pi}\ [X'_{\gamma+\alpha}(S,\theta)\ Y'_{\gamma-\alpha}(S,\theta) + X'_{\gamma+\alpha}(S+½,\theta)\ Y'_{\gamma-\alpha}(S+½,\theta)]. \quad (33)$$

Here $X'_{\gamma+\alpha} = X'_{\langle\gamma+\alpha\rangle}\ \exp(-i2k\pi)$, $Y'_{\gamma-\alpha} = Y'_{\langle\gamma-\alpha\rangle}\ \exp(-i2k\pi)$, and $\langle\gamma\pm\alpha\rangle = \gamma \pm\alpha - 2k\pi$, with $k = 0,\pm1$, such that $-\pi < \langle\gamma\pm\alpha\rangle \le \pi$. The amplitudes $X'_{\langle\gamma+\alpha\rangle}(S,\theta)$ and $Y'_{\langle\gamma-\alpha\rangle}(S,\theta)$ are obtained from $X'(S,\theta)$ and $Y'(S,\theta)$ according to Eq.(9).

ii) Product of extended poloidal variable representations

If $Z'_{\alpha''}(S,\theta) = X'_\alpha(S,\theta)\ Y'_{\alpha'}(S,\theta)$, then

$$Z'_{\alpha''}(S,\theta) = \sum_{m=-\infty}^{+\infty} [\hat{K}_\beta(\hat{\theta}) + \hat{K}_{\beta\pm\pi}(\hat{\theta})\ \exp(\pm i2\pi S)]\ \exp(-\ i2\pi m2S)\ , \quad (34)$$

where $\beta = \alpha''/2$ and $\hat{K}_\beta(\hat{\theta})$, with $\hat{\theta} = \theta+2\pi m$, is the extended poloidal variable representation of $K'_\beta(2S,\theta) = K_\beta(2S,\theta)\ \exp(i2S\theta)$, with $K_\beta(2S,\theta)$ from Eq.(30). Equation (34) is inverted by

$$\hat{K}_\beta(\hat{\theta}) = \int_{-½}^{½} dS\ Z'_{\alpha''}(S,\theta)\ \exp(i2\pi m2S)\ , \quad (35)$$

and

$$\hat{K}_{\beta\pm\pi}(\hat{\theta}) = \int_{-½}^{½} dS\ Z'_{\alpha''}(S,\theta)\ \exp[i2\pi\ (m \mp \tfrac{1}{2})\ 2S]\ . \quad (36)$$

The relationship between the extended poloidal variable representation of $Z'_{\alpha''}(S,\theta)$ and those of $X'_\alpha(S,\theta)$ and $Y'_{\alpha'}(S,\theta)$ is expressed by the convolution product

$$\hat{K}_\beta(\hat{\theta}) = \sum_{-\infty p}^{+\infty} \hat{X}_\alpha(\hat{\theta}^+)\, \hat{Y}_{\alpha'}(\hat{\theta}^-) \ , \qquad (37)$$

$$\hat{K}_{\beta\pm\pi}(\hat{\theta}) = \sum_{-\infty p}^{+\infty} \hat{X}_\alpha(\hat{\theta}^+ \mp 2\pi)\, Y_\alpha(\hat{\theta}^-) \ , \qquad (38)$$

where $\hat{\theta} = \theta + 2\pi m$, $\hat{\theta}^+ = \hat{\theta} + 2\pi p = \theta + 2\pi(m + p)$, and $\hat{\theta}^- = \hat{\theta} - 2\pi p = \theta + 2\pi(m - p)$.

iii) Product of extended radial variable representations

If $Z_{\alpha''}(S,\theta) = X_\alpha(S,\theta)\, Y_{\alpha'}(S,\theta)$, then

$$Z_{\alpha''}(S,\theta) = \sum_{-\infty \ell}^{+\infty} [\bar{K}_\beta(\overline{2S}) + (-)^\ell\, \bar{K}_{\beta\pm\pi}(\overline{2S}) \exp(\pm i\overline{2S}\pi)] \exp(i\ell\theta) \ , \qquad (39)$$

where $\beta = \alpha''/2$ and $\bar{K}(\overline{2S})$, with $\overline{2S} = 2S + \ell$, is the extended radial variable representation of $K_\beta(2S,\theta)$ from Eq.(30). Equation (39) is inverted by

$$\bar{K}_\beta(\overline{2S}) = \frac{1}{2} \int_{-\pi}^{+\pi} \frac{d\theta}{2\pi} [Z_{\alpha''}(S,\theta) + Z_{\alpha''}(S + ½,\theta) \exp(i\theta)] \exp(- i\ell\theta), \qquad (40)$$

and

$$\bar{K}_{\beta\pm\pi}(\overline{2S}) = \frac{1}{2} \int_{-\pi}^{+\pi} \frac{d\theta}{2\pi} [Z_{\alpha''}(S,\theta) - Z_{\alpha''}(S+½,\theta) \exp(i\theta)] \exp(\mp i2\pi S - i\ell\theta). \qquad (41)$$

The relationship between the extended radial variable representation of

$Z_{\alpha''}(S,\theta)$ and those of $X_\alpha(S,\theta)$ and $Y_{\alpha'}(S,\theta)$ is expressed by the convolution product

$$\bar{K}_\beta(\overline{2S}) = \frac{1}{2} \sum_{p=-\infty}^{+\infty} [\bar{X}_\alpha(\bar{S}^+)\ \bar{Y}_{\alpha'}(\bar{S}^-) + \bar{X}_\alpha(\bar{S}^+ + \tfrac{1}{2})\ \bar{Y}_{\alpha'}(\bar{S}^- - \tfrac{1}{2})] \qquad (42)$$

$$\bar{K}_{\beta\pm\pi}(\overline{2S}) = \frac{1}{2} \sum_{p=-\infty}^{+\infty} [\bar{X}_\alpha(\bar{S}^+)\ \bar{Y}_{\alpha'}(\bar{S}^-)$$

$$- \bar{X}_\alpha(\bar{S}^+ + \tfrac{1}{2})\ \bar{Y}_{\alpha'}(\bar{S}^- - \tfrac{1}{2})] \exp(\mp i2\pi S) \qquad (43)$$

where $\overline{2S} = 2S + \ell$, $\bar{S}^+ = S + \ell/2 + p$ and $\bar{S}^- = S + \ell/2 - p$.

The functions $\bar{K}_\beta(\overline{2S})$ and $K_\beta(\theta)$, and $\bar{K}_{\beta\pm\pi}(\overline{2S})$ and $K_{\beta\pm\pi}(\theta)$, are related by a Fourier transformation analogous to that in Eqs.(10) and (21).

CONCLUSIONS

We have shown that the extended variable representation, employed in the plasma physics literature for the description of short wavelength excitations in a toroidal axisymmetric plasma configuration, is related to the irreducible representation of an Abelian subgroup G of the Heisenberg group H. The Lie algebra of H commutes with the differential operators in the dispersion equation of the excitations. The subgroup G corresponds to discrete transformations in the poloidal and in the radial directions. The reduced invariance from H to G arises from the poloidal modulation of the coefficients of the dispersion equation and from the requirement that the excitation amplitude be a single valued function of the poloidal angle.

The action of G can be suitably extended so as to include products of the amplitudes of excitations with equal (or multiple) toroidal mode numbers. The expansion of these products into irreducible representations allows us to generalise the formalism of the extended variable representation to the description of the non linear evolution of the excitations.

In this paper we have explicitly developed this formalism in the case of quadratic terms. It can be extended to the general case along the same lines. In a future paper these results will be applied to the solution of the non linear dynamical equations for the magnetic excitations described in the introduction.

ACKNOWLEDGEMENT

This paper was written on the occasion of the 70th birthday of Prof. L.A. Radicati di Brozolo whose teaching first, inspiration and guidance afterwards, and criticism always, I valued greatly over the years. Indeed it was with some guilt that I realised, while preparing this paper, that a formalism I had contributed to develop some ten years ago, could have been formulated in a more transparent, and now I see, more complete way, had I used some of the mathematical ideas he introduced me to.

REFERENCES

[1] S. Chandrasekhar. Hydrodynamic and Hydromagnetic Stability, Clarendon Press, Oxford, 1961.

[2] C.M. Bender and S.A. Orszag, Advanced Mathematical Methods for Scientists and Engineers, McGraw-Hill Book Company, New York, 1978,.

[3] B. Coppi, Phys. Rev. Lett. 39, 939 (1977).

[4] B. Coppi, J. Filreis and F. Pegoraro, Ann. Phys. 121, 1 (1979).

[5] F. Pegoraro and T.J. Schep, presented at the International Atomic Energy Agency Advisory Group meeting on Thermal Conduction and Transport, Kiev (1977), also in Plasma Physics and Controlled Nuclear Fusion Research, Innsbruch 1978 (International Atomic Energy Agency, Vienna 1978), Vol.1, p.507.

[6] J.W. Connor, R.J. Hastie and J.B. Taylor, Phys. Rev. Lett. 40, 396 (1978) and in Proc. R. Soc. London Ser. A365, 1 (1979).

[7] Y.C. Lee and J.W. Van Dam in Proceedings of the Finite Beta Theory Workshop, edited by B. Coppi and W. Sadowski (U.S. Department of Energy, Washington, D.C. 1977) p.93.

[8] A.H. Glasser, ibid. p.55.

[9] F. Pegoraro and T.J. Schep, Phys. of Fluids 24, 478 (1981).

[10] F. Pegoraro and T.J. Schep, Plasma Phys. Controlled Fusion 28, 697 (1986).

[11] R. Hermann, Lie Groups for Physicists, W.A. Benjamin, Inc. New York (1966).

[12] H. Goldstein, Classical Mechanics, 2nd Ed., Addison-Wesley Publ., Reading, Massachusetts (1980).

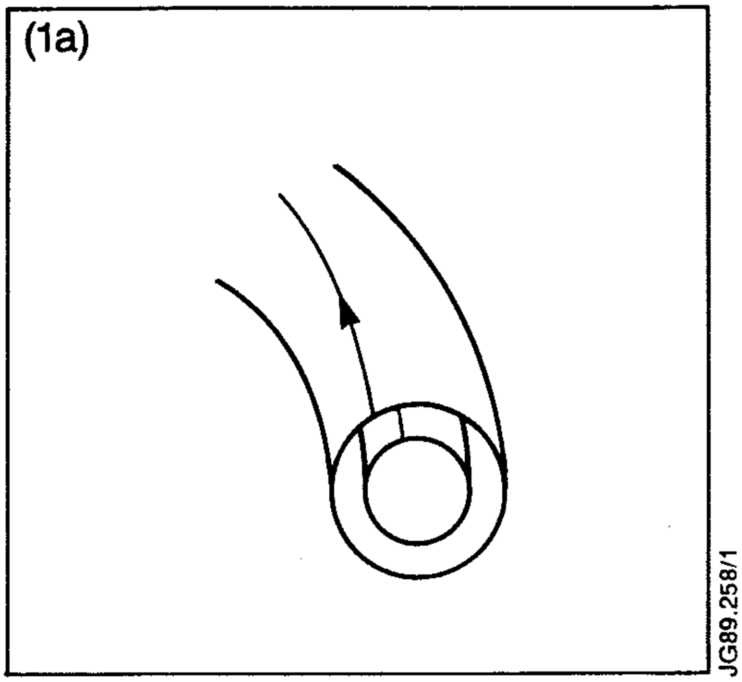

Fig. 1a Section of a toroidal magnetic equilibrium configuration. Two magnetic surfaces and two magnetic field lines (thin lines) are displayed.

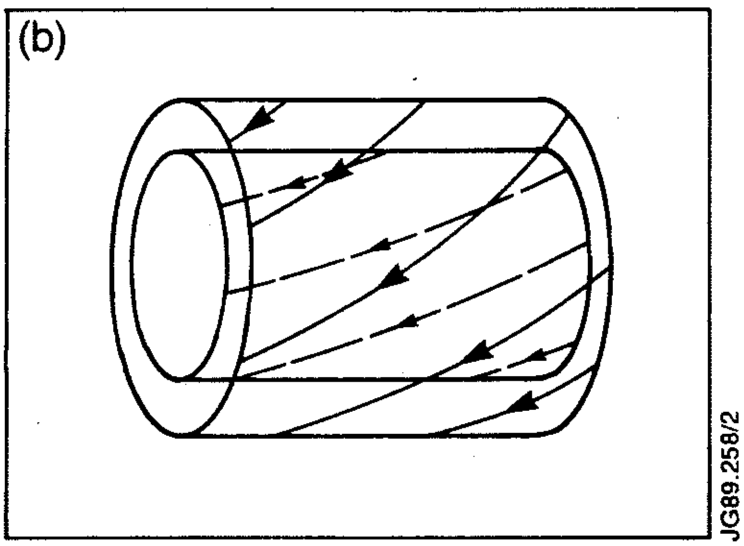

Fig. 1b Section of two (straightened) magnetic surfaces displaying the change in the pitch of the magnetic field lines (thin lines).

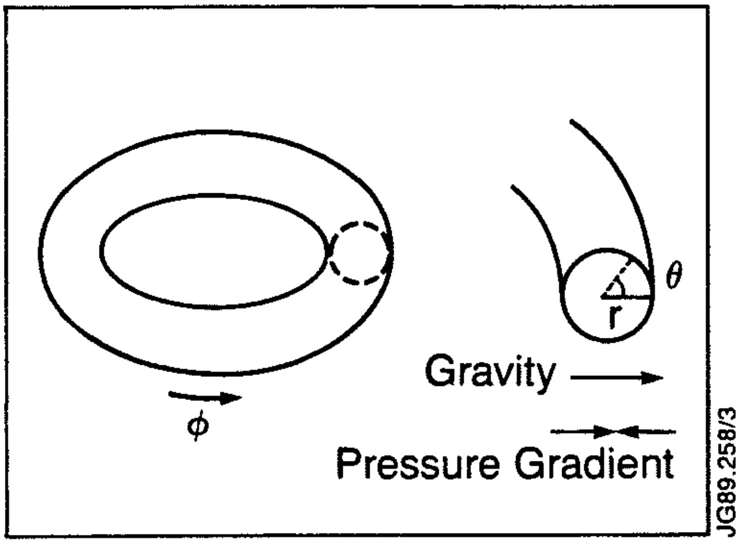

Fig.2 Toroidal and poloidal angles ϕ and θ and radial coordinate r. Relative orientation of the effective gravity and of the pressure gradient at the outside ($\theta = 0$, right) and at the inside ($\theta = \pi$, left) of the torus.