JOINT EUROPEAN TORUS

JET–P(87)31

M.F.F. Nave and J. Wesson

# Stability of the Ideal m = 1 mode in a Tokamak

# Stability of the Ideal m = 1 mode in a Tokamak

M.F.F. Nave and J. Wesson

*JET-Joint Undertaking, Culham Science Centre, OX14 3DB, Abingdon, UK*

Preprint of Paper to be submitted for publication in
Nuclear Fusion

STABILITY OF THE IDEAL m = 1 MODE IN A TOKAMAK

M.F.F. Nave and J.A. Wesson

## 1. Introduction

The observation of a very fast collapse phase in JET sawtooth oscillations [1] has led us to reconsider the stability of the ideal m = 1 mode. With a resonant surface in the plasma this mode is unstable in cylindrical geometry [2] but can be stable in toroidal geometry [3].

The condition for toroidal stability was calculated by Bussac et al. [4] in terms of the parameter

$$\beta_p = \frac{2\mu_o R^2}{r_1{}^4 B_\phi{}^2}\int_o^{r_1} (-dp/dr)\, r^2\, dr \tag{1}$$

where R is the major radius of the plasma, $B_\phi$ the toroidal magnetic field, $r_1$ the radius of the q = 1 surface and p is the plasma pressure. For current profiles of the form

$$j = j_o[1 - (r/a)^2]^\nu \tag{2}$$

they found that the m = 1 mode is stable for $\beta_p < \beta_{pc}$ where, for values of $r_1 \lesssim 0.3$ and $0 < \nu \lesssim 4$, $\beta_{pc}$ lies in the range 0.2 to 0.3. Sawtooth oscillations are observed in JET for values of $\beta_p$ much smaller than the critical value.

However there are theoretical arguments [5] and experimental results [6] which suggest that the profile of the safety factor q might be flattened inside the q = 1 surface.

The calculation described below uses the procedure given by Bussac et al. [4] and recalculates $\beta_{pc}$ for current profiles which, outside the q = 1 surface, take the form given by equation (2) but are flattened to a chosen degree inside this surface. The critical value of $\beta_p$ is determined as a function of the safety factor on axis, $q_o$, and the radius, $r_1$, of the q = 1 surface. It is found that the value of $\beta_p$ required to produce instability is much reduced and for $q_o < 1$ it is found that the critical $\beta_p \to 0$ as $1-q_o \to 0$.

These results lead to the possibility that the ideal m = 1 mode could be the cause of the rapid sawtooth collapse. The detailed implications of this result for the theory of sawtooth oscillations are explored elsewhere [5,7]. Here we describe the linear stability calculation.

## 2. The Cylinder

Although the tokamak problem has features which are fundamentally different from the cylindrical analogue it is useful, both as an introduction and to give a perspective, to recall the theory of the m = 1 mode in a cylinder.

The potential energy associated with an internal m = 1 radial displacement $\xi$ in a cylindrical plasma of length $2\pi R$, with a radius a such that the aspect ratio $\varepsilon = a/R \ll 1$, is given by

$$\delta W = 2\pi^2 R \int_0^a \left[\frac{B_\theta^2}{2\mu_0}(1-q)^2 \left(\frac{d\xi}{dr}\right)^2 + \frac{1}{R^2}\left\{rp' - \frac{B_\theta^2}{2\mu_0}(1-q)(1+3q)\right\}\xi^2\right] rdr$$

$$\sim \frac{2\pi^2 B_\phi^2}{\mu_0 R}\int_0^a \left[O(\varepsilon^2)\left(r\frac{d\xi}{dr}\right)^2 + O(\varepsilon^4)\,\xi^2\right] rdr$$

where $B_\theta$ and $B_\phi$ are the azimuthal and axial magnetic fields, p is the plasma pressure and $q = rB_\phi/RB_\theta$. If there is a q = 1 surface in the plasma, the $\varepsilon^2$ leading order term is minimised to zero for the displacement shown in Fig. 1. $\delta W$ then becomes

$$\delta W = \frac{2\pi^2\xi_0^2}{R}\int_0^{r_1}\left[rp' - \frac{B_\theta^2}{2\mu_0}(1-q)(1+3q)\right] rdr. \tag{3}$$

which gives instability for the conventional case of $p' < 0$ and $q < 1$ in the region $r < r_1$.

## 3. Summary of Large Aspect-Ratio Calculation of $\delta W$

The toroidal calculation is unfortunately very complicated. The analysis was first carried out by Bussac et al. [4], is given in a detailed form by Connor and Hastie [8].

As in the cylindrical case the leading order, $\varepsilon^2$, terms in $\delta W$ are minimised to zero by the function shown in Fig. (1) but there are now additional toroidal terms in $O(\varepsilon^4)$ which invalidate equation (3).

Physically there are three factors determining stability. The first is the normally destabilising effect of the pressure gradient which enters through $\beta_p$ defined in equation (1). The second is the stabilising effect of shear arising from the difference between q and unity in the region $r < r_1$ and appearing as

$$s = \frac{1}{r_1{}^4} \int_0^{r_1} \left(\frac{1}{q} - 1\right)^2 r^3 \, dr. \tag{4}$$

The third contribution comes from the interaction of the cosθ dependence of the m = 1 part of the displacement to the cosθ variation of the equilibrium arising from the toroidal geometry. This leads to a cos 2θ coupling to the m = 2 part of the displacement.

The resulting potential energy takes the form

$$\delta W = 2\pi^2 R \, \frac{B_\phi{}^2}{\mu_o} \, \xi_o^2 \left(\frac{r_1}{R}\right)^4 (\alpha_o + \alpha_1 \beta_p + \alpha_2 \beta_p{}^2)/\alpha_3 \tag{5}$$

where the α's represent the contributions described above in the form

$$\alpha_o = -4s^2(c+3)(b+3) - 6s(b-1)(c+3) + 8s(b-c) + \frac{9}{4}(b-1)(1-c)$$

$$\alpha_1 = -8s(c+3)(b+3) - 6(b-1)(c+3)$$

$$\alpha_2 = -4(c+3)(b+3)$$

$$\alpha_3 = 16(b-c)$$

The, as yet, undefined quantities b and c arise from the m = 2 coupling. They are determined from the solutions of the Euler equation

$$\frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q} - \frac{1}{m}\right)^2 \frac{d\xi^{(2)}}{dr}\right] - (m^2-1)\left(\frac{1}{q} - \frac{1}{m}\right)^2 \xi^{(2)} = 0 \text{ with } m = 2. \tag{6}$$

The solution of equation (6) in the region $0 < r < r_1$ satisfying $\xi^{(2)} = 0$ at $r = 0$ allows the determination of

$$b = \left( \frac{r}{\xi^{(2)}} \frac{d\xi^{(2)}}{dr} \right)_{r = r_1^-} \qquad (7)$$

and the solution in the region $r > r_1$ satisfying $\xi^{(2)} = 0$ at the radius of the $q = 2$ surface ($r = r_2$) if this surface lies in the plasma, or otherwise $\xi^{(2)} = 0$ at $r = a$, determines

$$c = \left( \frac{r}{\xi^{(2)}} \frac{d\xi^{(2)}}{dr} \right)_{r = r_1^+} . \qquad (8)$$

Thus for a given $p(r)$ and $q(r)$ the quantities $\beta_p$, s, b and c are determined by equations (1), (4), (7) and (8) and stability is determined by the resulting sign of $\delta W$ as given by equation (5).

## 4. Stability Calculation for Flattened Profiles

The case of current profiles $j = j_o(1 - r^2/a^2)^\nu$ was treated numerically by Bussac et al. [4]. In the limit $r_1 \ll a$ they obtained the analytic result that the m = 1 mode is stable if

$$\beta_p < \left( \frac{13}{144} \right)^{1/2}$$

that is $\beta_p \lesssim 0.3$. We shall now give the calculation for the case of flattened current profiles.

The j and q profiles used are illustrated in Fig. 2. The j profiles have the form

$$j = j_c(1 - r^2/a^2)^\nu \qquad r > r_1 \qquad (9)$$

$$j = \frac{2B_\phi}{Rq_o} \left[1 - 2\,(1 - q_o)\, \frac{r^2}{r_1^2}\right] \qquad r < r_1 \qquad (10)$$

where

$$j_c = \frac{2B_\phi}{R} \frac{(2 - 1/q_o)}{(1 - r_1{}^2/a^2)^\nu}$$

and the corresponding q profiles are

$$q = \frac{r^2/a^2}{\frac{r_1{}^2}{a^2} - \frac{2-1/q_o}{\nu+1}\left(\frac{(1-r^2/a^2)^{\nu+1}}{(1-r_1{}^2/a^2)^{\nu+1}} - 1\right)(1 - r_1^2/a^2)} \qquad r > r_1 \qquad (11)$$

$$q = \frac{q_o}{1 - (1 - q_o)(r^2/r_1{}^2)} \qquad r < r_1 \qquad (12)$$

We note that whereas in the case of the unflattened profiles $q_o$ is a function of $r_1$ for a given $\nu$, with the present profiles $q_o$ and $r_1$ are independent.

We now proceed to calculate the coefficients b and c for the case $r_1 << a$. It is convenient to transform equation (6) using

$$\psi = B_\theta(m-q)\ \xi^{(2)} \qquad (13)$$

to give

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) - \frac{4}{r^2}\psi - \frac{dj/dr}{B_\theta(1-q/2)}\psi = 0 \qquad (14)$$

Expanding $\psi$ in orders of $(r_1/a)^2$, $\psi = \psi_o + \psi_1 \ldots$, $\psi_o$ is determined by

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi_o}{dr}\right) - \frac{4}{r^2}\psi_o = 0$$

and the solution may be written

$$\psi_o = (r/r_1)^2 \qquad r < r_1 \qquad (15)$$

and

$$\psi_o = (r_1/r)^2 \qquad r > r_1 \qquad (16)$$

The next order term $\psi_1$ satisfies

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi_1}{dr}\right) - \frac{4}{r^2}\psi_1 = V \qquad (17)$$

where

$$V = \frac{dj/dr}{B_\theta(1-q/2)}\psi_o$$

and the solution of equation (17) is

$$\psi_1 = \frac{r^2}{4} \int \frac{V}{r}\,dr - \frac{1}{4r^2} \int Vr^3\,dr \qquad (18)$$

Using equations (9)-(12), (15) and (16) the appropriate limiting expressions for V are

$$V = -\,16(1-q_o)\,\frac{r^2}{r_1{}^4} \qquad r < r_1$$

and

$$V = -\,\frac{8\nu r_1{}^2}{a^2 r^2} \qquad r > r_1 .$$

The corresponding solutions of equation (17) given by equation (18) are

$$\psi_1 = -\,\frac{4}{3}\,(1-q_o)\left(\frac{r}{r_1}\right)^4 \qquad r < r_1$$

and

$$\psi_1 = 2\nu\left(\frac{r_1}{a}\right)^2 \qquad r > r_1 .$$

It is now possible to determine the required coefficients b and c defined by equations (7) and (8) from the relation

$$\frac{r}{\xi^{(2)}}\,\frac{d\xi^{(2)}}{dr} = \frac{r\psi'}{\psi} + 2q'r - 1 \qquad r = r_1$$

derived from equation (13). Thus using

$$\frac{\psi'}{\psi} = \frac{\psi_o{}' + \psi_1{}'}{\psi_o + \psi_1}$$

we obtain

$$b = 1 + \frac{4}{3}\,(1 - q_o) \qquad (19)$$

and

$$c = -\,3 + 4\nu\left(\frac{r_1}{a}\right)^2 + 4(1-q_o). \qquad (20)$$

In the same approximation equation (4) gives the remaining unknown

$$s = \frac{1}{6}(1 - q_o) \qquad (21)$$

and allows the determination of $\delta W$ from equation (5). The coefficients appearing in this equation are found to be

$$\alpha_o = \frac{52}{3}(1-q_o)$$

$$\alpha_1 = 0(\frac{r_1}{a})^4$$

$$\alpha_2 = - 64[1-q_o + \nu(\frac{r_1}{a})^2]$$

$$\alpha_3 = 64$$

The condition for instability, $\delta W < 0$, may therefore be written

$$\beta_p^2 > -\frac{\alpha_o}{\alpha_2}$$

that is the m = 1 mode is unstable if

$$\beta_p > \left(\frac{13}{144}\frac{3\Delta q}{\Delta q + 2\Delta q^*}\right)^{\frac{1}{2}} \qquad (q_o < 1) \qquad (22)$$

where

$$\Delta q = 1 - q_o$$

and

$$\Delta q^* = \frac{\nu}{2}(\frac{r_1}{a})^2,$$

$\Delta q^*$ being the value of $1 - q_o$ which would exist with no flattening of the profiles.

For the case where $q_o > 1$, there still being a q = 1 surface at $r = r_1$, $\delta W < 0$ and the condition for instability of the m = 1 mode is then

$$\beta_p > 0 \qquad (q_o > 1). \qquad (23)$$

In the limiting case of no flattening, that is $\Delta q = \Delta q^*$, inequality (22) takes the form given by Bussac et al. [4]

$$\beta_p > \left(\frac{13}{144}\right)^{\frac{1}{2}}$$

Graphs of the results for several values of $\nu$ for $r_1/a = 0.3$ are shown in Fig. 3.

## Discussion

From equations (22) and (23) and Fig.3 it is seen that whereas conventional q-profiles require a substantial $\beta_p$ in order to be unstable to the ideal m = 1 mode, a flattened q-profile allows instability for a low and even zero $\beta_p$. The significance of this is that the ideal m = 1 mode, which had previously been ruled out as the cause of the sawtooth collapse, can now be considered as a candidate for this role. It would also appear to offer an explanation of the short collapse time observed in some experiments.

In the calculation described here it was assumed that $1 - q_o$ was <u>not</u> as small as $r_1/R$. This allowed use of the approximate leading order eigenfunction $\xi$ = constant for $r < r_1$ and $\xi = 0$ for $r > r_1$. In order to use the derived results for flatter profiles with $1 - q_o \lesssim r_1/R$, the $\xi$ functions used must be regarded as trial functions. It is then known that the true eigenfunctions would give a lower value of $\delta W$. Thus the derived stability criteria can be regarded as necessary conditions for stability and the plasma will actually be more unstable than the conditions imply.

The significance of these results is more fully explored in references [5] and [7] where the relation of the results to experiment is discussed and the implications of the modification of the eigenfunction for small $1-q_o$ are described.

## References

[1] CAMPBELL, D.J., GILL, R.D., GOWERS, C.W., WESSON, J.A., BARTLETT, D.V., BEST, C.H., CODA, S., COSTLEY, A.E., EDWARDS, A., KISSEL, S.E., NIESTADT, R.M., PIEKAAR, H.W., PRENTICE, R., ROSS, R.T. and TUBBING, B.J.D. Nuclear Fusion Letters, 26 (1986).

[2] SHAFRANOV, V.D., Sov. Phys.-Tech. Phys. 15 (1970) 175.

[3] SYKES, A. and WESSON, J.A. Nucl. Fusion 14 (1974) 645.

[4] BUSSAC, M.N., PELLAT, R., EDERY, D. and SOULE, J.L. Phys. Rev. Letts. 35 (1975) 1638.

[5] WESSON, J.A. Plasma Physics and Controlled Fusion, 28 (1986) 243.

[6] EDWARDS, A.W., CAMPBELL, D.J., ENGELHARDT, W.W., FAHRBACH, H.-U., GILL, R.D., GRANETZ, R.S., TUBBING, B.J.D., WELLER, A., WESSON, J.A. and ZASCHE, D. Phys. Rev. Letts. 57 (1986) 210.

[7] WESSON, J.A., KIRBY, P. and NAVE, M.F.F. Proc. of the Eleventh International Conference on Plasma Physics and Controlled Nuclear Fusion Research, paper IAEA-CN-47/E-I-1-1 (1986).

[8] CONNOR, J.W. and HASTIE, R.J. Culham Lab. Rep. CLM-M106 (1985).

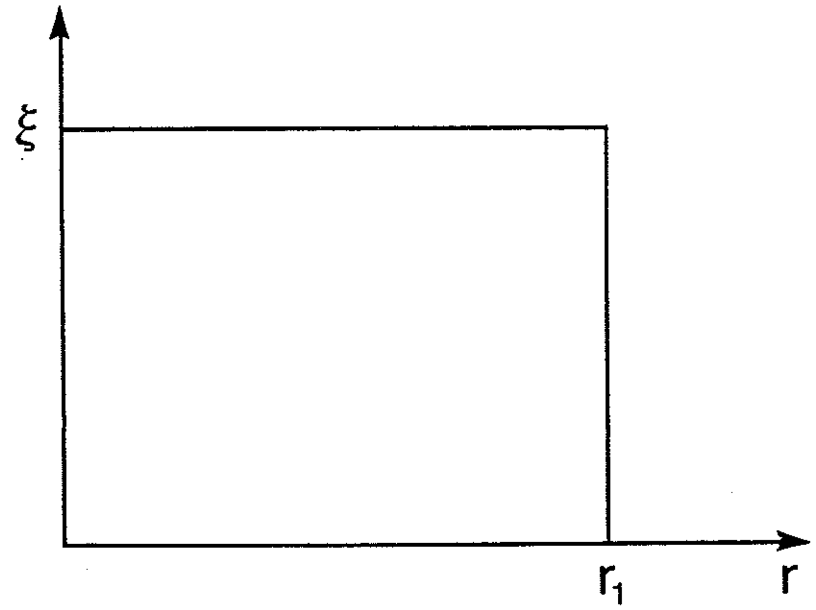


Figure 1  ξ(r) which minimises leading order contribution in δW to zero.

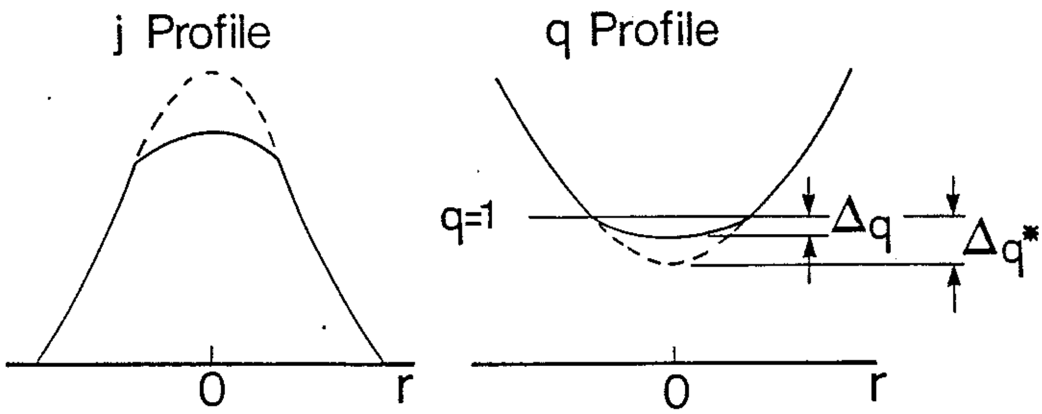


Figure 2  j(r) and q(r) profiles showing unflattened profile with 1 - q(0) = Δq* and flattened q profile with 1 - q(0) = Δq.

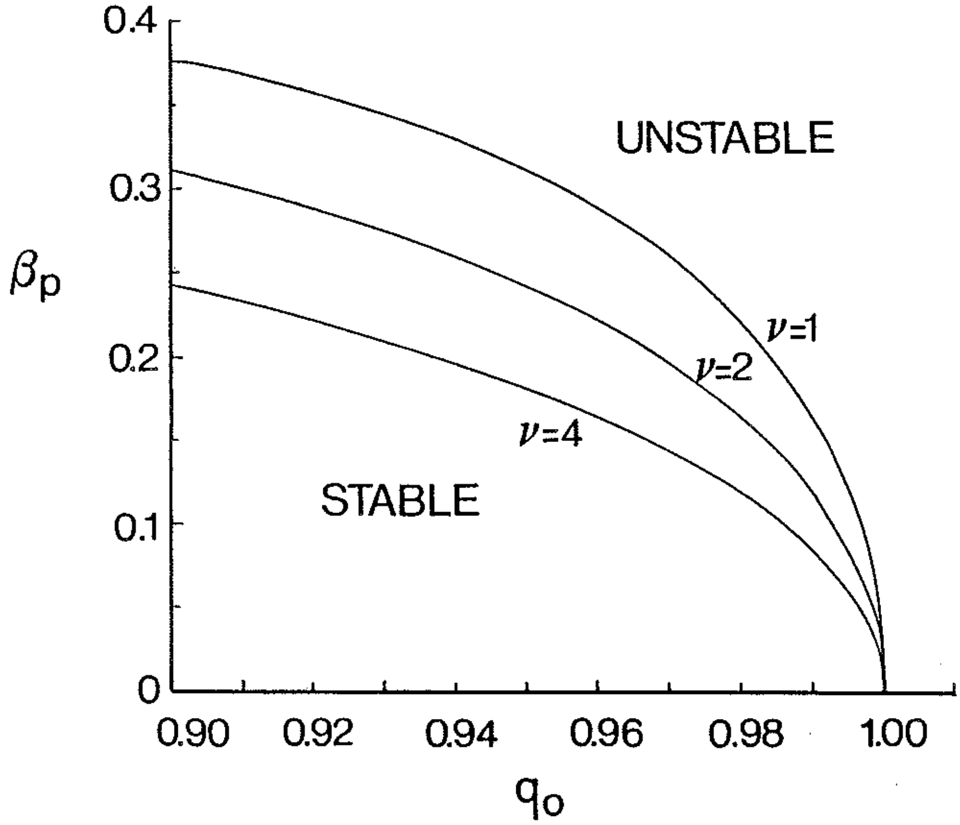


Figure 3 Graphs giving the initial values of $\beta_p$ as a function of q(0) for $\nu$ = 1,2 and 4. The m = 1 mode is unstable above the curves.