High β Tokamaks

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1. THE IMPORTANCE OF β

In principle the mhd limitation on β could be a constraint on the achievement of ignition. However it seems that an experiment designed to give ignition with the presently anticipated level of energy confinement would be within the simple form of β limit, $\beta_m \simeq 3\ I(MA)/Ba$.

With the achievement of ignition, attention turns to economics and it has always been recognised that in this context high β is desirable, most simply because the plasma pressure measures the benefit and the magnetic field the cost.

A more recent concern has been the need to drive the toroidal current. β also plays a role here in that the poloidal beta determines the amount of bootstrap current and hence the amount of current drive which must be provided.

The general analysis of these matters is of course complex but their contribution to the basic economics can be sketched.

The "benefit" of the reactor is the power it produces and since the power density is roughly proportional to n^2T^2, that is to p^2, a plasma volume V gives a benefit $\sim p^2V$. In the present context this benefit is reduced by the power required to drive the current. Since only the non-bootstrap fraction, $(1 - f_{BS})$, of the current requires power, this loss of benefit is proportional to $I(1 - f_{BS})$. Thus, using the approximation $f_{BS} = \frac{1}{3}\beta_p$ where $\beta_p = \int pdS / \left(\mu_o I^2 / 8\pi\right)$ the total benefit can be written

$$\text{Benefit} = c_1\, p^2V - c_2\, I(3 - \beta p).$$

The cost function is, in reality, complicated but we shall represent it by B^2V. This basically arises from taking the cost to be proportional to the volume of material, with the ratio of this volume to the plasma volume being given by ratio of the magnetic stress $B^2/2\mu_o$ to the material stress. This argument applies most obviously to the electromagnetic components but, for example, is also relevant for the forces on the vacuum vessel due to disruption currents.

Thus, using the above relations, the benefit/cost ratio takes the approximate form

$$\frac{\text{Benefit}}{\text{Cost}} = c_3 B^2\beta^2 - c_4\frac{I}{B^2V}\left(3-\beta_p\right), \qquad (1)$$

where $\beta = \bar{p} / \left(B^2 / 2\mu_o\right)$.

Although there are functional dependencies for I and V, equation (1) represents adequately the case for high β and high β_p.

The dependence of stability on β is complex, involving geometry, profiles, and the influence of a conducting shell. There are also uncertainties as to what physics should be included outside the ideal mhd model. Within the ideal model it is possible to resolve the stability issues by numerical calculation for each case. However, this does not readily lead to an insight as to the underlying behaviour. This aim of this talk is to outline the factors involved and to give simple models which, while not precise, represent the role of each factor.

2. MHD β-LIMITS - INTRODUCTION

When tokamaks first became thought of as providing the best route to a reactor their stability properties were only vaguely understood. Real progress toward understanding β-limits started with two almost simultaneous developments, the derivation of a theory of high-n ballooning modes[1-3] and the advent of 2-D toroidal computer codes[4] to study global modes. However the results of the computer calculations required some interpretation.

It was generally anticipated that the mhd β-limit would result from a balance between the pressure/curvature destabilising force and the line bending implied by ballooning. This gives

$$\frac{1}{R}\frac{dp}{dr} \sim \frac{B^2/\mu_o}{(qR)^2}$$

and

$$\beta \sim \frac{\varepsilon}{q^2},$$

where $q = 2\pi\, a\, \varepsilon B/\mu_o I$ and $\varepsilon = a/R$. In terms of I/aB

$$\beta \sim \frac{1}{\varepsilon}\left(\frac{I}{aB}\right)^2.$$

However when the first comprehensive calculations[5] were analysed[6] an approximately linear dependence on 1/q was found, so that β ~ I/aB. The calculations were for JET and the results are given in figure 1.

Further calculations were subsequently carried out by Sykes et al.[7], and Troyon et al.[8].

An interpretation of this result follows[9] from an analysis of the shear/ pressure gradient, or (s, α) ballooning mode diagram which relates

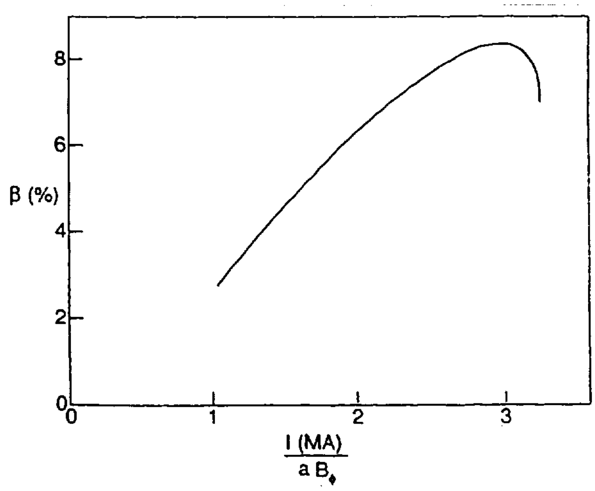

Fig.1: Optimised β-limit as a function of I/aB (from Wesson, J.A. in Stringer, T.E. Comp. Phys. Comm. 24, 337 (1981))

$$\alpha = -\frac{2\mu_o R q^2}{B^2}\frac{dp}{dr} \quad \text{and} \quad s = \frac{r}{q}\frac{dq}{dr}. \tag{2}$$

The definition of β is

$$\beta = \frac{2\int_o^a pr\,dr}{a^2 B^2 / 2\mu_o}$$

and through integration by parts this can be written in terms of α,

$$\beta = \frac{1}{Ra^2}\int_o^a \frac{\alpha r^2}{q^2}dr. \tag{3}$$

The essential results are obtained using the approximate stability boundary $\alpha = 0.6s$ which shows the benefit of shear for stability. Using this result in equation (3) leads to

$$\beta_m = -0.30\frac{1}{Ra^2}\int_o^a \frac{d}{dr}\left(\frac{1}{q^2}\right) r^3 dr. \tag{4}$$

This formulation gives a precise optimum β_m, but before turning to this optimum it is of interest to see the dependence of β_m on the current distribution.

Integration of equation (4) by parts leads to

$$\beta_m = 0.9\frac{R}{a^2 B^2}\left[\int_o^a B_\theta{}^2 dr - \frac{1}{3} a B_{\theta a}{}^2\right].$$

Thus β_m is almost a function of l_i, and with some approximations can be manouvred into the form[10]

$$\beta_m = c\frac{I}{aB}\left(\ell_i - \frac{1}{q_a}\right). \tag{5}$$

It is clear from equation (5) that it is advantageous to concentrate the current towards the axis thus producing a high ℓ_i However the axial value of q is limited by the sawtooth instability. This can be specifically allowed for by returning to the form of equation (4).

Since q_o is constrained by the internal kink mode to be above a critical value around one, the total available shear is limited by $q_a - q_o$. In equation (4) the optimal distribution of shear has all of the shear in the outer region. The result is that for maximum β, all of the current is carried in the central region and all of the pressure gradient lies in a current free outer region as shown in figure (2). This defines q(r) and, using equation (4), leads to a β-limit of the form

$$\beta_m = g\frac{I}{aB}.$$

The full optimisation gives g = 5.6 but from a practical point of view the value will depend upon the degree of optimisation and on other physics.

The results described above provided a clarification of the previously poorly understood factors governing stability. However subsequent analysis introduced two further important elements. The first was the recognition of another regime where stable high pressure gradients can be achieved at small shear, the so-called second region of stability. The second factor arises from the benefits of a large bootstrap current fraction in reducing the need for external current drive. Since the bootstrap current is driven by a plasma pressure gradient it introduces a linkage of the q-profile to the pressure profile in addition to that arising from stability.

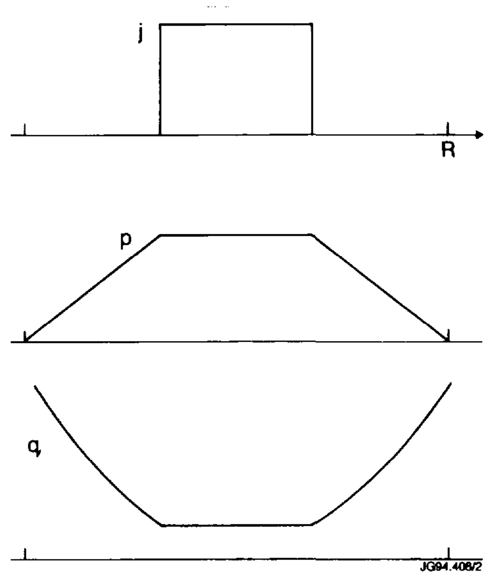

Fig. 2 Current, pressure and q profiles for optimised β in the first region of stability.

It was recognised that negative shear[11] gives ballooning mode stability, so that an inner region with dq/dr < 0 is favourable, but in addition it was found[12] that even for small dq/dr >0 there is a region of stability.

However for small shear the ballooning mode equation does not properly describe stability. The low-n modes then take a different form and if they are unstable when high-n ballooning theory predicts stability they are called "infernal" modes.

The bootstrap current complicates matters further. As explained above, the simple β optimisation concentrated the shear and pressure gradient in the outer part of the plasma by concentrating the current in the inner region. However the pressure gradient will drive a bootstrap current in the outer part of the plasma, in conflict with the narrow current channel requirements. Such configurations tend also to have current near to the surface and are therefore susceptible to external kink modes.

It is seen that with these additional features the overall stability problem is quite involved and there is not yet a consensus on the optimisation. In the following sections the various aspects of the problem will be described.

First we discuss the factors involved in the second region of stability. Then we return to the first stable region case and impose the constraint of a bootstrap current. Finally we consider the general case with access to the second region and a requirement of a large bootstrap current fraction.

3. SECOND STABLE REGION

In the simplest model with circular flux surfaces the potential energy for ballooning modes takes the form[3]

$$\delta W = \int \left\{ \left(1+\Lambda^2\right)\left(\frac{dF}{d\theta}\right)^2 - \alpha(\Lambda \sin\theta - \cos\theta)F^2 \right\} d\theta, \tag{6}$$

where

$$\Lambda = s\theta - \alpha \sin\theta$$

and the shear and pressure gradient variables s and α are defined by equations (2).

The term proportional to $(dF/d\theta)^2$ represents the stabilising effect of line bending and the $\alpha \cos\theta\, F^2$ term gives the destabilising effect of the pressure gradient coupled with the $\cos\theta$ dependence of the magnetic field. The $\alpha\Lambda \sin\theta\, F^2$ term gives the shear dependent contribution arising from the geodesic curvature, Λ being related to the shear by the equation

$$\frac{d\Lambda}{d\theta} = s - \alpha\cos\theta,$$

$\alpha\cos\theta$ giving the θ dependence of the shear around the surface. The stability boundary resulting from equation (6) is shown in figure 3. The origin of the second stable region can be seen by taking the limit of small s and large α in equation (4). δW then takes the positive definite form

$$\delta W = \int \left\{ \alpha^2 \sin^2\theta \left(\frac{dF}{d\theta}\right)^2 + F^2 \right\} d\theta. \tag{7}$$

The two terms in equation (7) have arisen from the local shear as represented by the $\alpha \cos\theta$ contribution to $d\Lambda/d\theta$. Thus the

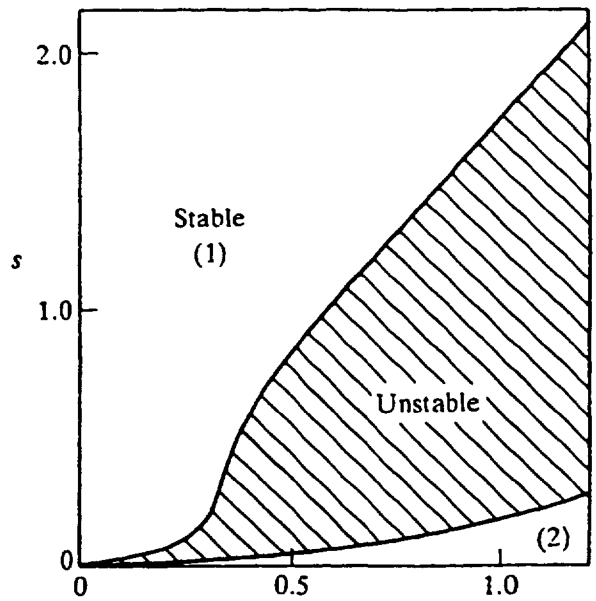

Fig. 3 The (s, α) stability diagram for circular flux surgaces showing the first (1) and second (2) stability regions.

second region of stability is generated by a modification of the toroidal equilibrium at low shear caused by the pressure gradient.

The above (s, α) treatment is inadequate at small α and s, but Pogutse and Yurchenko[13] have carried out an extended calculation which gives an analytic stability criteria requirement for stability is

$$s^2 + 4\alpha\varepsilon\left(1-\frac{1}{q^2}\right) - s\alpha^2 - 3\,\alpha\,\exp\left(-\frac{1}{|s|}\right) > 0, \tag{8}$$

where $\varepsilon = r/R$. The term $s^2 + 4\alpha\varepsilon$ represent the balance of shear and pressure gradient in the cylindrical case first analysed by Suydam. The term $4\alpha\varepsilon/q^2$ term is the toroidal stabilising term introduced by Mercier's analysis and first given by Ware and Haas [14]. The term $-s\alpha^2$ represents a normally destabilising effect which is reversed with negative shear. For a given shear s(r), criterion (8) gives a critical value of the pressure gradient factor $\alpha(r)$. The result depends upon the particular configuration but the general form is illustrated in figure 4.

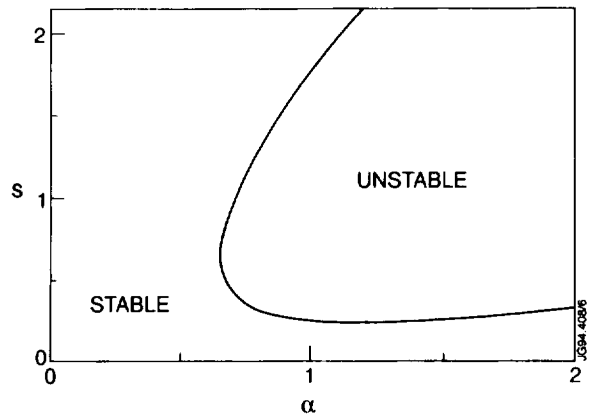

Fig. 4 Sketch of typical (s, α) stability diagram.

4. BOOTSTRAP CURRENT IN FIRST STABLE REGION

The bootstrap current density is given approximately by

$$j_B = c_B(r/R)^{\frac{1}{2}}\frac{1}{B_\theta}\frac{dp}{dr}. \tag{9}$$

This immediately shows an inconsistency in the calculation of the first stable region limit introduced in section 2. In the optimised configuration shown in figure 2 the pressure gradient is concentrated in the outer region where j = 0. In the absence of a negative current drive, equation (9) imposes a limit on the smallness of j in the pressure gradient region. To see the effect the β limit calculation can be repeated with the constraint by equation (9)[14]. Ampère's law gives

$$\mu_o j = \frac{B}{Rq}(2-s). \tag{10}$$

Thus in equation (9), we use equation (10) to substitute for $j_B = j$, and on the right hand side dp/dr is given by the relation $\alpha = 0.6s$.

The result is

$$s = \frac{2}{1+\mu},$$

where the factor $\mu = 0.3/c_B\ \varepsilon^{\frac{1}{2}}$ introduces the reduction in shear arising from the bootstrap current. Since we are concerned here with the outer region we can use the approximation $\varepsilon^{\frac{1}{2}} = (a/R)^{\frac{1}{2}}$. The solution for q is then

$$q = q_a\left(\frac{r}{a}\right)^{\frac{2}{1+\mu}}$$

and substitution into equation (4) gives a maximum β

$$\beta_m = 1.2\frac{\varepsilon}{q_a^2}\frac{q_a^{\frac{1-3\mu}{2}}-1}{1-3\mu}.$$

Taking $\mu = \frac{1}{2}$, the fractional loss of β_m arising from the bootstrap current is

$$\frac{\Delta\beta_m}{\beta_m} = -\frac{q_a^{1/4}+3-q_a^{-1/4}}{q_a^{1/2}-1} \qquad (11)$$

and this is plotted in figure 5. Equation (11) is represented quite well by the approximation

$$\frac{\Delta\beta_m}{\beta_m} = \frac{1}{4}(q_a - 1)^{1/2} \qquad (2 < q_a < 5). \quad (12)$$

Thus the bootstrap current leads to a loss of β_m in the first stable region as a result of the diminished shear in the outer region of the plasma. However it is possible to combine a high bootstrap current with a configuration in the second stable region, and we shall now consider this case.

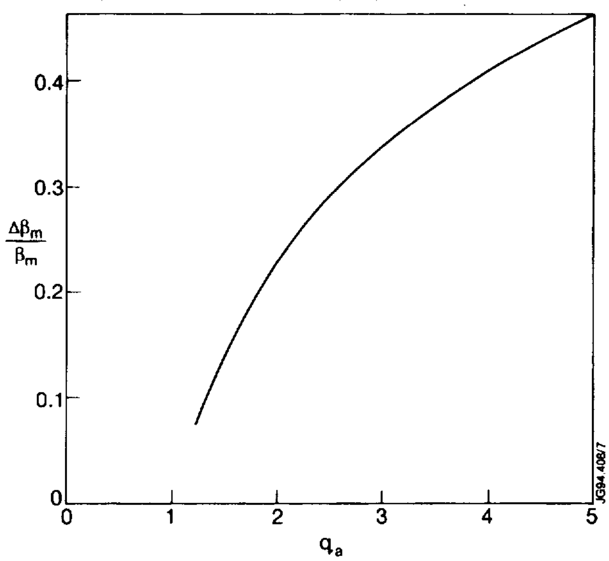

Fig. 5 Showing the fractional reduction in the first region β-limit resulting from the bootstrap current ($\mu = \frac{1}{2}$).

5. SECOND STABLE REGION WITH BOOTSTRAP CURRENT

The results of the last three sections show that

i) Optimisation of β considering only the first stable regime leads to low shear in the central region and high pressure gradients in the outer region.

ii) In the first regime the pressure gradient driven bootstrap current appears preferentially in the outer region, consistently with the low shear in the inner region.

iii) The low shear of the inner region allows us to use the second stable regime.

We are thus led to the concept of a high β, high β_p plasma with the central region in the second regime of stability and the bootstrap current carrying a large fraction of the total current. We shall shortly analyse this model, but it is first necessary to discuss two complications. The first is kink modes and the second is infernal modes.

The conventional account of kink modes is concerned with avoiding current gradients at the edge of the plasma. Such current gradients with a resonance close to the surface can lead to instability. However at high β the pressure gradient becomes important, but not necessarily destabilising because for curvature averaging perturbations the average curvature is stabilising.

Infernal modes are pressure gradient driven instabilities which are not adequately treated by the ballooning mode approximation. Ballooning mode theory is applicable at large n and the need for a separate treatment arises at low n. In such cases it is necessary to resort to 2-D stability calculations. The inadequacy of the ballooning mode approximation is associated with the presence of low shear, and the infernal modes occur when a region of small shear occurs close to low n resonant surface. This situation arises in the presence of a flat or hollow q profile. Figure 6 shows a stability diagram calculated by Ozeki et al. [16] for such a case. It is seen from this figure that it is possible to arrange conditions such that complete stability is acheived.

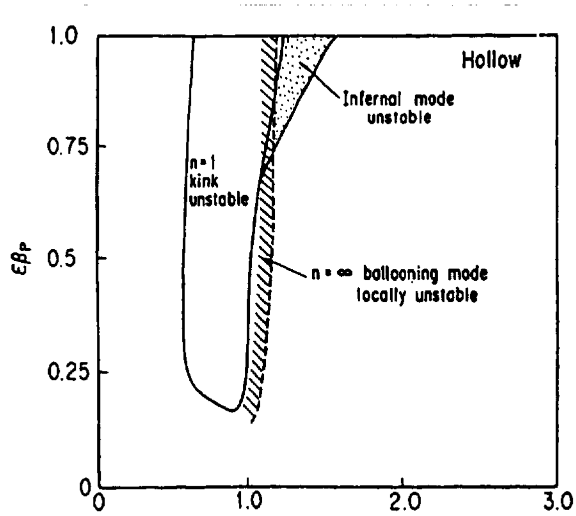

Fig. 6 Stability diagram in ($\varepsilon\beta_p$, q_{min}) space (Ozeki, T. et al. Nuc. Fus. 33 1025 (1993)).

The strategy, therefore, is to optimise the first region stability in the outer region as in the conventional treatment outlined in section 2 and to make the most of the low shear second stable inner region by avoiding infernal modes.

What limits β in the inner region? If ballooning and infernal modes are stabilised, a limit on the internal β_p will be imposed by equilibrium requirements. There is of course no limit to the total β within ideal mhd. However for the practical case with a *given plasma current* both β_p and β are limited by the formation of an X-point. We shall now consider the consequences of such a β-limitation.

Let us separate the plasma into an inner second stable region $0 < r < r_b$ and an outer first stable region $r_b < r < a$. Then

$$\beta = \frac{4\mu_o}{a^2B^2}\left[\int_o^{r_b} prdr + \int_{r_b}^{a} prdr\right]$$

$$= \frac{4\mu_o}{a^2B^2}\left[\int_o^{r_b} (p - p_b)rdr + \frac{\pi r_b^2}{2}p_b + \int_{r_b}^{a} prdr\right] \tag{13}$$

where $p_b = p(r_b)$. The last two terms in equation (13) give the conventional first regime β limit which, when the reduction due to the bootstrap current given by equation (12) is included, takes the form

$$\beta_l = c_l\frac{\varepsilon}{q_c}\left[1 - \frac{1}{4}(q_c - 1)^{\frac{1}{2}}\right]. \tag{14}$$

The first term in equation (13) gives the contribution from the second regime in the central region

$$\beta_2 = \frac{4\mu}{a^2B^2}\int_o^{r_b} (p - p_b)rdr,$$

so that

$$\beta = \beta_l + \beta_2, \tag{15}$$

the contribution of β_2 being limited by the total β available at the equilibrium limit. This limit is given by

$$\beta_p = c_e\frac{R}{a},$$

where c_e is around unity. The corresponding value of β is $(a/Rq_c)^2\beta_p$, so that

$$\beta = c_e \frac{\varepsilon}{q_c^2} \tag{16}$$

Substitution of equations (14) and (16) into equation (13) gives the scope for an enhancement from the second region of stability

$$\frac{\beta}{\beta_l} = \frac{c_e}{c_l} \frac{1}{q_c\left[1 - \frac{1}{4}(q_c - 1)^{\frac{1}{2}}\right]}.$$

SUMMARY

The economics of a steady state fusion reactor calls for high β to make efficient use of the magnetic field, and a high β_p for the bootstrap current to reduce the cost of current drive.

The original beta limit is restricted to the first stability regime and requires the current to be carried centrally. The pressure gradient drives a bootstrap current in the outer region reducing the β limit.

The β value can be improved by using the second stability regime in the central low shear region. Within this framework it is possible under some conditions to increase β and β_p until the equilibrium limit is reached. The total β limit falls with increasing q_c as does the fractional contribution of the second regime.

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