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ABSTRACT.

The problem of resonant ion tail formation during ICRF heating of tokamak plasmas is considered. Using the fact that the RF diffusion and Fast-ion Fokker-Planck equations admit simple solution procedures, a method is given for the determination of the solution of the time dependent ICRF Fokker-Planck equation in the small Larmor-radius approximation ($\rho.k_\perp \ll 1$). The solution procedure leads to a Volterra integral equation for the determination of the effect of the Coulomb scattering processes on tail formation. Then using an iterative-perturbation technique, the problem of tail formation during heating at the fundamental ($n = 1$) and second harmonic ($n = 2$) frequencies is addressed.

I INTRODUCTION

In recent years there has been considerable interest in the problem of the determination of the time dependent behaviour of resonant ions in ICRF heated tokamak plasmas, especially since a detailed knowledge of the distribution function is essential to the theoretical and experimental study of the heating process. As is well known, the collisionless interaction of the resonant ions with the cyclotron wave fields leads to anisotropic diffusion in velocity space, and the evolution of the ion distribution function due to the collective effects of particle-wave interaction and Coulomb scattering is determined by a quasi-linear Fokker-Planck equation [1,2]. To date, the problem of finding useful solution procedures has proved a formidable task. Although numerical solutions to the complete system have been obtained [3], analytical model treatments have only been partially successful. Stix (1975), using a small Larmor-radius approximation ($\rho_i k_\perp \ll 1$), derived a lowest order steady-state solution to the equation describing minority ion heating at the fundamental frequency. From this solution the principal features of the ion distribution function were identified. Further, perturbation solutions in the weak RF interaction limit have been obtained and used to investigate tail distortion [4], and ICRF beam hybrid heating schemes [5]. The "collisionless" RF-diffusion equation has also received considerable attention in the literature, particularly in connection with enhanced resonant ion loss processes [6,7]. These authors justified the use of the RF-diffusion equation in their investigations of particle losses by assuming that the early time development of the distribution function is dominated by particle wave interactions. However, Coulomb scattering processes are an important component in the evolution of the resonant ions and even a small degree of scattering can be expected to significantly modify the "collisionless" distribution function. Accordingly, a consistent treatment would appear to be necessary, even for the initial phase of the evolution.

In the treatment presented in this work, a mathematical technique based upon the observation that the RF-diffusion and "Fast Ion" Fokker-Planck equation separately admit simple solution procedures, is used to determine the complete solution of the problem. Consideration is restricted to the development of a solution procedure for the determination of the evolution of the resonant ions in the tail of the distribution function. For this class of particles, the Coulomb scattering proceses are primarily collisions with the plasma thermal ions and electrons, and the relatively infrequent energetic ion-ion encounters can be negelcted. Further, it is assumed that the resonant ions in the bulk of the distribution do not depart appreciably from thermal equilibrium with the non-resonant plasma species, and in accordance with [2], orbit effects are neglected, and a small Larmor-radius approximation is used. The loss in generality implicit in the use of the ($\rho_i k_\perp << 1$) approximation, although undesirable, does lead to a significant gain in tractability.

The organisation of this work is a follows:

In section II, details of a model Fokker-Planck equation which determines the evolution of the resonant ions in the tail of the distribution are presented. The equation includes the dominant Coulomb scattering processes of dynamical friction, pitch angle scattering, and energy diffusion, with a simplified quasi-linear RF-operator.

Then in section III, the aforementioned technique of interaction separation is used to reformulate the problem in terms of an integral equation of the Volterra type. In this formulation it is analytically convenient, for reasons which will become apparent, to derive the source or driving term for the system from the solution to the RF-diffusion, or collisionless Fokker-Planck equation, and determine the influence of the Coulomb scattering processes on tail formation from the solution of the integral equation. Following an expansion of the solution in a series of Legendre polynomials the system is reduced to an integral equation in the time domain, which is solved using an interactive-perturbation method.

Finally, in section IV, as an application of the solution procedure, the problem of tail formation during ICRF heating at the fundamental and second harmonic frequencies is addressed. The validity of the "collisionless" model is also discussed.

II DERIVATION OF THE FOKKER-PLANCK EQUATION

The Fokker-Planck equation describing the evolution of resonant ions subject to the collective effects of Coulomb scattering and particle-wave interactions can be formally written as [2]:

$$\frac{\partial f}{\partial t} = C(f) + Q(f) \qquad (1)$$

where f is the resonant ion distribution function, C(f) is the Coulomb scattering operator, and Q(f) is the quasi-linear operator describing the collisionless interaction of the ions with the RF wave fields.

For resonant ions in the tail of the distribution function, the Coulomb operator simplifies considerably and in terms of the normalised velocity $v(= v/v_{th})$ takes the form [8]:

$$C(f) = \frac{1}{\tau_s} \; D_c \frac{1}{v^2} \frac{\partial^2}{\partial v^2} \left\{ v^2 \left(1 + \frac{\sigma}{v^3}\right) f \right\} + \frac{1}{v^2} \frac{\partial}{\partial v} \left\{ (v^3 + v^3{}_c)\, f \right\}$$

$$+ \beta \left\{\frac{v_c}{v}\right\}^3 \frac{\partial}{\partial \xi} \; (1 - \zeta^2) \frac{\partial f}{\partial \zeta} \qquad ,$$

where

$$\tau_s = \frac{3 m_e v^3{}_e \, m}{16\pi^{1/2} \, e^2 \, Z n_e \, \ell n \, \Lambda}$$

$$D_c = \frac{Te}{2T}$$

$$\sigma = \frac{3\pi^{1/2} \, v_e}{2 \, n_e v^3_{th}} \sum \frac{n_j Z_j^2 \, T_j}{m_j}$$

$$v_c^3 = \frac{3\pi^{1/2} \, v_e^3}{4 n_e v^3_{th}} \sum \frac{n_j Z_j^2}{m_j}$$

$$\beta = \frac{1}{2m} \frac{\sum n_j Z_j^2}{\sum \frac{n_j Z_j}{m_j}}$$

$$v_e = \sqrt{2T_e/m_e} \,,$$

$$v_{th} = \sqrt{2T/m},$$

$$1 << v << v_e/v_{th},$$

$\zeta(= v_{\parallel}/v)$ is the pitch variable, $v_{\parallel}$ is the projection of the normalised velocity along the magnetic field line; m, T, Z are the resonant ion mass, temperature and charge; m_j, n_j, T_j, Z_j, are the background plasma ion species (j) mass, density, temperature, and charge; m_e, n_e, T_e, are the electron mass, density and temperature respectively, and finally $\ell n\Lambda$ is the Coulomb logarithm.

According to [2], the velocity space diffusion due to the resonant interaction of the ions with the wave fields is well described by the quasi linear theory of Kennel and Engelmann, and for cyclotron absorption of the fast wave energy at the n^{th} harmonic their operator reads

$$Q_n(f) = \frac{\pi Z^2 e^2}{2m^2 k_{\parallel}} \left|E_+\right|^2 \frac{1}{v_{\perp}} \frac{\partial}{\partial v_{\perp}} v_{\perp}^2 \left|J_{n-1} \frac{k_{\perp} v_{\perp}}{\omega_{ci}}\right|^2$$

$$\cdot\, \delta\left\{v_{\parallel} - \frac{\omega - n\omega_{ci}}{k_{\parallel}}\right\} \frac{1}{v_{\perp}} \frac{\partial f}{\partial v_{\perp}} \,, \qquad (2)$$

where E_+ is the component of the wave electrical field in phase with the ion gyro-rotation; $k_{\parallel}$, $k_{\perp}$ are the parallel and the perpendicular components of the wave vector; $v_{\perp}$ is the perpendicular component of the ion velocity; and ω, ω_{ci} are the wave and ion gyro-angular frequencies respectively.

Following the usual procedure of averaging over a magnetic surface to eliminate the delta function in Eq.(2) we obtain for wave propagation across the magnetic field ($k_{\parallel}\approx 0$), and into systems where $k_{\perp} v_{\perp} << \omega_{ci}$, the simplified operator

$$Q_n(f) = \frac{1}{\tau_{nh}} \frac{1}{v_{\perp}} \frac{\partial}{\partial v_{\perp}} v_{\perp}^{2n-1} \frac{\partial f}{\partial v_{\perp}} \,, \qquad (3)$$

with

$$\tau_{nh}^{-1} = \frac{1}{\Gamma(n)^2} \left\{ \frac{k_\perp v_{th}}{2\omega_{ci}} \right\}^{2n-2} \frac{Z^2 e^2 R}{8m^2 n\omega_{ci} r \sin \Theta_o \; v_{th}^2} \left|E_+\right|^2 ,$$

where here, and in the following $v_\perp (v=v_\perp/v_{th})$ is the normalised perpendicular component of the ion velocity, R is the major radius of the toroidal axis, r is the minor radius of the particular magnetic surface over which the averaging is taken, and Θ_o is the corresponding angle of intersection with the resonance layer.

III FORMAL SOLUTION PROCEDURE AND REDUCTION OF INTEGRAL EQUATION

3.1 Formal Solution Procedure

The process of obtaining a solution to Eq. (1) is relatively straightforward. We first write the solution in the form:

$$f(\underline{v},t) = f_w(\underline{v},t) + f_c(\underline{v},t) \ ,$$

where $\underline{v} = (v,\zeta)$, and $f_w(\underline{v},t)$ is a particular solution of RF diffusion equation:

$$\frac{\partial f_w}{\partial t} = Q_n(f_w) \qquad (4)$$

and the part of the distribution function denoted by $f_c(\underline{v},t)$ then satisfies the inhomogenous equation:

$$\frac{\partial f_c}{\partial t} - C_1(f_c) = C(f_w) + (Q_n + C_2)(f_c) \qquad (5)$$

and, for analytical convenience the Coulomb scattering operator has been written as a sum of operators:

$$C = C_1 + C_2 \ ,$$

where

$$C_1(f) = \frac{1}{\tau_s}\left[\frac{1}{v^2}\frac{\partial}{\partial v}\left\{(v^3 + v^3_c)f\right\} + \beta\left\{\frac{v_c}{v}\right\}^3 \frac{\partial}{\partial \zeta}\ (1 - \xi^2)\frac{\partial f}{\partial \zeta}\right] \ ,$$

is the operator describing resonant ion dynamical friction on the plasma ions and electrons together with pitch angle scattering on the plasma ions and

$$C_2(f) = \frac{1}{\tau_s}\ D_c\frac{1}{v^2}\frac{\partial^2}{\partial v^2}\left\{v^2\ (1 + \frac{\sigma}{v^3})\ f\right\} \ ,$$

is the operator describing energy diffusion due to collisions with the bulk plasma ions and electrons.

The solution to Eq.(1) can be formally written as:

$$f_c = \int d\underline{v}' \int dt' \; G(\underline{v},\underline{v}', t-t') \left[C(f_w) + \left\{ Q_n + C_2 \right\} (f_c) \right] \quad (6)$$

where $G(\underline{v},\underline{v}', t-t')$ is the usual Greens function which satisfies the equation:

$$\frac{\partial G}{\partial t} - C_1(G) = - \delta\ (\underline{v}-\underline{v}')\delta(t-t') \ . \quad (7)$$

It follows that once the Greens function and the solution to Eq.(4) f_w have been found, the above integral equation can be formally solved by the method of succesive approximation.

3.2 The Determination of the Greens Function G(v,v'.t-t')

The Greens function equation, Eq. (7) in spherical coordinates (v,ξ) in velocity space is:

$$\tau_s \frac{\partial G}{\partial t} - \frac{1}{v^2} \frac{\partial}{\partial v} (v^3 + v_c^3)\ G - \beta \left\{ \frac{v_c}{v} \right\}^3 \frac{\partial}{\partial \zeta} \ (1 - \zeta^2) \ \frac{\partial G}{\partial \zeta} =$$

$$- \tau_s \ \frac{\delta(v-v')}{v^2} \ \delta(\zeta - \zeta) \ \delta(t - t'),$$

and the general technique involved in the determination of the solution is well known. The solution is first expanded as a Laplace integral-Legendre series:

$$G(v,\zeta,\tau,) = \sum_{m=o}^{\infty} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dp\ e^{pt}\ g_m\ (v,\zeta,p)\ P_m\ (\zeta),$$

where $P_m(\zeta)$ is the Legendre polynomial, and for t>0 the p-integration is closed in the half space Re(p)<c. The equation for the functions $g_m(v,\zeta,p)$ then reads:

$$\frac{1}{v^2}\frac{d}{dv}\left\{(v^3 + v_c^3)\, g_m\right\} - \left[\beta\left\{\frac{v_c}{v}\right\}^3 m(m+1) + \tau_s p\right] g_m =$$

$$\tau_s \left(m + \frac{1}{2}\right) P_m (\zeta')e^{-pt'} \frac{\delta(v - v')}{v^2} \quad ,$$

this equation is readily integrated, and after some straightforward reduction, followed by a further integration in the complex p-plane, the following Greens function is obtained:

$$G(v,v',\zeta,\zeta' t-t') = \frac{\tau_s}{v^3 + v_c^3} \sum_{m=o}^{\infty} \left\{\frac{1 + (v_c/v')^3}{1 + (v_c/v)^3}\right\}^{\beta m(m+1)/3}$$

$$\cdot(m + \tfrac{1}{2})P_m(\zeta)P_m(\zeta')\delta\left\{t - t' - \frac{\tau_s}{3}\ln\frac{v'^3 + v_c}{v^3 + v_c^3}\right\} \quad ; v<v'. \qquad (8)$$

3.3 Reduction of the Integral Equation

In order to reduce the integral equation to a tractable form it is first assumed that the solution to the RF-diffusion equation, f_w, and the solution to the integral equation f_c admit expansions in the form of series in Legendre polynomials:

$$f_w(v,\zeta,t) = \sum_{m=o}^{\infty} S_m(v,t)\, P_{2m}(\zeta),$$

$$f_c(v,\zeta,t) = \sum_{m=o}^{\infty} A_m(v,t)\, P_{2m}(\zeta).$$

Then substituting these expansions and the Greens function Eq.(8) into Eq.(6), integrating over the pitch variable ζ, and using the δ-function to effect the integration over the velocity v' gives finally the integral equation:

$$A_m(v,t) = \frac{1}{\tau_s}\int_0^t d\tau\ G_m(v,t-\tau)\left[D_c \frac{1}{u^2}\frac{\partial^2}{\partial u^2}\left\{u^2\left(1 + \frac{\sigma}{u^3}\right)\right\}\right.$$

$$\left. + \frac{1}{u^2}\frac{\partial}{\partial u}(u^3 + v_c^3) - \beta\left\{\frac{v_c}{u}\right\}^3 2m(2m+1)\right] S_m(u,\tau)$$

$$+ \sum_{\ell=0}^{\infty} \frac{2}{4m+1}\int_0^t d\tau\ G_m(v,t-\tau)$$

$$\cdot \int_{-1}^{+1} d\zeta \left\{Q_n + C_2\right\} A_\ell\ (u,\tau) P_{2m}\ (\zeta)\ P_{2\ell}\ (\zeta) \qquad (9)$$

where, here and in the following,

$$G_m(v,\tau) = e^{3\tau/\tau_s}\left[1 + \left\{\frac{v_c}{v}\right\}^3\left\{1 - e^{-3\tau/\tau_s}\right\}\right]^{-2m\beta(2m+1)/3},$$

and $$u^3 = (v^3 + v_c^3)\, e^{3(t-\tau)/\tau_s} - v_c^3 .$$

This equation gives an exact formal solution for the functions $A_m(v,t)$; in order to determine the actual solution in a particular case, it is necessary to introduce an explicit form for the source functions $S_m(v,t)$, and the expansion of the RF interaction operator Q in the velocity space coordinate system (v,ζ) into Eq. (9). To obtain the required expansions in the case of heating at the fundamental (n=1), and the second harmonic frequencies is a relatively easy task. For the source functions $S_m(v,t)$, it is first necessary to derive the solution to the RF diffusion equation, Eq.(4), subject to the appropriate initial conditions, the expansion of the resulting solution in a series of Legendre polynomials can then be obtained in a straightforward manner, and the source functions identified. The determination of the expansion of the operator Q and the integration over the pitch variable ζ present no particular difficulties.

In the following section the problems of the determination of the time dependent behaviour of resonant ion distribution functions during heating at the fundamental, and second harmonic frequencies is undertaken.

IV APPLICATIONS

4.1 Tail Formation During Heating at the Fundamental Frequency (n=1)

To determime the evolution of the resonant ions in the tail of the distribution function during ICRF heating at the fundamental frequency is is first necessary to obtain the solution of the RF diffusion equation:

$$\tau_{1h} \frac{\partial f}{\partial t} = \frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} v_\perp \frac{\partial f}{\partial v_\perp} ,$$

when initially the velocity distribution of the ions is the Maxwellian:

$$f(v,t=o) = \frac{1}{\pi^{3/2}} \exp\left\{- v^2\right\}.$$

Following the application of the Laplace transform technique, the solution subject to the initial condition is easily obtained. We have:

$$f(v_\parallel,v_\perp t) = \frac{1}{\pi^{3/2}} \int_0^\infty v_\perp' dv_\perp' e^{-v'^2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dp\, e^{pt}\, I_o(\lambda v_\perp)\, K_o(\lambda v_\perp') ,$$

where $\lambda^2 = \tau_{1h}p, v'^2 = v_\perp'^2 + v_\parallel'^2$, I_o, K_o are the modified Bessel functions, and for t>o the p-integration contour is closed in the half plane Re(p)<c.

The indicated integrations are easily carried through and we obtain:

$$f(v_\parallel,v_\perp,t) = \pi^{3/2}\left\{(1 + 4t/\tau_{1h})\right\}^{-1} \exp -\left\{ v_\parallel^2 + v_\perp^2/(1 + 4t/\tau_{1h})\right\}.$$

This solution is identical to the result obtained in Ref [6] and provides a clear description of the time development of the resonant ion distribution function in the absence of Coulomb scattering. Resonant ions which are initially Maxwellian and isotropic in velocity space are progressivly deformed into highly anisotropic distributons with the most energetic ions concentrated deep in the trapped region of velocity space. The scale of the deformation and the time of particle build-up in the trapped region $v_{\parallel}\approx 0$ is given by the ratio $T_{\perp}/T_{\parallel} = 1 + 4t/\tau_h$, where $T_{\parallel}$ and $T_{\perp}$ are the parallel and perpendicular temperatures of the resonant ion distribution function.

In order to obtain a representation in the form of an infinite series of Legendre polynomials, we proceed as follows.

First we set $v_{\parallel} = v\zeta$, and $v_{\perp} = v\sqrt{1-\zeta^2}$, then on using the integral representation

$$e^{-\alpha^2\beta^2} = \frac{1}{\pi^{1/2}\beta}\int_{-\infty}^{\infty} dk\ e^{i2\alpha k - k^2/\beta^2}, \tag{10}$$

we obtain:

$$f(v,\zeta,t) = \frac{e^{-v^2/(1 + 4t/\tau_{1h})}}{\pi^2 v(1 + 4t/\tau_{1h})^{1/2}}\int_{-\infty}^{\infty} dk\ e^{i4(t/\tau_{1h})^{1/2}\zeta k - k^2(1 + 4t/\tau_{1h})/v^2}$$

Invoking the Baur [9] plane wave expansion formula:

$$e^{i4(t/\tau_{1h})^{1/2}\zeta k} = \sum_{m=0}^{\infty} i^m(2m + 1)\ j_m\{4(t/\tau_{1h})^{1/2}k\}P_m(\zeta), \tag{11}$$

where $j_m(x)$ is the spherical Bessel function of order m, and using the Hankel formula [9]

$$\int_0^{\infty} dt\ t^{\mu-1}\ J_\nu(at)\ \exp - (pt)^2 = \frac{\Gamma(\frac{\mu + \nu}{2})}{\Gamma(\nu + 1)}\ \frac{a^\nu}{2^{\nu+1}p^{\nu+\mu}}\ {}_1F_1\left\{\frac{\mu + \nu}{2},\ \nu+1,\ -\frac{a^2}{4p^2}\right\} \tag{12}$$

where, ${}_1F_1$ is the confluent Hypogeometric function gives

$$f(v,\zeta,t) = \frac{e^{-v^2/(1 + 4t/\tau_{1h})}}{2\pi^{3/2}(1 + 4t/\tau_{1h})}\sum_{m=0}^{\infty}(-1)^m\ \frac{(4m + 1)}{\Gamma(m + 1)}\ H_m P_{2m}(\zeta),$$

where

$${}_mH\ (v,t)\ \frac{1}{\sqrt{\alpha}}\int_0^{\alpha} dx\ x^{m-1/2}\left\{1 - \frac{x}{\alpha}\right\}^m\ \exp\{-x\},$$

and, $\alpha = v^2 4t/\tau_{1h}/(1 + 4t/\tau_{1h})$.

From this solution representation, the source functions are readily identified. We have:

$$S_m(v,t) = \frac{e^{-v^2/(1 + 4t/\tau_{1h})}}{2\pi^{3/2}(1 + 4t/\tau_{1h})} (-1)^m \frac{(4m + 1)}{\Gamma(m + 1)} H_m(v,t), \qquad (13)$$

$$m = 0,1...$$

Next, a transformation of the RF operator Eq. (3) from cylindrical coordinates $(v_{\parallel},v_{\perp})$ in velocity space to the spherical coordinate system (v,ζ) is required. For n=1, the operator transforms to:

$$Q_1 = (1 - \zeta^2) \frac{\partial^2}{\partial v^2} + \frac{(1 + \zeta^2)}{v} \frac{\partial}{\partial v} - \frac{2}{v} \frac{\partial}{\partial v} - \frac{1}{v^2} \zeta(1 - \zeta^2) \frac{\partial}{\partial \zeta}$$

$$+ \frac{\zeta^2}{v^2} \frac{\partial}{\partial \zeta} (1 - \zeta^2) \frac{\partial}{\partial \zeta}. \qquad (14)$$

Finally, substituting Eq.(13) and Eq.(14) into the original integral equation, Eq.(9), invoking the orthogonal properties of the Legendre polynomials [10] and completing the integration over the pitch variable ζ yields the integral equation for the functions $A_m(v,t)$. The equation reads:

$$A_m(v,t) = \frac{1}{\tau s} \int_0^t d\tau\, G_m(v,t-\tau) \Bigg[D_c \frac{1}{u^2} \frac{\partial^2}{\partial u^2} u^2 \left(1 + \frac{\sigma}{u^3}\right)$$

$$+ \frac{1}{u^2} \frac{\partial}{\partial u} (u^3 + v_c^3) - \beta \left\{\frac{v_c}{u}\right\}^3 2m(2m+1) \Bigg]$$

$$\cdot (-1)^m \frac{e^{-u^2/(1 + 4\tau/\tau_{1h})}}{2\pi^{3/2} (1 + 4\tau/\tau_{1h})} \frac{(4m + 1)}{\Gamma(m + 1)} H_m(u,\tau)$$

$$+ \frac{1}{\tau_s} \int_0^t d\tau\, G_m (v, t-\tau)\, D_c \frac{1}{u^2} \frac{\partial^2}{\partial u^2} u^2 \left(1 + \frac{\sigma}{u^3}\right) A_m(u,\tau) + ..$$

$$+ \frac{1}{\tau_{1h}} \int_{o}^{t} d\tau \; G_m \,(v,t-\tau) \sum_{\ell=-1}^{1} \left[W^2_{m+2\ell} \frac{\partial^2}{\partial u^2} + W^1_{m+2\ell} \frac{1}{u} \frac{\partial}{\partial u} \right.$$

$$\left. + W^o_{m+2\ell} \frac{1}{u^2} \right] A_{m+2\ell}(u,\tau), \qquad (15)$$

where the functional form of the particle-wave coupling coefficients $W^i_{m+2\ell}(m)$; i=0,1,2 are presented in Table I.

Further analytic progress at this point towards the determination of a closed form solution without further simplifications represents considerable difficulties. Accordingly, Eq.(15) completes the analytical treatment of the solution to the problem of heating at the fundamental frequency n=1. In a particular case the actual solution may be obtained through numerical iteration.

4.2 Tail Formation During Heating at the Second Harmonic Frequency (n=2)

In this case the RF diffusion equation to be solved is:

$$\tau_{2h} \frac{\partial f}{\partial t} = \frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} v_\perp^3 \frac{\partial f}{\partial v_\perp} \quad ,$$

subject to the initial condition that the resonant ions be the Maxwellian:

$$f(v,t=0) = \frac{1}{\pi^{3/2}} \exp\left\{-v^2\right\}.$$

The procedure for solving this equation is similar to the case of heating at the fundamental frequency n=1, and a solution using the Laplace transform technique is again indicated. This procedure gives after some straightforward reduction the solution:

$$f(v_\parallel,v_\perp,t) = \frac{e^{-v_\parallel^2}}{\sqrt{4\pi^4 t/\tau_{2h}}} \int_{-\infty}^{\infty} dx \exp - \left\{ (x - 2t/\tau_{2h})^2/4(t/\tau_{2h}) + v_\perp^2 \, e^{2x} \right\} \qquad (16)$$

While the integral appearing in Eq.(16) cannot be evaluated directly, a simple application of the Method of Steepest Descents does enable a closed form expression to be obtained. Following this standard procedure, we get

$$f(v_{\parallel},v_{\perp},t) = \frac{e^{-v_{\parallel}^2}}{\pi^{\frac{3}{2}}\sqrt{1-2\eta}} \quad \exp - (\bar{\eta}^2 - \bar{\eta}) /4(t/\tau_{2h}) \quad ,$$

and $\eta = \eta(v_{\perp},t)$ is a root of the equation $x \exp\{-x\} + \lambda = 0$, where $x = 2\eta$ and $\lambda = 8(t/\tau_{2h})\ v_{\perp}^2 \exp(4t/\tau_{2h})$. This result is useful in the investigation of the initial phase of the tail development under conditions of "strong" RF interaction and $\tau_{2h} << \tau_s$.

In order to obtain an expansion of the above distribution function, as a series of Legendre polynomials, we first set $v_{\parallel} = v\zeta$ and $v_{\perp}\sqrt{1-\zeta^2}$, in Eq.(16), and then following the procedure used previously, gives the representation:

$$f(v,\zeta,t) = \frac{1}{\pi^{\frac{3}{2}}} \sum_{m=o}^{\infty} (-1)^m \frac{(4m+1)}{\Gamma(m+1)} H_m(v,t)\, P_{2m}(\zeta) \ ,$$

where

$$H_m(v,t) = \frac{1}{\sqrt{4\pi t/\tau_{2h}}} \int_{-\infty}^{\infty} dx \exp - \left\{(x - 2t/\tau_{2h})^2/4(t/\tau_{2h}) + v^2 e^{2x}\right\}$$

$$\cdot \frac{1}{\sqrt{\alpha}} \int_{o}^{\alpha} dz z^{m-\frac{1}{2}} \left\{1 - \frac{z}{\alpha}\right\}^m \exp(-z) \ ,$$

and

$$\alpha = v^2(1 - e^{2x}).$$

The source functions are:

$$S_m(v,t) = (-1)^m \frac{(4m+1)}{\Gamma(m+1)} H_m(v,t). \qquad (17)$$

$m = 0,1$ and the RF operator in spherical coordinates (v,ζ) in velocity space is:

$$Q_2 = (1-\zeta^2)^2 v^2 \frac{\partial^2}{\partial v^2} + \left\{(3 - 2\zeta^2 - \zeta^4) - 2\zeta(1-\zeta^2)^2 \frac{\partial}{\partial \zeta}\right\} v \frac{\partial}{\partial v}$$

$$- \zeta(1-\zeta^4)\frac{\partial}{\partial \zeta} + \zeta^2 (1-\zeta^2)\frac{\partial}{\partial \zeta}(1-\zeta^2)\frac{\partial}{\partial \zeta} \qquad (18)$$

Substituting Eq.(17) and Eq. (18) into the integral equation, Eq. (9), invoking the orthogonal properties of the Legendre polynomials, and completing the simple integration over the variable ζ yields the integral equation:

$$A_m(v,t) = \frac{1}{\tau_s}\int_0^t d\tau\, G_m(v,t-\tau)\left[D_c \frac{1}{u^2}\frac{\partial^2}{\partial u^2}\left\{u^2\left(1+\frac{\sigma}{u^3}\right)\right\}\right.$$

$$\left. + \frac{1}{u^2}\frac{\partial}{\partial u}(u^3 + v_c^3) - \beta\left\{\frac{v_c}{u}\right\}^3 2m(2m+1)\right] S_m(u,t)$$

$$+ \frac{1}{\tau_s}\int_0^t d\tau\, G_m(v,t-\tau)\, D_c \frac{1}{u^2}\frac{\partial^2}{\partial u^2}\left\{u^2\left(1+\frac{\sigma}{u^3}\right)\right\} A_m(u,\tau)$$

$$+ \frac{1}{\tau_s}\int_0^t d\tau\, G_m(v,t-\tau)\sum_{\ell=-2}^{2}\left[W^2_{m+2\ell}\, u^2 \frac{\partial^2}{\partial u^2}\right.$$

$$\left. + W^1_{m+2\ell}\, u\frac{\partial}{\partial u} + W^0_{m+2\ell}\right] A_{m+2\ell}(u,\tau), \qquad (19)$$

and the particle-wave coupling coefficients $W^i_{m+2\ell}$; i=0,1,2 are presented in Table II.

Equations (15) and (19) are almost identical in form, and as previously mentioned, further progress towards the determination of the actual solution in a particular case requires the intervention of a suitable numerical procedure.

V DISCUSSION AND NUMERICAL RESULTS

In the previous section a semi-analytic procedure for the determination of the solution of the time dependent ICRF Fokker-Planck equation has been presented. The solution procedure leads to the integral equation, Eq.(9), which in a particular heating configuration, describes the effect of the Coulomb scattering processes on the evolution of the resonant particles in the tail of the distribution function.

The efficiency of the solution procedure, and in particicluar, that for the coefficients $A_m(v,t)$ depends on the form of the solution to the RF-diffusion equation, Eq.(4), and its development in a series of Legendre polynomials. The possibility of obtaining suitable solutions is crucially dependent on the form of the RF-diffusion coefficient; hence the use of the small Larmor-radius approximation in this work. However, the $(\rho_i k_\perp << 1)$ approximation effectively limits the method to tail particle energies $E_n \leq 0.5m\ n\ \omega_{ci}^2/k_\perp^2$, which for fundamental (n=1) heating of minority hydrogen (H) in JET with f = 45MHz, and $k_\perp = 0.3cm^{-1}$ is typically $E_n \leq 400keV$.

From an inspection of the derived integral equations, it is evident that while the expressions for the source functions $S_m(v,t)$ and coupling coefficients $W^i_{m+2\ell}$ are readily calculated for heating at the fundamental frequency, the corresponding formulas for (n>1) become increasingly cumbersome, nevertheless, the method is far simpler to implement than alternative schemes. For second harmonic heating (n=2) the formulas for $S_m(v,t)$, Eq.(17) and $W^i_{m+2\ell}$, Table II, are appreciably more complicated than those for (n=1), and without further simplification involves further numerical evaluations. However, in view of the limitation of the $(\rho_i k_\perp <<1)$ approximation for this configuration the method is restricted to the initial phase of the tail formation process $t \leq \tau_{2h} < \tau_s$, or the situation of weak RF interaction, where $\tau_s << \tau_{2h}$, the contribution of the integral appearing in Eq. (19) can then be negelcted. Furthermore, the method of Steepest Descents can be gainfully employed to reduce the amount of numerical computation required. A further point to note is that direct second harmonic heating of the bulk plasma ions can give rise to substantial tail populations, with an increase in the frequency of the self collisions of the energetic tail ions. These energetic ion-ion encounters will be an important component in the tail forming process and for the final stage of the evolution $t=\tau_s$ a full non linear treatment of the problem is probably required.

Before considering an actual application of the method there are a few points of practical utility worth mentioning. The integral equation is of the Volterra type and a well known method of obtaining a solution is the method of succesive approximations which is essentially identical to a Neumann series expansion. The zeroth approximation, valid for evolution time $t<<\tau_s$, and the first Born approximation are readily obtained. Unfortunately, the determination of the validity of these approximations is not an easy task and in a particular case has to be assessed numerically. While higher order Born approximations can in principle be formulated they are far too complex for efficient numerical computation. Thus, the treatment is limited to the zero, and first Born approximations. A further point to note is that for the determination of the plasma quantities which are of experimental interest, such as fusion reactivity and bulk plasma heating rates, it is only necessary to solve the integral equations for $A_o(v,t)$.

Finally, under conditions of "strong" RF interaction, where $\tau_{nh}<<\tau_s$, and when $t<<\tau_s$, the Coulomb scattering processes are inconsequential, and the "collisionless" theory can be used to estimate the parameters of interest.

In order to discuss the validity of the "collisionless" model in ICRF studies, a problem of energetic tail formation during minority heating at the fundamental frequency in a tokamak configuration that is typical of the present and the next generation machine is examined. For the representative configuration, the following scheme of minority hydrogen (H) in a deuteruim (D) plasma is considered; Bulk plasma ion and electron temperatures T_D, T_e = 2.5keV; initial minority temperature T_H = 2.5keV; electron density $n_e = 3\times10^{13}cm^{-3}$, with a 10% minority hydrogen. The characteristic time scales for this data are are $\tau_{1h} = 1.32\times10^{-2}s$, and $\tau_s = 1.32\times10^{-1}s$. For this configuration, a "Born approximation" to the solution of the integral equation, Eq.(15) has been calculated, and using a 15 Legendre polynomial expansion, the resonant ion distribution function has been constructed at times given by t/τ_s = 0.25, 0.5 and 1.0, after the onset of the heating. The pertinent results are shown in Fig. 1.

It is interesting to note that for $t/\tau_s \leq 0.25$ ($t \leq 2.5\tau_{RF}$) little deviation from the "Collisionless" distribution function is indicated. For these system time scales, and for the early time development of the distribution function, the Coulomb scattering processes are not important, and the tail formation process is well described by the RF-diffusion equation.

VI SUMMARY

In this work it has been shown that, by applying the method of interaction separation, the problem of solving the time-dependent ICRF Fokker-Planck equation can be reduced to that of obtaining a solution to an ancilliary integral equation in the time domain. The integral equation which is of the Volterra type is amenable to an efficient numerical solution procedure, and for evolution times $t \leq \tau_s$, a "Born approximation" is readily obtained. The method is particularly effective in the determination of energetic tail formation in fundamental heated ICRF tokamak systems.

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APPENDIX

The particle-wave coupling coefficients appearing in Eq. (15) and Eq. (16), $W^{i}_{m+2\ell}$ and their dependence on m are readily obtained using suitable frictional relationships between the Legenre polynomials $P_\nu(\zeta)$. The recurrence formulas needed for the calculations are:

$$(2\nu+1)\,\zeta P_\nu = (\nu+1)\,P_{\nu+1} + \nu P_{\nu-1},$$

$$(1-\zeta^2)\,P'_\nu = (\nu+1)\,\zeta P_\nu - (\nu+1)\,P_{\nu+1},$$

and

$$\zeta^2 P_\nu = \frac{(\nu+1)(\nu+2)}{(2\nu+1)(2\nu+3)}\,P_{\nu+2} + \frac{\nu^2}{(2\nu-1)(2\nu+1)} + \frac{(\nu+1)^2}{(2\nu+1)(2\nu+3)}\,P_\nu + \frac{(\nu-1)\nu}{(2\nu-1)(2\nu+1)}\,P_{\nu-2}\,.$$

The particle-wave coupling coefficients are shown below in Tables I and II.

TABLE I

Fundamental Frequency (n=1) Particle-Wave Coupling Coefficients $w^{i}_{m+2\ell}$

ℓ	$\ell \quad W^{0}_{m+2\ell}$
-1	$\frac{(2m-1)(2m)}{(4m-3)(4m-1)}$
0	$1 - \frac{(2m)^2}{(4m-3)(4m-1)} - \frac{(2m+1)^2}{(4m+1)(4m+3)}$
1	$-\frac{(2m+1)(2m+2)}{(4m+3)(4m+5)}$
ℓ	$W^{1}m+2\ell$
-1	$\frac{(2m-1)(2m)}{(4m-1)}$
0	$1 - \frac{(2m)^2}{(4m-1)} + \frac{(2m+1)^2}{(4m+3)}$
1	$-\frac{(2m+1)(2m+2)}{(4m+3)}$
ℓ	$W^{2}_{m+2\ell}$
-1	$-\frac{(2m-2)(2m-1)(2m)^2}{(4m-3)(4m-1)}$
0	$-\frac{2m}{(4m+1)}\left\{\frac{(2m-1)(2m)(2m+1)}{(4m-1)} + \frac{(2m+1)^2(2m+2)}{(4m+3)}\right\}$
1	$-\frac{(2m+1)^2(2m+2)(2m+3)}{(4m+3)(4m+5)}$

TABLE II

Second Harmonic Frequency (n=2) Particle Wave Coupling Coefficients $W^{i}_{m+2\ell}$

ℓ	$W^{o}_{m+2\ell}$
-2	$\frac{(2m-3)(2m-2)(2m-1)(2m)}{(2m-7)(4m-5)(4m-3)(4m-1)}$
-1	$-\frac{(2m-1)(2m)}{(4m-3)(4m-1)}\{2 - \frac{(2m-2)^2}{(4m-3)(4m-1)} - \frac{(2m-1)^2}{(4-1)(4m-3)} - \frac{(2m)^2}{(4m-1)(4m+1)}\}$
0	$\{1 - \frac{(2m)^2}{(4m-1)(4m+1)} - \frac{(2m+1)^2}{(4m+1)(4m+3)}\}^2 + \frac{(2m-1)^2(2m)^2}{(4m+1)(4m+3)^2(4m+5)} + \frac{(2m-1)^2(2m)^2}{(4m-3)(4m-1)(4m+1)}$
1	$-\frac{(2m+1)(2m+2)}{(4m+3)(4m+5)}\{2 - \frac{(2m)^2}{(4m-1)(4m+3)} - \frac{(2m+1)^2}{(4m+1)(4m+3)} - \frac{(2m+2)^2}{(4m+3)(4m+5)} - \frac{(2m+3)^2}{(4m+5)(4m+5)(4m+7)}$
2	$\frac{(2m+1)(2m+2)(2m+3)(2m+4)}{(4m+3)(4m+5)(4m+7)}$
ℓ	$W^{1}_{m+2\ell}$
-2	$\frac{(2m-3)(2m-2)(2m-1)(2m)}{(4m-5)(4m-3)(4m-1)}$
-1	$\frac{(2m-1)(2m)}{(4m-3)(4m-1)}\{(4m-3) - \frac{(2m-2)^2}{(4m-5)} - \frac{(2m-1)^2}{(4m-1)}\} - \frac{(2m-1)(2m)}{(4m-1)(4m+1)}\{\frac{(2m)^2}{(4m-1)} + \frac{(2m+1)^2}{(4m3)}\}$
1	$-\frac{(2m+1)^2(2m+2)^2}{(4m+3)^2(4m+5)^2} + \{\frac{(2m)^2}{(4m-1)} + \frac{(2m+1)^2}{(4m+3)}\}\{1 - \frac{(2m+1)^2}{(4m+1)(4m+3)} + \frac{(2m-1)^2(2m)^2}{(4m-1)(4m-3)}\}$
2	$-\frac{(2m+5)(2m+6)(2m+7)(2m+8)}{(4m+3)(4m+5)(4m+7)}$

(Continued ...)

TABLE II (continued)

ℓ	$W^2_{m+2\ell}$
-2	$\frac{(2m-4)(2m-3)(2m-2)^2(2m-1)(2m)}{(4m-7)(4m-5)(4m-3)(4m-1)}$
1	$\frac{(2m-1)(2m)}{4m-7)(4m-5)} \{- (2m-2)^2 - \frac{(2m-2)(2m-1)}{(4m-5(4m-3)} + \frac{(2m-1)^2}{(4m-3)(4m-1)} + \frac{(2m-2)(2m)^2}{(4m-1)(4m+1)} + \frac{(2m-2)(2m)(2m+1)^2}{(4m+1)(4m+3)}$
0	$\frac{(2m)^2}{(4m-1)} + \frac{(2m)2m+1)^2(2m+2)^3}{(4m+)(4m+3)^2(4m+5)} + \{\frac{(2m+1)^2}{(4m-1)(4m+1)} + \frac{(2m+1)^2}{(4m+1)(4m+3)}\}\{-(2m)^2 - \frac{(2m)(2m+1)}{(4m-1)(4m+1)} + \frac{(2m+1)}{(4m+1)(4m+3)}\} + \frac{(2m-1)^3(2m)^2(2m+1)}{(4m-1)^2(4m+1)(4m+3)}$
1	$\frac{(2m+1)(2m+2)}{(4m+3(4m+5)} \{- (2m+3)^2 - \frac{(2m+2)(2m+3)}{(4m+3)(4m+5)} + \frac{(2m+3)^2}{(4m+5)(4m+7)} + \frac{(2m+1)^3(2m+3)}{(4m+1)(4m+3)} + \frac{(2m+1)(2m)^2(2m+3)}{(4m-1)(4m+1)}\}$
2	$\frac{(2m+1)(2m+2)(2m+3)^2(2m+6)(2m+5)}{(4m+1)(4m+2)(4m+7)(4m+9)}$

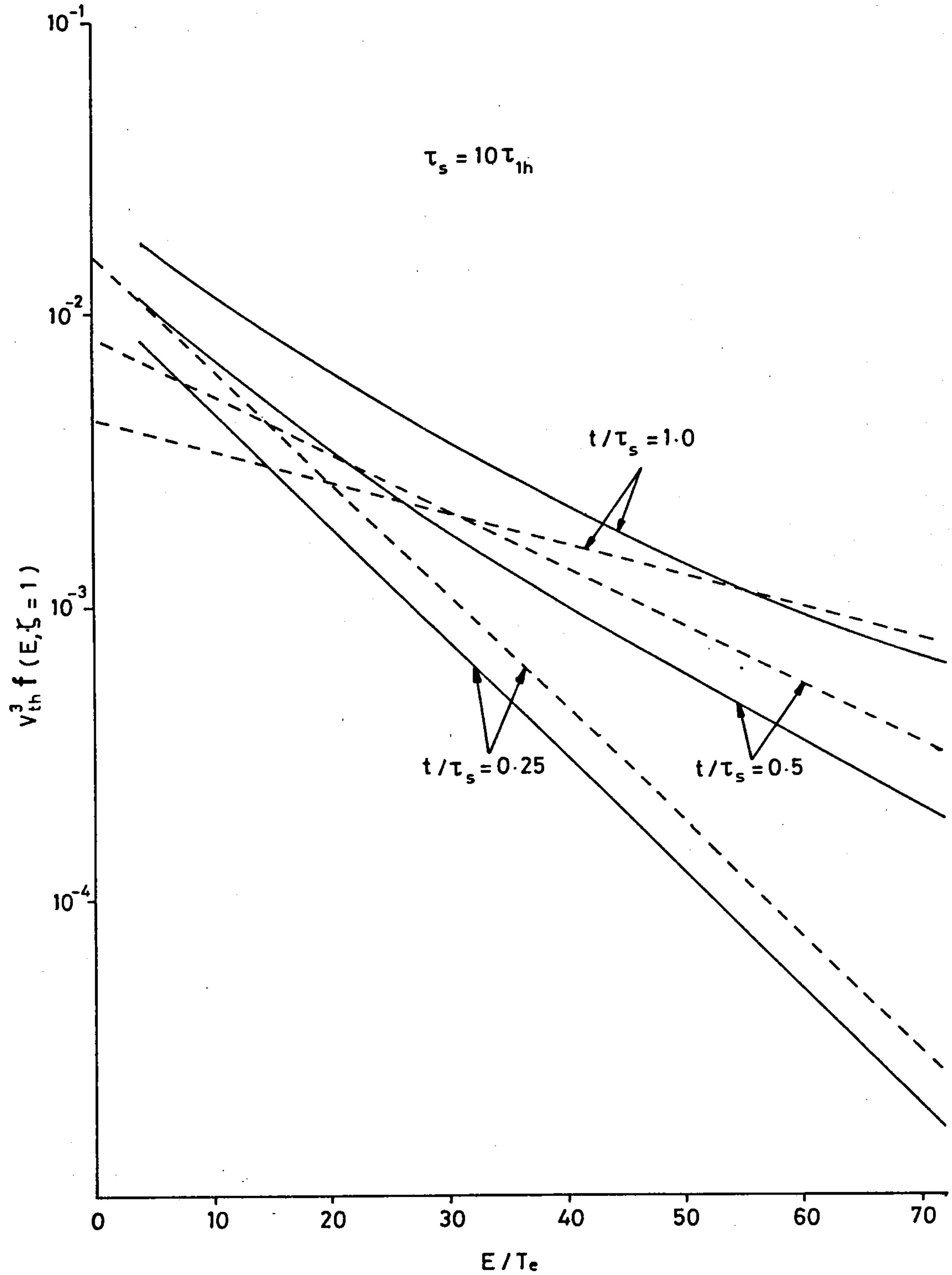

Fig. 1 The formation of a minority hydrogen (H) tail in a deuterium plasma (D). The broken line is the 'collisionless' distribution function. Bulk Plasma temperatures $T_e, T_D = 2.5$ keV, initial minority temperature $T_H = 2.5$ keV, electron density $n_e = 3 \times 10^{13}$ cm/, minority concentration 10% and $\tau_1/\tau_s = 0.1$.