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P. Sandquist, S.E. Sharapov, P. Helander, M. Lisak
and JET-EFDA Contributors

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F.P. Sandquist[1], S.E. Sharapov[2], P. Helander[2], M. Lisak[1]
and JET-EFDA Contributors*

[1]*EURATOM-VR Association, and Department of Radio and Space Science,*
Chalmers University of Technology, SE-412 96 Göteborg, Sweden
[2]*EURATOM/UKAEA Fusion Association, Culham Science Centre, Abingdon, OX14 3DB, UK*
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ABSTRACT.
In recent sawtooth experiments at JET fast electron bremsstrahlung was found to be significantly enhanced in low-density JET plasmas with grassy sawteeth [1]. Similar observations were found in T−10 where the appearance of beams of suprathermal electrons accompanied the sawteeth [2]. Estimates for grassy sawteeth JET discharges show that the on-axis inductive electric field is close to the critical electric field corresponding to the minimum of the friction force on a relativistic electron, while the electric field induced by sawteeth in the reconnection region could approach the Dreicer electric field. Under such conditions, suprathermal electrons are generated at each sawtooth crash while a near-threshold inductive electric field prevents a rapid deceleration of these electrons. In this paper, a corrected relativistic collision operator is used to derive a Fokker- Planck equation for the distribution function of relativistic suprathermal electrons in a weakly relativistic plasma, $T_e / m_e c^2 \sim 10^{-2} << 1$, which is then solved by a procedure similar to that employed in [3] and [4]. Special emphasis is placed on a near-critical electric field case typical for plasmas with grassy sawteeth on JET. It is found that the main result concerning the runaway rate is still valid. In addition [4], new near-critical electric field regions are considered. In the weakly relativistic region a numerical solution enables matching to the high momentum analytical asymptote. The form of the electron velocity distribution function up to the relativistic region, where suprathermal electrons are present, is thus accurately described and can be used for investigations of fast electron bremsstrahlung from suprathermal electrons in hot tokamak plasmas.

1. INTRODUCTION

It has long been recognised that electric fields generated during reconnection events in magnetised plasmas may accelerate electrons and ions of the plasma. One type of these reconnection events, the sawtooth instability, induces electric fields, E_{saw}, which can accelerate electrons repeatedly at each sawtooth crash throughout the discharge. Suprathermal electrons with energies up to 20–100keV were recently observed during magnetic reconnection at sawtooth crashes in the T−10 tokamak [1]. Fast electron bremsstrahlung was also found to be significantly enhanced in low-density JET plasmas with short-period, chaotic sawteeth, so-called grassy sawteeth [2], [3]. Estimates show that the on-axis inductive electric field was close to the critical field, i.e. the critical field parameter $\alpha = E/E_c$ satisfies $\alpha = 1$ [2], [3] throughout these discharges, while electric fields induced by each sawtooth, E_{saw}, could in the reconnection layer be much higher, $E_{saw} >> E_c$. Here, the critical electric field is given by

$$E_c = \frac{n_e e^3 \ln \Lambda}{4\pi \epsilon_0^2 m_e c^2} \tag{1}$$

and standard notation has been used for thermal plasma quantities. Under such conditions, suprathermal electrons are generated at each sawtooth crash while a near-threshold inductive electric field prevents a rapid deceleration of these electrons.

To accurately describe suprathermal electrons in sawtoothing plasmas one would need to solve the time-dependent Fokker-Planck equation for the electron distribution function. The present paper reports how a general Fokker-Planck theory for the steadystate relativistic electron velocity distribution function of a plasma in a near-critical electric field has been developed.

2. GOVERNING EQUATION

The governing equation for a steady-state velocity distribution function *f* subjected to an external electric field E is the Fokker-Planck equation

$$-e\mathbf{E} \cdot \nabla_p f = C\,(f), \tag{2}$$

where *C*(*f*) is the collision operator, p is the (relativistic) momentum and e the absolute value of the electron charge. Previous authors [4] had a collision operator not conserving number of particles, as was pointed out by Karney and Fisch [5]. However, instead of using a symbolic algebra package as Karney and Fisch did, one can derive the relativistic electron collision operator analytically and the Fokker-Planck equation becomes

$$\begin{aligned} \alpha\left(\mu\frac{\delta f}{\delta q}+\frac{1+\mu^2}{q}\frac{\delta f}{\delta\mu}\right) &= \frac{\sqrt{q^3+1}}{2q^3}\left(1+Z-\in\frac{2q^2+1}{q^2\,(q^2+1)}\right)\frac{\delta}{\delta\mu}(1-\mu^2)\frac{\delta f}{\delta\mu} \\ &+\frac{1}{q^2}\left(q^2+1-\in\frac{(1-2q^2)\sqrt{q^2+1}}{q^2}\right)\frac{\delta f}{\delta q}+\in\frac{(q^2+1)^{3/2}}{q^3}\frac{\delta^2 f}{\delta q^2}+\frac{2f}{q}\,. \end{aligned} \tag{3}$$

Here, $\in = E_c/E_D = T_e/m_ec^2$ is the small expansion parameter satisfying $\in \gtrsim 10^{-2}$ in the grassy sawtooth discharges on JET [2], q = p/mec is the normalised momentum, *Z* is the effective charge number of the plasma ions and terms of order $O(\in^2)$ have been neglected. From this Fokker-Planck equation one can derive all the necessary analytical expressions for the steady-state relativistic electron distribution function.

3. SOLUTION OF THE FOKKER-PLANCK EQUATION

In a weakly relativistic plasma with $\in$ = 1, Eq. (3) can be solved using asymptotic techniques, devised by Kruskal and Bernstein [6], who first solved the corresponding nonrelativistic problem. As in their analysis as well as in [4], different asymptotic expansions must be used in five different regions of velocity space.

In region I, one considers two small quantities, $\in$ and q, and makes use of the nonrelativistic limit of the Fokker-Planck equation (3) to solve the expansion $f_\mathrm{I} = f_0 + \in f_1$. To leading order one obtains the Maxwellian distribution. The next order term, f_I, can be analytically obtained in two limiting cases but is not needed for the subsequent analysis.

The ordering in region I breaks down at $q^2 \sim \in^{1/2}$ and region II is considered. An expansion of $F = \ln f_{II}$ in $\sqrt{\in}$ is performed and the leading order term is Maxwellian. For the next order term, $F^{(1)} = u^4/8 + F'$, a high momenta analytical asymptote is obtained with a matching constant $b(\alpha, Z)$ which was previously left undetermined. A numerical algorithm, which essentially consists of first performing a Legendre polynomial decomposition of $h(u, \mu) = \exp(F')$ and then solving the parabolic equation as an initial value problem, can be formulated and used to obtain this matching constant for any plasma parameters. The matching at $\mu = 1$ is illustrated in Fig.1 (a), which depicts how the numerical solution evolves towards the high momenta analytical asymptote for large u. Figure 1 (b) illustrates the directionality of the distribution function in μ space.

In region III, the electrons are fully relativistic and the electron velocity distribution function starts to deviate strongly from the Maxwellian distribution. Moreover, region III either extends to infinity in q-space ($\alpha \leq 1$) in which case only exponentially few energetic electrons are generated, or its solution breaks down for $\alpha > 1$, and a region of runaway acceleration appears for $q > 1/\sqrt{\alpha > 1}$. One expands $F = \ln f_{III}$ in $\sqrt{\in}$ and the analysis for $\alpha > 1$ more or less reproduces the results of [4], whereas the analysis for $\alpha < 1$ presents new results. The analysis for $\alpha > 1$ breaks down in an α boundary layer of width $\delta\alpha = \alpha - 1 \sim \in^{1/2}$, which was previously not noticed. A first order perturbation solution, $\tilde{F} = F_0 + \delta\alpha F_1$, can be obtained in this boundary layer.

Returning to the case $\alpha > 1$ one considers a boundary layer at $q = q_c = 1/\sqrt{\alpha > 1}$, region IV. An inner variable x is introduced according to $q = q_c(1 + \in^{1/3} x)$ and the distribution function in region IV is then obtained by expanding $F = \ln f_{IV}$ in $\in^{1/3}$. The numerical scheme for determining the matching constant $b = b(\alpha, Z)$ between regions II and III enables a calculation of the runaway rate to order $O(1)$. Therefore, one takes the matching of region IV onto region III to order $O(1)$, whereas previous authors only took the matching to order $O(\ln \in)$ (since previously, $b \sim O(1)$ was unknown).

In order to find the runaway flux one finally considers the runaway region, region V. In a plasma with a slightly super-critical electric field, $0 < \alpha - 1 << 1$, the characteristic value of q in the runaway region satisfies $q > q_c = 1/\sqrt{\alpha - 1} >> 1$, and the distribution function fV can then be described analytically throughout region V. Requiring a beam-like solution, $q_{\|} >> q_{\perp}$ and $>> \delta/\delta q_{\|}$, one obtains the equation

$$(\alpha - 1) = \frac{\delta f_V}{\delta q_{\|}} \frac{(1 + Z)}{2} \frac{1}{q_{\perp}} \frac{\delta}{\delta q_{\perp}} q_{\perp} \frac{\delta f_V}{\delta q_{\perp}} + \frac{q_{\perp}}{q_{\|}} \frac{\delta f_V}{\delta q_{\perp}} + \frac{2 f_V}{q_{\|}} \cdot \tag{4}$$

Separation of variables can be undertaken once the variable transformation

$$\xi = \frac{q_{\perp}}{\sqrt{q_{\|}(1 + Z)/2}} \; ; \; \eta = \frac{q_{\|}(1 + Z)}{2} \tag{5}$$

has been performed. Writing $f_V = \Phi(\xi)\,\Psi(\eta)$, one obtains the equations

$$\frac{\delta^2 \Phi}{\delta \xi^2} + \left(\frac{1}{\xi} + \frac{\alpha + 1}{2} \xi \right) \frac{\delta \Phi}{\delta \xi} + C_s \Phi = 0, \tag{6}$$

$$\frac{1}{\Psi}\frac{\delta\Psi}{\delta\eta}+\frac{C_S-2}{\alpha-1}\frac{1}{\eta}=0, \tag{7}$$

where C_s is a separation constant. Equation (7) is easy to solve, and Eq. (6) can be transformed into the Kummer equation, which has the confluent hypergeometric function, ${}_1F_1$ $(a, b; x)$ (a, b parameters), as a solution. Finally, the solution in region V is given by

$$f_V=\frac{A}{q_{\|}^{(C_s-2)/(\alpha-1)}}\exp\left(-\frac{(\alpha+1)\,q_{\perp}^2}{2(1+Z)\,q_{\|}}\right){}_1F_1=\left(1-\frac{C_s}{\alpha+1},1;\frac{(\alpha+1)\,q_{\perp}^2}{2(1+Z)\,q_{\|}}\right), \tag{8}$$

where A is a source strength constant, related to the runaway electron source at the boundary layer, $q = q_c$. Equation (8) is a generalisation of Eq. (60) in [4], and is valid for all q in region V in the case of a near-critical electric field. A condition for $f_V \to 0$ as $q_{\|} \to \infty$ is that $C_s > 2$, which holds for this near-critical electric field case. Connor and Hastie [4] took the special case $C_s = \alpha + 1$. However, this choice of the separation constant precludes asymptotic matching and is therefore incorrect.

One finally determines the constants A and C_s in Eq. (8) to O(1) by matching the solution in region V to the region IV solution, after which the electron velocity distribution function is fully described in all five regions of momentum space.

4. RUNAWAY RATE

The region V solution, Eq. (8), which is valid for $\in^{1/2} \ll \alpha - 1 \ll 1$, gives the final expression for the runaway rate to $O(1)$ in $\in$. The scaling of the runaway rate with the electric field, given by Eq. (61) of Ref. [4], is reproduced to $O(\ln \in)$:

$$S_R \propto \exp\left(\frac{2}{\in}\left\{\sqrt{\alpha(\alpha-1)}-\alpha+\frac{1}{2}\right\}-\frac{1}{2}\sqrt{\frac{\alpha(1+Z)}{\in(\alpha-1)}}\left\{\frac{\pi}{2}-\sin^{-1}\left[1-\frac{2}{\alpha}\right]\right\}\right.$$
$$\left.-\frac{\ln\in}{16(\alpha-1)}\left\{\alpha(1+Z)-Z+7+2(1+Z)(\alpha-2)\sqrt{\frac{\alpha}{\alpha-1}}\right\}\right) \tag{9}$$

However, Connor and Hastie had a remaining unknown proportionality constant, $C_R(\alpha, Z) \sim O(1)$. The present analysis facilitates a determination of this proportionality constant. Table I presents results for typical near-critical electric field plasma parameters.

CONCLUSIONS

The present paper reports how the electron velocity distribution function of a plasma in a near-critical electric field is obtained by solving the Fokker-Planck equation in five regions of momentum space by an asymptotic expansion technique. In addition to previous analysis [4], the solutions are determined to O(1) in $\in$. This enables a determination of the previously unknown runaway rate proportionality constant.

The results can be used to investigate experimentally observed phenomena with a significant suprathermal electron population, e.g. enhanced bremsstrahlung from suprathermal electrons during grassy sawtooth discharges on JET [2], [3].

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α	Z = 1	Z = 2	Z = 3
1.3	11.2	1.67	0.400
1.4	12.2	4.01	1.67
1.5	10.8	5.74	3.47

Table I: The runaway rate proportionality constant CR(α ,Z) for plasmas with critical field parameters α = 1.3, 1.4 and 1.5, and with effective charge numbers Z = 1, 2 and 3.

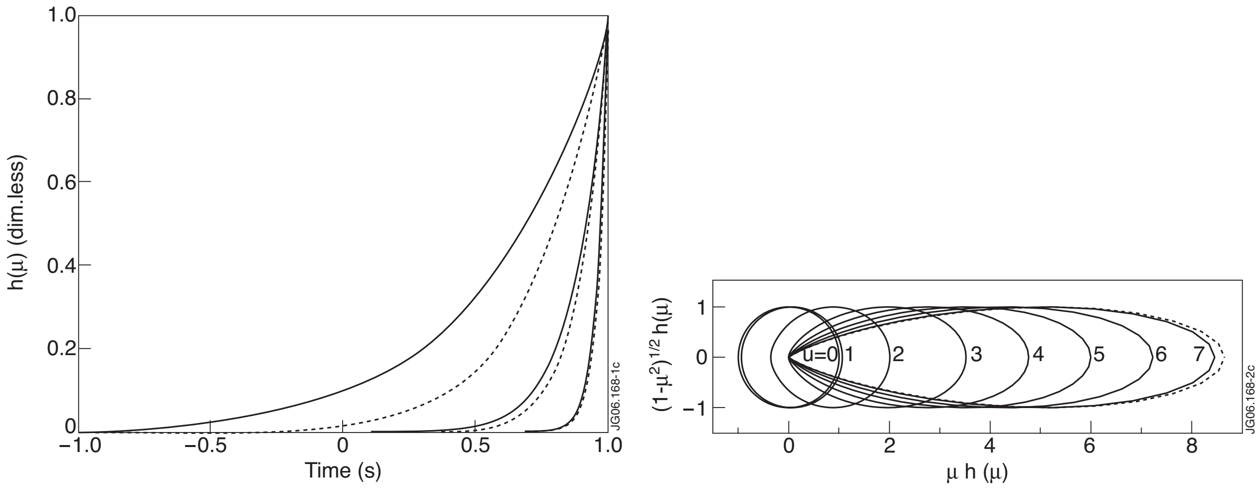

Figure 1: In (a) numerical solutions (solid lines) and high momenta analytic asymptotes (broken lines) in region II as functions of pitch angle μ for normalised momenta μ = [2.25, 4, 7] is depicted. The matching constant is b(α = 0.9, Z = 2.1) = 0.69 in this case. In (b) a polar plot of normalised function h(u, μ) for normalised momenta u = [0, 1, 2, 3, 4, 5, 6, 7] (solid lines) and the high momentum analytic asymptote for u = 7 (broken line) is illustrated. The plot shows how the low momenta solutions are isotropic as compared to the high directionality of the high momenta solutions and the analytic asymptote. The same plasma parameters as in (a) are used.